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More R-sequenceable groups

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Abstract

A group is called R-sequenceable if its nonidentity elements can be listed as $a_1, a_2, ..., a_{m-1}$ such that $a_1^{-1}a_2, a_2^{-1}a_3, ..., a_{m-2}^{-1}a_{m-1}, a_{m-1}^{-1}a_1$ are all distinct. It is called R^*-sequenceable if, further, $a_i = a_{i-1}a_{i+1} = a_{i+1}a_{i-1}$ for some i. In this paper, we prove that (1) the direct product of a cyclic group of order p, where p is a prime, with a cyclic group of order m, where m is any odd integer greater than 3, is R-sequenceable, and (2) the direct product of an R^*-sequenceable group G with an R^*-sequenceable group H of odd order is R^*-sequenceable, with the possible exception of the case when G is of odd order and the order of H is divisible by 3.

1 Introduction

Sequences in finite groups have been used for many combinatorial designs. R-sequenceability of groups was introduced by Paige [7] in 1951 when he observed that the R-sequenceability of a group is a sufficient condition for the group having a complete mapping. Ringel [8] came across the same concept in 1974 in his process of solving the Heawood map coloring problem. Friedlander et al. [3] studied the R-sequenceability of abelian groups in great depth and introduced following definition motivated by sequences used by Ringel.

Definition 1.1 A sequence $a_1, a_2, ..., a_{m-1}$ containing all non-identity elements of a finite group G of order m is called an R-sequencing of G if $a_1^{-1}a_2, a_2^{-1}a_3, ..., a_{m-2}^{-1}a_{m-1}$, $a_{m-1}^{-1}a_1$ are all distinct. A finite group G is called R-sequenceable if it has an R-sequencing.

An equivalent definition motivated by sequences used by Paige can be found in [6]. From now on, we are going to use additive notation when we discuss abelian groups. It was proved in [3] that the following types of abelian group are R-sequenceable: cyclic groups of order relatively prime to 6; abelian groups of odd order whose Sylow 3-sugroup is cyclic; elementary abelian p-groups except Z_2; abelian groups of type $G = Z_2 \bigoplus Z_{4k}$; abelian groups whose Sylow 2-subgroup is mZ_2 where $m \neq 3$ and

$m > 1$; abelian groups whose Sylow 2-subgroup S is $Z_2 \oplus Z_{2^k}$where k is odd or $k > 1$ is even and G/S has a direct cyclic summand of order congruent 2 modulo 3. Headley [4] proved that non-cyclic abelian 2-groups are R-sequenceable. The complete determination of R-sequenceable abelian groups is still an open question. For nonabelian groups, it was proved that dihedral groups of order $4n$ are R-sequenceable [5], dicyclic groups of order $8n$ are R-sequenceable for $n > 2$, and the nonabelian groups of order pq, where $p < q$ are primes, are R-sequenceable [9]. We refer to [6] for a survey on this topic.

As a variation of R-sequenceability, Beals et al. [1] introduced the following:

Definition 1.2. A group G of order m is called harmonious if its elements can be listed as $a_1, a_2, \ldots, a_{m-1}, a_m$ such that $a_1a_2, a_2a_3, \ldots, a_{m-1}a_m, a_ma_1$ are all elements of G. The sequence $a_1, a_2, \ldots, a_{m-1}, a_m$ is called a harmonious sequence.

Let $G^\sharp = G - \{e\}$, where e is the identity of G. Beals et al. [1] also discussed harmonious sequences in $G^\sharp$. It is convenient to call a harmonious sequence in $G^\sharp$ a $\sharp$-harmonious sequence of G, so we introduce the following:

Definition 1.3 A group G of order m is called $\sharp$-harmonious if all nonidentity elements of G can be listed as $a_1, a_2, \ldots, a_{m-1}$ such that elements $a_1a_2, a_2a_3, \ldots,$ $a_{m-2}a_{m-1}, a_{m-1}a_1$ are all nonidentity elements of G. The sequence $a_1, a_2, \ldots, a_{m-1}$is called a $\sharp$-harmonious sequence.

It was proved in [1] that an abelian group G having either a non-cyclic or trivial Sylow 2-subgroup is $\sharp$-harmonious unless $G = Z_2$.

