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A correction in the formula for the number of spanning trees in threshold graphs

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Abstract

A graph G is a threshold graph if, for all pairs of nodes u and v in G, $N(u) - v \subseteq N(v) - u$ whenever $\deg(u) \leq \deg(v)$. The 1996 paper by Hammer and Kelmans, "Laplacian spectra and spanning trees of threshold graphs" that appeared in Discrete Applied Math. 65 (1996), 255–273, presents a formula for the number of spanning trees that depends solely on the degree sequence. However, their formula does not work in all cases. We present a correction of this formula, as well as a demonstration of its equivalence to a formula that appeared in the 1985 doctoral thesis of Zbigniew Bogdanowicz.

1 Introduction

In this paper we consider only simple graphs. Let $G = \langle V(G), E(G)\rangle$ denote a graph on n nodes and e edges having node set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{x_1, x_2, ..., x_e\}$. For all graph theoretic terminology not described here, we refer to [3].

Computing the All-Terminal Reliability (ATR) of a probabilistic graph, where the nodes represent stations which are perfectly reliable, and the edges represent links which operate with equal and independent probabilities p, has been demonstrated to be intractable [1]. Thus, it is advantageous to bound the ATR of a network represented by a graph in a particular class $\Omega(n, e)$ (i.e., all graphs on n nodes and e edges). It has been conjectured that the best such lower bound would be provided by threshold graphs [7].

If $N(u)$ denotes the neighborhood of node u, then we have the following definition:

Definition 1.1 *(Threshold Graph) A graph G is a threshold graph if, for all pairs of nodes u and v in G, $N(u) - v \subseteq N(v) - u$ whenever deg$(u) \leqslant$ deg(v).*

Any graph can be transformed into a threshold graph by a finite sequence of applications of a reliability-reducing graph surgery, the swing surgery [2].

If a graph minimizes the number of spanning trees, it will minimize reliability for small values of p. To calculate the number of spanning trees of G, we first denote by $A(G)$, or A, the *adjacency matrix* of G, which has the rows (and columns) corresponding to the nodes, and entries

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between nodes } i \text{ and } j \\ 0, & \text{otherwise} \end{cases}$$

In general, for $i \neq j$, entry a_{ij} denotes the number of edges between nodes i and j. In addition, let $D(G)$ represent the diagonal matrix of the degrees of the nodes of G. We denote by $H(G)$, or H, the *Laplacian matrix* (also known as the nodal admittance matrix) of G. From $H(G) = D(G) - A(G)$, we see that:

$$h_{ij} = \begin{cases} \text{degree of node i}, & \text{if } i = j \\ -a_{ij}, & \text{if } i \neq j \end{cases}$$

Next, we state the well-known theorem of Kirchhoff.

Theorem 1.2 *(Kirchhoff's Matrix-Tree Theoerem) All cofactors of H are equal and their common value is the number of spanning trees.*

A well-known corollary states the relationship between the number of spanning trees of a graph and the eigenvalues of its nodal admittance matrix [5].

Corollary 1.3 *The value of $t(G)$, the number of spanning trees of a graph, is related to the eigenvalues of its nodal admittance matrix as follows:*

$$t(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i(G), 0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n.$$

An alternate equivalent description of a threshold graph is given in the next result.

Proposition 1.4 *A graph G on n nodes is a threshold graph if and only if the node set V of G is the disjoint union of sets U and W such that $\langle U \rangle = K_{n-c}, \langle W \rangle = cK_1$, and if $u, v \in W$ such that deg$(u) \leqslant$ deg(v), then $N(u) \subseteq N(v)$. In other words, G consists of a clique and an independent set of nodes (the so-called "cone points") such that the neighborhoods of the nodes of the independent set are contained in one another, i.e., they are "nested in the clique."*

A threshold graph on n nodes having c cones of degrees $q_1, q_2, ..., q_c$, where $q_i \geq q_{i+1}$ and clique K_{n-c} will be denoted $TG_{n;q_1,q_2,...,q_c}$. For example, the threshold graph on 7 nodes with clique K_5 and two cones having degrees 3 and 2 is denoted by $TG_{7;3,2}$ [Figure 1].

Other information about threshold graphs can be found in [6].