2 The direct product of Z_p with Z_m

Lemma 2.1 *There is an R-sequencing $a_1, a_2, ..., a_{m-1}$ of Z_m for odd integers $m > 7$ such that $a_1 - 2a_{m-1} = 2$.*

Proof. In [3], Theorem 4, it was proved that Z_{2k+1} has an R-sequencing $a_1, a_2, \ldots,$ a_{m-1} which is given by the formula

$$a_i = \begin{cases} (i+1)/2 & \text{if } i \text{ is odd} \\ k+1-i/2 & \text{if } i \text{ is even} \end{cases}$$

$$a_{k+i} = a_{k-i+1} + k$$

for $i = 1, 2, ..., k$.

Case 1. $m = 12t + 1$. In this case, $a_{8t-1} = 10t$ and $a_{8t} = 20t + 2$. We rewrite the R-sequencing as $a_{8t}, a_{8t+1}, ..., a_{m-1}, a_1, a_2, \ldots, a_{8t-1}$.

Case 2. $m = 12t + 3$. In this case, $a_{4t-1} = 2t$ and $a_{4t} = 4t + 2$. We rewrite the R-sequencing as $a_{4t}, a_{4t+1}, \ldots, a_{m-1}, a_1, a_2, \ldots, a_{4t-1}$.

Case 3. $m = 12t + 5$. In this case, $a_{4t+2} = 4t + 2$ and $a_{4t+3} = 2t + 2$. We rewrite the R-sequencing as $-a_{4t+2}, -a_{4t+1}, \ldots, -a_2, -a_1, -a_{m-1}, -a_{m-2} \ldots, -a_{8t-1}$.

Case 4. $m = 12t + 7$. In this case, $a_{4t+3} = 2t + 2$ and $a_{4t+4} = 4t + 2$. We rewrite the R-sequencing as $-a_{4t+4}, -a_{4t+5}, \ldots, -a_{m-1}, -a_1, -a_2, \ldots, -a_{4t+3}$.

Case 5. $m = 12t + 9$. In this case, $a_{8t+7} = 10t + 8$ and $a_{8t+8} = 20t + 14$. We rewrite the R-sequencing as $-a_{8t+8}, -a_{8t+9}, \ldots, -a_{m-1}, -a_1, -a_2, \ldots, -a_{8t+7}$.

Case 6. $m = 12t + 11$. In this case, $a_{4t+4} = 4t + 4$ and $a_{4t+5} = 2t + 3$. We rewrite the R-sequencing as $-a_{4t+4}, -a_{4t+3}, \ldots, -a_2, -a_1, -a_{m-1}, -a_{m-2}, \ldots, -a_{4t+5}$. □

By Lemma 2.1, we see that Z_m also has an R-sequencing $a_1, a_2, ..., a_{m-1}$ such that $a_1 - 2a_{m-1} = -2$.

Lemma 2.2. *There is a ♯-harmonious sequence $b_1, b_2, ..., b_{m-1}$ of Z_m such that for every $b_i \neq \pm 2, \pm(2k-1)$ when $m = 4k + 1$, and for every $b_i \neq \pm 2, \pm(2k+1)$ when $m = 4k + 3$, we have $b_i - b_{i-1} = 2$ or $b_{i-1} - b_i = 2$.*

Proof. (Beals et al. [1]) The ♯-harmonious sequence of Z_m from [1] defined by

$$\begin{aligned} b_i &= 2k + 2i, \text{ for } i = 1, 2, \ldots, k \\ b_{k+i} &= 2i, \text{ for } i = 1, 2, \ldots, k \\ b_{2k+i} &= 2k - 2i + 1, \text{ for } i = 1, 2, \ldots, k \\ b_{3k+i} &= 4k - 2i + 1, \text{ for } i = 1, 2, \ldots, k \end{aligned}$$

if $m - 1 = 4k$, and as

$$\begin{aligned} b_i &= 2k + 2i, \text{ for } i = 1, 2, \ldots, k+1 \\ b_{k+1+i} &= 2i, \text{ for } i = 1, 2, \ldots, k \\ b_{2k+1+i} &= 2k - 2i + 3, \text{ for } i = 1, 2, \ldots, k+1 \\ b_{3k+2+i} &= 4k - 2i + 3, \text{ for } i = 1, 2, \ldots, k \end{aligned}$$

if $m - 1 = 4k + 2$, has the property described in the lemma. □

Lemma 2.3. *For any odd integer $m > 3, Z_m$ has an R-sequencing $a_1, a_2, \ldots, a_{m-1}$ and an #-harmonious sequence $b_1, b_2, \ldots, b_{m-1}$ such that $b_1 = a_{m-1}$ and $b_{m-1} = a_1 - a_{m-1}$*

Proof. When $m > 7$, by Lemma 2.1, we may construct an R-sequencing $a_1, a_2, \ldots, a_{m-1}$ such that the difference between a_1 and $2a_{m-1}$ is 2.

Case 1. $m = 12t + 1$. By the proof of Lemma 1, we may assume $a_{m-1} = 10t$ and $a_1 - a_{m-1} = 10t + 2$.

Case 2. $m = 12t + 3$. We may assume $a_{m-1} = 2t$ and $a_1 - a_{m-1} = 2t + 2$

Case 3. $m = 12t + 5$. We may assume $a_{m-1} = 10t + 3$ and $a_1 - a_{m-1} = 10t$

Case 4. $m = 12t + 7$. We may assume $a_{m-1} = 10t + 5$ and $a_1 - a_{m-1} = 10t + 7$.

Case 5. $m = 12t + 9$. We may assume $a_{m-1} = 2t + 1$ and $a_1 - a_{m-1} = 2t + 3$.

Case 6. $m = 12t + 11$. We may assume $a_{m-1} = 10t + 8$ and $a_1 - a_{m-1} = 10t + 10$.

In each case, we see that $a_{m-1} \neq \pm 2, \pm(2k-1)$ when $m = 4k + 1$, and $a_{m-1} \neq \pm 2, \pm(2k+1)$ when $m = 4k + 3$. Hence, a_{m-1} and $a_1 - a_{m-1}$ are two adjacent terms in the ♯-harmonious sequence given in Lemma 2.2. Hence we may rewrite the #-harmonious sequence beginning at a_{m-1} and ending with $a_1 - a_{m-1}$.

If $m = 5$, we have an R-sequencing 4,2,1,3 and ♯-harmonious sequence 3,4,2,1.

If $m = 7$, we have an R-sequencing $6, 4, 1, 3, 2, 5$ and $\sharp$-harmonious sequence $5, 4, 6, 2, 3, 1$. □

Now we are going to present the main result in this section.

Theorem 2.4 *The direct sum $Z_p \oplus Z_m$ is R-sequenceable for any odd prime p and any odd integer $m > 3$.*

Proof. By Cohen and Mullen [2], for any prime $p > 3$, there are two primitive roots α_1 and α_2 of p such that $\alpha_1 + \alpha_2 = 1$. If we write $\beta = 1/\alpha_1$,then $1/\alpha_2 = 1/(1-\alpha_1) = \beta/(\beta-1)$. Therefore both β and $\beta/(\beta-1)$ are primitive roots of p. For $p = 3$, we can take $\beta = 2$.

Let $a_1, a_2, ..., a_{m-1}$ be an R-sequencing of Z_m and let $b_1, b_2, ..., b_{m-1}$ be an $\sharp$-harmonious sequence of Z_m such that $b_1 = a_{m-1}$ and $b_{m-1} = a_1 - a_{m-1}$. We claim that the following sequence is an R-sequencing of $Z_p \oplus Z_m$:

$$(0,a_1),(0,a_2),\ldots,(0,a_{m-1}),(1,b_1),(\beta^{p-2},-b_1),(\beta^{p-3},b_1),$$
$$(\beta^{p-4},-b_1),\ldots,(\beta^2,b_1),(\beta,-b_1),(1,b_2),$$
$$(\beta^{p-2},-b_2),(\beta^{p-3},b_2),(\beta^{p-4},-b_2),\ldots,(\beta^2,b_2),(\beta,-b_2),$$
$$(1,b_3),\ldots,(\beta^{p-2},-b_{m-2}),(\beta^{p-3},b_{m-2}),(\beta^{p-4},-b_{m-2}),\ldots,$$
$$(\beta^2,b_{m-2}),(\beta,-b_{m-2}),(1,b_{m-1}),(\beta^{p-2},-b_{m-1}),(\beta^{p-3},b_{m-1}),$$
$$(\beta^{p-4},-b_{m-1}),\ldots,(\beta^2,b_{m-1}),(\beta,-b_{m-1}),$$
$$(\beta,0),(\frac{\beta^2}{(\beta-1)},0)(\frac{\beta^3}{(\beta-1)^2},0),\ldots,(\frac{\beta^{p-2}}{(\beta-1)^{p-3}},0),((\beta-1),0)$$