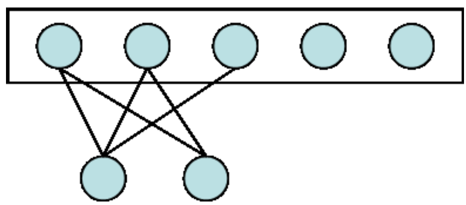

Figure 1: Threshold Graph $TG_{7;3,2}$. Note that the rectangular box containing 5 nodes is a clique.

2 The number of spanning trees in threshold graphs, formula 1

The 1996 paper by Hammer and Kelmans [4] asserts that threshold graphs have Laplacian spectra that closely resemble their degree sequence, i.e., the list of eigenvalues can be predicted from the degree sequence, and vice versa, in magnitude and multiplicity. Unfortunately, the formula they present to produce the eigenvalues given the degree sequence is not valid in all possible cases. Let s be the number of distinct terms in the degree sequence of a threshold graph, and $P(G, \lambda)$ represent the characteristic polynomial of a the Laplacian matrix of a graph G. Their theorem is as follows:

Theorem 2.1 (Theorem 5.4 in [4]) *Let G be a connected threshold graph, and let $\boldsymbol{d}(G) = (v_1^{(n_1)}, v_2^{(n_2)}, \ldots, v_s^{(n_s)})$ be the degree sequence of G, where $v_i \leqslant v_{i+1}$ for $i = 1, \ldots, s-1$, and n_i is the multiplicity of v_i in the degree sequence. Then*

$$P(\lambda, G) = (\prod_{i=1}^{k}(\lambda - v_i)^{n_i} \prod_{i=k+2}^{s} (\lambda - v_i - 1)^{n_i})(\lambda - v_{k+1} - 1)^{n_{k+1}-1}$$

and

$$t(G) = (\prod_{i=1}^{k}(v_i)^{n_i} \prod_{i=k+2}^{s-1} (v_i + 1)^{n_i})(v_{k+1} + 1)^{n_{k+1}-1}(v_s + 1)^{n_s-1}$$

where $k = \lfloor \frac{s-1}{2} \rfloor$.

Degree sequences of threshold graphs can usually be divided into two types of terms: those exclusively representing the cone points, and those exclusively representing the points of the clique. For $TG_{7;3,2}$ in Figure 1, the degree sequence can be expressed as $\mathbf{d}(TG_{7;3,2}) = 2^{(1)}, 3^{(1)}, 4^{(2)}, 5^{(1)}, 6^{(2)}$, which is defined as an *odd degree set* because it has an odd number of distinct terms, 5. The cone nodes have degrees 2 and 3, while the clique nodes have degrees 4, 5, 6.

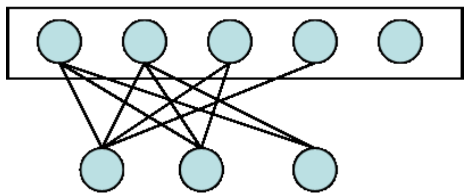

Figure 2: Threshold Graph $TG_{7;4,2}$.

Figure 2 shows the graph $TG_{7;4,2}$, another threshold graph on 7 nodes with clique K_5. Its degree sequence can be expressed as $\mathbf{d}(TG_{7;4,2}) = 2^{(1)}, 4^{(2)}, 5^{(2)}, 6^{(2)}$, which has four unique terms, making it an *even degree set.* The graph has two nodes of degree 4, one of which is a cone point and one of which is a clique point, thus making it impossible to classify all terms of the degree sequence as either one or the other. Whenever this ambiguity occurs, the threshold graph has an even degree set. The notion of odd and even degree sets is summarized in the following proposition:

Proposition 2.2 *(i) A threshold graph in which the terms of the degree sequence are distinguishable as either clique or cone degrees has an odd number of distinct terms.*

(ii) A threshold graph in which there is a single term indistinguishable as either clique or cone degree has an even number of unique terms in the degree sequence.

Proof. (i) Let there be l distinct values in the degree sequence that correspond to cone points. Each of these causes a different degree in the degree sequence among the clique points, so there are l distinct values in the degree sequence that correspond to clique points that are adjacent to cone points. In addition, there is exactly one term of the degree sequence that corresponds to clique points not adjacent to any cone points. Thus the number of distinct values in the sequence is equal to $l+l+1 = 2l+1$, which is necessarily odd.