Since $\beta/(\beta-1)$ is a primitive root of p, so $\frac{\beta}{(\beta-1)}, \frac{\beta^2}{(\beta-1)^2}, \ldots, \frac{\beta^{p-2}}{(\beta-1)^{p-2}}$ consists of all nonzero elements in Z_p except 1. Hence, $\frac{\beta^2}{(\beta-1)}, \frac{\beta^3}{(\beta-1)^2}, \ldots, \frac{\beta^{p-2}}{(\beta-1)^{p-3}}, \frac{\beta^{p-1}}{(\beta-1)^{p-2}}, \beta$ are all nonzero elements in Z_p. Note that $\frac{\beta^{p-1}}{(\beta-1)^{p-2}} = \beta - 1$. Observing that $\beta, \beta^2, \ldots, \beta^{p-1} = 1$ are all nonzero elements in Z_p and the elements b_i for $i = 1, 2, \ldots, m-1$ are all nonzero elements in Z_m, we conclude the above sequence contains all nonzero elements of $Z_p \bigoplus Z_m$.

The differences $x_h - x_{h-1}$ between each element x_h and its predecessor in the above sequence are:

$$(1-\beta,a_1),(0,a_2-a_1),\ldots,(0,a_{m-1}-a_{m-2}),(1,0),$$

$$(\beta^{p-2}-1,-2b_1),(\beta^{p-3}-\beta^{p-2},2b_1),\ldots,(\beta-\beta^2,-2b_1),$$
$$(1-\beta,b_2+b_1),(\beta^{p-2}-1,-2b_2),(\beta^{p-3}-\beta^{p-2},2b_2),\ldots,(\beta-\beta^2,-2b_2),$$
$$(1-\beta,b_3+b_2),\ldots,(\beta^{p-2}-1,-b_{m-2}),(\beta^{p-3}-\beta^{p-2},2b_{m-2}),\ldots,$$
$$(\beta-\beta^2,-2b_{m-2}),(1-\beta,b_{m-1}+b_{m-2}),(\beta^{p-2}-1,-2b_{m-1}),$$

$$(\beta^{p-3}-\beta^{p-2}, 2b_{m-1}), \ldots, (\beta-\beta^2, -2b_{m-1}), (0, b_{m-1}),$$
$$(\frac{\beta^2}{\beta-1}-\beta, 0), (\frac{\beta^3}{(\beta-1)^2}-\frac{\beta^2}{\beta-1}, 0), \ldots, (\frac{\beta^{p-2}}{(\beta-1)^{p-3}}-\frac{\beta^{p-3}}{(\beta-1)^{p-4}}, 0), ((\beta-1)-\frac{\beta^{p-2}}{(\beta-1)^{p-3}}, 0).$$

Since $\frac{\beta^{i+1}}{(\beta-1)^i}-\frac{\beta^i}{(\beta-1)^{i-1}}=\frac{\beta^i}{(\beta-1)^i}$, so $\frac{\beta^{i+1}}{(\beta-1)^i}-\frac{\beta^i}{(\beta-1)^{i-1}}, i=1,2,\ldots,p-2$, are all nonzero elements in Z_m except 1. We conclude that all nonzero elements of $Z_p \bigoplus Z_m$ are contained in these differences by observing that

(1) $a_1=b_{m-1}+b_1$ and $b_{m-1}=a_1-a_{m-1}$;
(2) $b_i, i=1,\ldots,m-1$ is a $\sharp$-harmonious sequence of Z_m ;
(3) $a_i, i=1,\ldots,m-1$ is an R-sequencing of Z_m ;
(4) $2b_i, i=1,2,\ldots,m-1$ are all nonzero elements of Z_m

□

This theorem gives infinitely many abelian R-sequenceable groups with order of a multiple of three which are not included in the abelian R-sequenceable groups found in [3].

Example. For $p=7$ and $m=9$, we have $\beta=3$ and $\frac{\beta}{\beta-1}=5$ are two primitive roots of 7. In Z_9 we have an R-sequencing $4,8,5,7,6,2,3,1$ and a $\sharp$-harmonious sequence $1,7,5,6,8,2,4,3$ with $a_{m-1}=b_1=1$ and $b_{m-1}=3=4-1=a_1-a_8$. Hence, by the construction in the proof of the theorem we have an R-sequencing of $Z_7 \bigoplus Z_9$ as follows.

$$\begin{gathered}
(0,4),(0,8),(0,5),(0,7),(0,6),(0,2),(0,3),(0,1)\\
(1,1),(5,8),(4,1),(6,8),(2,1),(3,8),\\
(1,7),(5,2),(4,7),(6,2),(2,7),(3,2),\\
(1,5),(5,4),(4,5),(6,4),(2,5),(3,4),\\
(1,6),(5,3),(4,6),(6,3),(2,6),(3,3),\\
(1,8),(5,1),(4,8),(6,1),(2,8),(3,1),\\
(1,2),(5,7),(4,2),(6,7),(2,2),(3,7),\\
(1,4),(5,5),(4,4),(6,5),(2,4),(3,5),\\
(1,3),(5,6),(4,3),(6,6),(2,3),(3,6),\\
(3,0),(1,0),(5,0),(4,0),(6,0),(2,0).
\end{gathered}$$

3 A product theorem

Definition 3.1. An R-sequencing $a_1, a_2, \ldots, a_{m-1}$ of a group G is called an R^*-sequencing if $a_i=a_{i-1}a_{i+1}=a_{i+1}a_{i-1}$ for some i. A group possessing an R^*-sequencing is called R^*-sequenceable.

This concept was introduced in [3] for abelian groups in order to establish a direct product theorem. In [10] the R^*-sequenceability of nonabelian groups was introduced and discussed. In this section, we are going to consider the direct product of two R^*-sequenceable groups.

First we point out that the R-sequencing given in Lemma 2.1 is R^*-sequencing when $m = 12t+1$, $12t+3$, $12t+7$, $12t+9$. Suppose that the R-sequencing $a_1, a_2, \ldots, a_{m-1}$ in the proof of Theorem 1.4 is the R-sequencing given in Lemma 1. Then when $m = 12t+1$, we have three consecutive terms $(0, 2t), (0, 4t+1), (0, 2t+1)$. When $m = 12t+3$, there are three consecutive terms $(0, 10t+3), (0, 8t+2), (0, 10t+2)$. When $m = 12t+7$ there are three consecutive terms $(0, 2t+1), (0, 4t+3), (0, 2t+1)$. When $m = 12t+9$, there are three consecutive terms $(0, 10t+8), (0, 8t+6), (0, 10t+7)$. Hence the R-sequencings of $Z_p \bigoplus Z_m$ that we obtained in the proof of Theorem 2.4 are R^*-sequencings when $m = 12t+1$, $12t+3$, $12t+7$, $12t+9$.

R^*-sequenceability is stronger than R-sequenceability. For example, R-sequenceable groups $Z_5, Z_3 \bigoplus Z_3$ are not R^*-sequenceable and the quaternion group is R-sequenceable but not R^*-sequenceable. Technically, we may regard Z_3 as being R^*-sequenceable, because there are only two non-zero elements in Z_3 so the condition of R^*-sequencing cannot apply to it. Since neither Z_3 nor Z_5 has R^*-sequencing, we shall assume from now on that any reference to an R^*-sequenceable group of odd order is to a group of order greater than 5.

Theorem 3.2 *The direct product of an R^*-sequenceable group G of order m with an R^*-sequenceable groups H of odd order $2k+1$ is R^*-sequenceable with the possible exception of the case when m is odd and $2k+1$ is divisible by 3.*

Proof. Assume that $a_1, a_2, \ldots, a_{m-1}$ is an R^*-sequencing of G with $a_1 = a_2 a_{m-1} = a_{m-1} a_2$. We write a $m \times (2k+1)$ matrix A as follows where $2k+1$ is the order of H.

$$A = \begin{pmatrix} - & a_1 & a_1 & \ldots & a_1 & e & e & \ldots & e \\ a_1 & a_1 & a_1 & \ldots & a_1 & e & e & \ldots & e \\ a_2 & a_2 & a_2 & \ldots & a_2 & a_2 & a_2 & \ldots & a_2 \\ \ldots \\ a_{m-1} & a_{m-1} & a_{m-1} & \ldots & a_{m-1} & a_{m-1} & a_{m-1} & \ldots & a_{m-1} \end{pmatrix}$$