(ii) Let there be l distinct values in the degree sequence that correspond to cone points. Since there is one cone degree that represents both cone and clique points, this clearly corresponds to a single clique point not adjacent to any cone points. This term in the degree sequence has already been counted in the cones point values. Once again, each of the cone point values causes a different degree in the degree sequence among the clique points, so there are l distinct terms in the degree sequence that correspond to clique points, and the number of distinct values in the sequence is given by $l + l = 2l$, which is necessarily even. ■

Proposition 2.2 is at the heart of where the spanning tree formula in Theorem 2.1 goes awry. The degree sequence of the threshold graph in Figure 3 is $\mathbf{d}(TG_{8;4,3,2}) =$

$2^{(1)}, 3^{(1)}, 4^{(2)}, 5^{(1)}, 6^{(1)}, 7^{(2)}$. According to the formula in Theorem 2.1, $k = \lfloor \frac{s-1}{2} \rfloor = \lfloor \frac{6-1}{2} \rfloor = \lfloor \frac{5}{2} \rfloor = 2$, and

$$P(\lambda, G) = (\prod_{i=1}^{k} (\lambda - v_i)^{n_i} \prod_{i=k+2}^{s} (\lambda - v_i - 1)^{n_i})(\lambda - v_{k+1} - 1)^{n_{k+1}-1} \Longrightarrow$$

$$\begin{aligned} P(\lambda, TG_{8;4,3,2}) &= (\prod_{i=1}^{2} (\lambda - v_i)^{n_i} \prod_{i=4}^{6} (\lambda - v_i - 1)^{n_i})(\lambda - v_3 - 1)^{n_3-1} \\ &= (\lambda - 2)^1 (\lambda - 3)^1 (\lambda - 6)^1 (\lambda - 7)^1 (\lambda - 8)^2 (\lambda - 5)^1. \end{aligned}$$

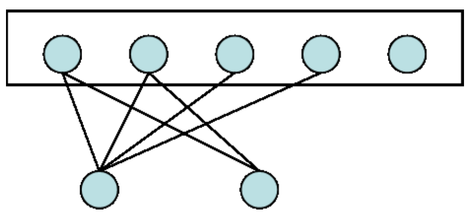

Figure 3: Threshold Graph $TG_{8;4,3,2}$.

Thus, the eigenvalues are, by this formula, $2, 3, 5, 6, 7, 8, 8$ (plus the zero eigenvalue, which does not factor into the spanning tree computations for connected graphs). But the sum of the eigenvalues, which is supposed to equal $2e = 2(19) = 38$, is 39, which would be impossible. In fact, direct calculation reveals that the eigenvalues are actually $0, 2, 3, 4, 6, 7, 8, 8$. This impacts their spanning tree formula as well, because the number of spanning trees for the graph would be 10,080 by this formula, but by direct application of Corollary 1.2 to the actual eigenvalues, the number of spanning trees is 8064.

The eigenvalue and spanning tree formulas for even degree sets should read as follows:

$$P(\lambda, G) = (\prod_{i=1}^{k} (\lambda - v_i)^{n_i} \prod_{i=k+2}^{s} (\lambda - v_i - 1)^{n_i})(\lambda - v_{k+1})^{n_{k+1}-1}$$

and

$$t(G) = (\prod_{i=1}^{k} (v_i)^{n_i} \prod_{i=k+2}^{s-1} (v_i + 1)^{n_i})(v_{k+1})^{n_{k+1}-1}(v_s + 1)^{n_s-1}$$

where $k = \lfloor \frac{s-1}{2} \rfloor$. In other words, the $(v_{k+1} + 1)$ factor in the formulas in Theorem 2.1 is actually (v_{k+1}) in the even case.

Application of these formulas to the degree sequence of $TG_{8;4,3,2}$ yields the same non-zero eigenvalues as direct computation. The formulas in Theorem 2.1 are valid for threshold graphs with odd degree sets. Consider once again the threshold graph in Figure 1. It has degree sequence $\mathbf{d}(TG_{7;3,2}) = 2^{(1)}, 3^{(1)}, 4^{(2)}, 5^{(1)}, 6^{(2)}$, and, by the formula of Kelmans and Hammer, non-zero eigenvalues $2, 3, 5, 6, 7, 7$. This matches the eigenvalues obtained by direct calculation. Application of Corollary 1.2 indicates that there are 1260 spanning trees for this graph. The spanning tree formula of Kelmans and Hammer yields the same value. We present the extended version of their theorem to now handle even degree sets:

Theorem 2.3 (Corrected Hammer-Kelmans Formula (CHK)) *Let G be a connected threshold graph, and let $\boldsymbol{d}(G) = (v_1^{(n_1)}, v_2^{(n_2)}, ..., v_s^{(n_s)})$ be the degree sequence of G, where $v_i \leqslant v_{i+1}$ for $i = 1, ..., s-1$, and n_i is the multiplicity of v_i in the degree sequence, with $k = \lfloor \frac{s-1}{2} \rfloor$. Then, for s odd,*

$$P(\lambda, G) = (\prod_{i=1}^{k}(\lambda - v_i)^{n_i} \prod_{i=k+2}^{s}(\lambda - v_i - 1)^{n_i})(\lambda - v_{k+1} - 1)^{n_{k+1}-1}$$

and

$$t(G) = (\prod_{i=1}^{k}(v_i)^{n_i} \prod_{i=k+2}^{s-1}(v_i + 1)^{n_i})(v_{k+1} + 1)^{n_{k+1}-1}(v_s + 1)^{n_s - 1}.$$

For s even,

$$P(\lambda, G) = (\prod_{i=1}^{k}(\lambda - v_i)^{n_i} \prod_{i=k+2}^{s}(\lambda - v_i - 1)^{n_i})(\lambda - v_{k+1})^{n_{k+1}-1}$$

and

$$t(G) = (\prod_{i=1}^{k}(v_i)^{n_i} \prod_{i=k+2}^{s-1}(v_i + 1)^{n_i})(v_{k+1})^{n_{k+1}-1}(v_s + 1)^{n_s - 1}.$$

3 Number of spanning trees in threshold graphs, formula 2

In his 1985 doctoral thesis, "Spanning trees in undirected simple graphs," Zbigniew Bogdanowicz used Kirchhoff's matrix tree theorem to develop a formula for the number of spanning trees for threshold graphs. As demonstrated above, threshold graphs have very predictable spectra, based on the total number of nodes in the graph as well as the number of cone points. In Bogdanowicz's formula, the cone degrees $q_i, 1 \leqslant i \leqslant c$, are arranged in non-increasing order. His formula is summarized in the next theorem:

Theorem 3.1 (Bogdanowicz, 1985) *For $TG_{n;q_1,q_2,...,q_c}$, the number of spanning trees is given by the formula*

$$t(TG_{n;q_1,q_2,...,q_c}) = q_c n^{q_c - 1}(n-c)^{n-q_1-c-1} \prod_{i=1}^{c-1} q_i(n-c+i)^{q_i - q_{i+1}}$$

.

We note that the Nodal Admittance Matrix for a general threshold graph will take on the form of this matrix:

$$H(TG_{n;q_1,q_2,\ldots,q_c}) = \begin{bmatrix} A & * & * \\ * & B & * \\ * & * & C \end{bmatrix}$$

where block A corresponds to clique nodes that are not adjacent to any cones, block B corresponds to those clique nodes with cone adjacencies, and block C corresponds to cone nodes. The full proof can be found in Lemma 4.1 of Bogdanowicz's doctoral thesis [8]. It involves performing a battery of 13 separate row and column operations on a cofactor of the matrix $H(TG_{n;q_1,q_2,\ldots,q_c})$ derived by eliminating the row and column corresponding to the cone of degree q_c. With judicious application of factorization, he obtains the determinant, which, by Theorem 1.2, yields the formula

$$t(TG_{n;q_1,q_2,\ldots,q_c}) = q_c n^{q_c-1}(n-c)^{n-q_1-c-1}\prod_{i=1}^{c-1} q_i(n-c+i)^{q_i-q_{i+1}}.$$

We remark that this formula holds in the case of the complete graph as well, if one considers a complete graph to be a threshold graph with one cone of degree $n-1$ having multiplicity 1. Thus, the value of q_1 is $n-1$, and $c=1$. The formula yields $t(G) = n^{n-2}$, which is the well-known formula for the number of spanning trees for the complete graph. The individual elements of Bogdanowicz's spanning tree formula are equivalent to those in the CHK formula. In the sequel, we show that the CHK formula is equivalent to the Bogdanowicz formula, term-by-term.