Assume that $b_1, b_2, \ldots, b_{2k}$ is an R^*-sequencing of H with $b_1 = b_2 b_{2k} = b_{2k} b_2$. If m is even, we write a matrix B as follows.

$$B = \begin{pmatrix} - & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ e & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} & b_1 & b_3 & \ldots & b_{2k-3} & b_{2k-1} \\ e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ e & b_3^{-1} & b_5^{-1} & \ldots & b_{2k-1}^{-1} & b_1^{-1} & b_2^{-1} & b_4^{-1} & \ldots & b_{2k-2}^{-1} & b_{2k}^{-1} \\ e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ \ldots \\ \ldots \\ e & b_3^{-1} & b_5^{-1} & \ldots & b_{2k-1}^{-1} & b_1^{-1} & b_2^{-1} & b_4^{-1} & \ldots & b_{2k-2}^{-1} & b_{2k}^{-1} \\ e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ b_2 & b_4 & h_6 & \ldots & b_{2k} & b_1 & b_3 & b_5 & \ldots & b_{2k-1} & e \end{pmatrix}$$

If m is odd, we just insert an extra row $e, b_3^2, b_5^2, \ldots, b_{2k-1}^2, b_1^2, b_2^2, b_4^2, \ldots, b_{2k-2}^2, b_{2k}^2$ and let

$$B = \begin{pmatrix} - & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ e & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} & b_1 & b_3 & \ldots & b_{2k-3} & b_{2k-1} \\ e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ e & b_3^{-1} & b_5^{-1} & \ldots & b_{2k-1}^{-1} & b_1^{-1} & b_2^{-1} & b_4^{-1} & \ldots & b_{2k-2}^{-1} & b_{2k}^{-1} \\ e & b_3^2 & b_5^2 & \ldots & b_{2k-1}^2 & b_1^2 & b_2^2 & b_4^2 & \ldots & b_{2k-2}^2 & b_{2k}^2 \\ e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ \ldots \\ \ldots \\ e & b_3^{-1} & b_5^{-1} & \ldots & b_{2k-1}^{-1} & b_1^{-1} & b_2^{-1} & b_4^{-1} & \ldots & b_{2k-2}^{-1} & b_{2k}^{-1} \\ e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\ b_2 & b_4 & h_6 & \ldots & b_{2k} & b_1 & b_3 & b_5 & \ldots & b_{2k-1} & e \end{pmatrix}$$

We claim that the sequence (x_i, y_i), where x_i and y_i are obtained by reading down the successive columns of matrices A and B respectively, is an R^*-sequencing of $G \times H$. Observe that (1) every row of B contains distinct elements of H, (2) every element of H appear exactly once among the entries of columns 1 to $k+1$ of the first two rows of B and these entries correspond to the entry a_1 in the matrix A, (3) every non-identity element of H appears exactly once among the entries from the $(k+2)$th column to the last column of the first two rows of B and these entries correspond to the entry e in the matrix A. Hence, when we look also at entries in the remaining rows of B, we see that the pairs (x_i, y_i) include all non-identity elements of $G \times H$.

Now we construct matrices A' and B' by assigning $a_{i-1,j}^{-1} a_{ij}$ (we use a_{ij} to denote the (i,j) entry of A) as the (i,j) entry of A' and $b_{i-1,j}^{-1} b_{ij}$ as the (i,j) entry of B'. Therefore,

$$A' = \begin{pmatrix} - & a_{m-1}^{-1}a_1 & \ldots & a_{m-1}^{-1}a_1 & a_{m-1}^{-1} & \ldots & a_{m-1}^{-1} \\ a_{m-1}^{-1}a_1 & e & \ldots & e & e & \ldots & e \\ a_1^{-1}a_2 & a_1^{-1}a_2 & \ldots & a_1^{-1}a_2 & a_2 & \ldots & a_2 \\ a_2^{-1}a_3 & a_2^{-1}a_3 & \ldots & a_2^{-1}a_3 & a_2^{-1}a_3 & \ldots & a_2^{-1}a_3 \\ \ldots \\ a_{m-2}^{-1}a_{m-1} & a_{m-2}^{-1}a_{m-1} & \ldots & a_{m-2}^{-1}a_{m-1} & a_{m-2}^{-1}a_{m-1} & \ldots & a_{m-2}^{-1}a_{m-1} \end{pmatrix}$$

Notice that $a_{m-1}^{-1}a_1 = a_2$ and $a_1^{-1}a_2 = a_{m-1}^{-1}$.

When m is odd, the first row of B' is (with the first entry empty)

$$b_2^{-1}b_3, b_4^{-1}b_5, \ldots, b_{2k-2}^{-1}b_{2k-1}, b_{2k}^{-1}b_1, b_1^{-1}b_2, b_3^{-1}b_4, \ldots, b_{2k-1}^{-1}b_{2k}.$$

Since $b_1, b_2, \ldots, b_{2k}$ is R^*-sequencing of H, so this row contains all nonidentity elements of H.

The second row of B' is

$$e, b_3^{-1}b_2, b_5^{-1}b_4, \ldots, b_{2k-1}^{-1}b_{2k-2}, b_1^{-1}b_{2k}, b_2^{-1}b_1, b_4^{-1}b_3, \ldots, b_{2k-2}^{-1}b_{2k-3}, b_{2k}^{-1}b_{2k-1}.$$

Since $b_{2k}, b_{2k-1}, \ldots, b_1$ is an R^*-sequencing of H, so the second row of B' contains all elements of H. The third row is the same as the first row except that its first entry is e. Notice that corresponding to $a_{m-1}^{-1}a_1$ in A' the entries in B' are all distinct, and corresponding to $a_1^{-1}a_2$ in A' the entries in B' are all distinct. The fourth row of B' is $e, b_3^{-2}, b_5^{-2}, \ldots, b_{2k-1}^{-2}, b_1^{-2}, b_2^{-2}, b_4^{-2}, \ldots, b_{2k-1}^{-2}, b_{2k}^{-2}$. This row contains all elements of H, because H is of odd order. The fifth row of B' is $e, b_3^3, b_5^3, \ldots, b_{2k-1}^3, b_1^3, b_2^3, b_4^3, \ldots, b_{2k-1}^3, b_{2k}^3$, which contains all elements of H, because when m is odd we assume the order of H is not divisible by 3.

The sixth row of B' is $e, b_3^{-1}, b_5^{-1}, \ldots, b_{2k-1}^{-1}, b_1^{-1}, b_2^{-1}, b_4^{-1}, \ldots, b_{2k-1}^{-1}, b_{2k}^{-1}$. Then from the seventh row, every row of odd order except the last row is

$$e, b_3^{-2}, b_5^{-2}, \ldots, b_{2k-1}^{-2}, b_1^{-2}, b_2^{-2}, b_4^{-2}, \ldots, b_{2k-1}^{-2}, b_{2k}^{-2}$$

and every row of even order is $e, b_3^2, b_5^2, \ldots, b_{2k-1}^2, b_1^2, b_2^2, b_4^2, \ldots, b_{2k-1}^2, b_{2k}^2$.

The last row of B' is $b_2, b_3^{-1}b_4, b_5^{-1}b_6, \ldots, b_{2k-1}^{-1}b_{2k}, e, b_2^{-1}b_3, b_4^{-1}b_5, \ldots, b_{2k-2}^{-1}b_{2k-1}, b_{2k}^{-1}$. Since $b_2 = b_{2k}^{-1}b_1$ and $b_{2k}^{-1} = b_1^{-1}b_2$, the last row contains all elements of H.

When m is even, the first three rows and the last row are the same as in the previous case. From the fourth row, every row of even order except the last row is $e, b_3^{-2}, b_5^{-2}, \ldots, b_{2k-1}^{-2}, b_1^{-2}, b_2^{-2}, b_4^{-2}, \ldots, b_{2k-1}^{-2}, b_{2k}^{-2}$ and every row of odd order is $e, b_3^2, b_5^2, \ldots, b_{2k-1}^2, b_1^2, b_2^2, b_4^2, \ldots, b_{2k-1}^2, b_{2k}^2$.

The sequence (x_i', y_i'), where x_i' and y_i' are obtained by reading down the successive columns of matrices $A'$and B' respectively, contains all nonidentity elements of $G \times H$. Therefore (x_i, y_i) is an R-sequencing. Notice that $(a_{m-1,}e), (a_1, e), (a_2, e)$ are three consecutive elements in the sequence, so (x_i, y_i) is an R^*-sequencing. □

The above theorem is parallel to a result in [11]. A group H of odd order $2k+1$ is called symmetric harmonious if it has a harmonious sequence $e, a_{1,}a_2, \ldots, a_k, a_{k+1}, \ldots, a_{2k}$ such that $a_i^{-1} = a_{k+i}$ for $i = 1, 2, ..., k$. It was proved in [11] that the direct product of an R*-sequenceable group G with a symmetric harmonious group H of odd order is R*-sequenceable with the possible exception of the case when the order of G is odd and the order of H is divisible by 3.