Recall that in the CHK formula, the degree sequence is arranged in increasing order, and in Bogdanowicz's formula, the individual cone degrees arranged in non-increasing order. In these comparisons, the parts under consideration are underlined.

1. CHK: for s odd, $t(G) = (\underline{\prod_{i=1}^{k}(v_i)^{n_i}} \prod_{i=k+2}^{s-1}(v_i+1)^{n_i})(v_{k+1}+1)^{n_{k+1}-1}(v_s+1)^{n_s-1}$.

For s even, $t(G) = (\underline{\prod_{i=1}^{k}(v_i)^{n_i}} \prod_{i=k+2}^{s-1}(v_i+1)^{n_i})(v_{k+1})^{n_{k+1}-1}(v_s+1)^{n_s-1}$.

Bogdanowicz: $t(G) = \underline{q_c} n^{q_c-1}(n-c)^{n-q_1-c-1}\prod_{i=1}^{c-1} \underline{q_i}(n-c+i)^{q_i-q_{i+1}}$.

For odd sets, these represent the product of the degrees of all the cones. For even sets, the cone of largest degree is not included in this term of the CHK formula.

2. CHK: for s odd, $t(G) = (\prod_{i=1}^{k}(v_i)^{n_i} \prod_{i=k+2}^{s-1}(v_i+1)^{n_i})(v_{k+1}+1)^{n_{k+1}-1}\underline{(v_s+1)^{n_s-1}}$.

For s even, $t(G) = (\prod_{i=1}^{k}(v_i)^{n_i} \prod_{i=k+2}^{s-1}(v_i+1)^{n_i})(v_{k+1})^{n_{k+1}-1}\underline{(v_s+1)^{n_s-1}}$.

Bogdanowicz:

$t(G) = q_c\underline{n^{q_c-1}}(n-c)^{n-q_1-c-1}\prod_{i=1}^{c-1} q_i(n-c+i)^{q_i-q_{i+1}}$.
In CHK,$(v_s+1) = n$.

The exponents in both formulas correspond to one less than the smallest cone degree, which is the same as one less than the number of full nodes.

3. CHK: for s odd, $t(G) = (\prod_{i=1}^{k}(v_i)^{n_i}\underline{\prod_{i=k+2}^{s-1}(v_i+1)^{n_i}})(v_{k+1}+1)^{n_{k+1}-1}(v_s+1)^{n_s-1}$.

For s even, $t(G) = (\prod_{i=1}^{k}(v_i)^{n_i}\underline{\prod_{i=k+2}^{s-1}(v_i+1)^{n_i}})(v_{k+1})^{n_{k+1}-1}(v_s+1)^{n_s-1}$.

Bogdanowicz:
$t(G) = q_c n^{q_c-1}(n-c)^{n-q_1-c-1}\prod_{i=1}^{c-1} q_i\underline{(n-c+i)^{q_i-q_{i+1}}}$.

Both formulas represent the product of 1 plus the degree of each clique node that is adjacent to cones, except for the full nodes.

4.(a) CHK: for s odd, $t(G) = (\prod_{i=1}^{k}(v_i)^{n_i}\prod_{i=k+2}^{s-1}(v_i+1)^{n_i})\underline{(v_{k+1}+1)^{n_{k+1}-1}}(v_s+1)^{n_s-1}$.

Bogdanowicz:
$t(G) = q_c n^{q_c-1}\underline{(n-c)^{n-q_1-c-1}}\prod_{i=1}^{c-1} q_i(n-c+i)^{q_i-q_{i+1}}$.

For odd sets, these formulas represent one more than the degree of the clique nodes of smallest degree, raised to one less than their multiplicity.

4.(b) CHK: For s even,
$t(G) = (\prod_{i=1}^{k}(v_i)^{n_i}\prod_{i=k+2}^{s-1}(v_i+1)^{n_i})\underline{(v_{k+1})^{n_{k+1}-1}}(v_s+1)^{n_s-1}$.
Bogdanowicz:
$t(G) = q_c n^{q_c-1}\underline{(n-c)^{n-q_1-c-1}}\prod_{i=1}^{c-1} q_i(n-c+i)^{q_i-q_{i+1}}$.

For even sets, Bogdanowicz's formula produces a term raised to the zero power, i.e., no contribution to the product. The CHK formula produces a term representing the shared cone and clique degree. This was not counted as a cone in the $\prod_{i=1}^{k}(v_i)^{n_i}$ term.

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