4 Remarks

The case when G is of odd order and the order of H is divisible by 3 is not covered by the above construction. In this case if we further assume that H is R_2-sequenceable (as defined below), then we show next that we may claim the same conclusion.

The following concept was introduced by Keedwell [5].

Definition 4.1. Suppose G is a group of order m. A sequence $a_1, a_2, ..., a_{m-1}$ of all nonidentity elements of G is called an R_h-sequencing if $a_i^{-1}a_{i+j}$, where $i = 1, 2, ..., m-1$, are distinct for any $1 \leq j \leq h$ (index arithmetic is modulo $m-1$). A group possessing R_h-sequencing is called R$_h$-sequenceable.

It was shown in [5] that the elementary abelian group of order p^n is R_h-sequenceable for $h = 1, 2, \ldots, p^{n-2}$.Except for this result, little is known on R_h-sequenceability for $h > 1$.

Now, with the same assumptions as for Theorem 3.2, we further assume $b_1, b_2, \ldots, b_{2k}$ is an R_2-sequencing. We keep the matrix A the same as in the proof of Theorem 3.2 and let:

$$B = \begin{pmatrix}
- & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\
e & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} & b_1 & b_3 & \ldots & b_{2k-3} & b_{2k-1} \\
e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\
e & b_1 & b_3 & \ldots & b_{2k-3} & b_{2k-1} & b_{2k} & b_2 & \ldots & b_{2k-4} & b_{2k-2} \\
e & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} & b_1 & b_3 & \ldots & b_{2k-3} & b_{2k-1} \\
e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\
\ldots \\
\ldots \\
e & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} & b_1 & b_3 & \ldots & b_{2k-3} & b_{2k-1} \\
e & b_3 & b_5 & \ldots & b_{2k-1} & b_1 & b_2 & b_4 & \ldots & b_{2k-2} & b_{2k} \\
b_2 & b_4 & h_6 & \ldots & b_{2k} & b_1 & b_3 & b_5 & \ldots & b_{2k-1} & e
\end{pmatrix}$$

From the fifth row, every row of odd order except the last row is the same as the second row and every row of even order is the same as the third row. We still assign $b_{i-1,j}^{-1}b_{ij}$ to be the (i, j) entry of B'. Then the first three rows and the last row are the same as in the proof of Theorem 3.2. The fourth row of B' becomes $e, b_3^{-1}b_1, b_5^{-1}b_3, ..., b_{2k-1}^{-1}b_{2k-3}, b_1^{-1}b_{2k-1}, b_2^{-1}b_{2k}, b_4^{-1}b_2, ..., b_{2k-2}^{-1}b_{2k-4}, b_{2k}^{-1}b_{2k-2}$. This row contains all elements of H because we assume $b_1, b_2, ..., b_{2k}$ is also an R_2-sequencing. The fifth row of B' becomes $e, b_1^{-1}b_2, b_3^{-1}b_4, ..., b_{2k-3}^{-1}b_{2k-2}, b_{2k-1}^{-1}b_{2k},$

$b_{2k}^{-1}b_1, b_2^{-1}b_3, ..., b_{2k-4}^{-1}b_{2k-3}, b_{2k-2}^{-1}b_{2k-1}$, which contains all elements of H. From the sixth row, every row of even order is the same as the third row and every row of odd order except the last row is the same as the second row. We can verify that the sequence (x'_i, y'_i), where x'_i and y'_i are obtained by reading down the successive columns of matrices A' and B' respectively, contains all nonidentity elements of $G \times H$. Hence we may state the following

Theorem 4.2. Suppose that G is R*-sequenceable of odd order and H of order $2k + 1$ has an R*-sequencing $b_1, b_2, .., b_{2k}$ which is also an R$_2$-sequencing, then $G \times H$ is R*-sequenceable.

However, to find an R$_2$-sequencing is very difficult. According to [5], Z_9 is not R$_2$-sequenceable while Z_{15} and Z_{21} are.

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