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On certain 2-rotational cycle systems of complete graphs

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Abstract

We exhibit 2-rotational k-cycle systems of K_v for all v, k satisfying the two necessary conditions $3k/2 \leq v$, $2k|v(v-1)$, and the two further conditions $v < 3k$, $\mathrm{GCD}((v-1)/2, k) = 2$. For our purposes we extend the notions of *partial difference* and *type* to the 2-rotational case. The required terminology, as well as the basic properties and techniques, still survive if the two last conditions are dropped. Therefore, the present method is expected to yield 2-rotational k-cycle systems of K_v for other pairs (v, k) in the near future.

1 Introduction

A *k-cycle system* of a graph $G = (V, E)$, also known as a (G, C_k)-*design*, is a set of k-cycles whose edges partition E. Several authors have been so far engaged in studying particular types of cycle systems, such as the ones which partition complete graphs. Remarkably, the existence question for k-cycle systems of K_v was settled by Alspach and Gavlas [1] in the case of k odd (see also [7]) and by Šajna [24] in the even case. Many challenging existence problems arise by further requiring that cycles be preserved by certain automorphisms of the vertices. For example, regarding V as $\mathbf{Z}_n$ for some fixed n naturally leads to consider the translation group, that is $\mathbf{Z}_n$ itself; any cycle system preserved by such automorphism group is usually termed *cyclic*. Exhaustive results on cyclic cycle systems have been recently achieved in the case $v \equiv 1 \pmod{2k}$ [3, 4, 9, 10] and $v \equiv k \pmod{2k}$ [8, 25], whereas some pioneering works in this field are [15, 17, 21, 22, 23] and a very recent paper is [11].

Some authors have also analysed cycle systems preserved by more sophisticated automorphism groups, such as the 1-*rotational* and 2-*rotational* systems. In the former case, the vertex set of K_v is regarded as $\mathbf{Z}_{v-1} \cup \{\infty\}$ equipped with the automorphism group $\mathbf{Z}_{v-1}$, fixing ∞ and acting as the translation group on the complement. In the latter case, the vertex set of K_v is instead regarded as $(\mathbf{Z}_{(v-1)/2} \times$

$\mathbf{Z}_2) \cup \{\infty\}$ while the related group is $\mathbf{Z}_{(v-1)/2}$, fixing ∞ and acting as the translation group on each of the two copies isomorphic to $\mathbf{Z}_{(v-1)/2}$, namely $\mathbf{Z}_{(v-1)/2} \times \{0\}$ and $\mathbf{Z}_{(v-1)/2} \times \{1\}$. In this case, all cycles decomposing K_v must be transformed into cycles under the above action of $\mathbf{Z}_{(v-1)/2}$.

The main object of study in this paper is a particular class of 2-rotational (K_v, C_k)-designs with k even. In details, we prove the existence of such systems for all v, k satisfying $3k/2 \le v$, $2k|v(v-1)$ (these conditions being necessary for the 2-rotational designs under examination – see in particular the lines preceding Property 2.5) and with the further property that $v < 3k$ and $\mathrm{GCD}((v-1)/2, k) = 2$. This work is hoped to be the first step towards the construction of a larger class of 2-rotational (K_v, C_k)-designs with k even.

We recall that the study of 1-rotational (K_v, C_k)-designs traces back to the 25-year-old paper [18] by Phelps and Rosa. These authors completely settled the existence problem for $k = 3$. Instead, the solution of Kirkman's schoolgirl problem (dated 1844, published in 1971 [19]) provided the first example of a 2-rotational 3-cycle system. In both cases, cycles were regarded as block designs of size 3. Recent papers such as [6, 7] have focused on the 1-rotational case with $k \ge 4$.

2 Partial differences in the 2-rotational case

In the sequel we assume that v, k are integers such that $v \ge k \ge 3$ and v is odd, the last hypothesis obviously being a necessary condition for the existence of a cycle system of a complete graph. The method of partial differences can be suited with few difficulties to the 2-rotational context. In this section we provide some basic definitions and related properties.

Notation 2.1. The generic vertex set of a 2-rotational (K_v, C_k)-design is denoted by $(\mathbf{Z}_{(v-1)/2} \times \mathbf{Z}_2) \cup \{\infty\}$. From now on vertices will be assumed to belong to the above set. Every element of the form $(z, 0)$ [resp. $(z, 1)$] will be shortly denoted by z [resp. $\langle z \rangle$]. For any $a, \langle b \rangle$ we extend the standard $\pm$ operation by introducing the symbol $\uparrow$ and defining $a - \langle b \rangle$ as $\uparrow (a - b)$. Further, we introduce the symbol $\downarrow$ and postulate that $- \uparrow z = \downarrow -z$ for all $z \in \mathbf{Z}_{(v-1)/2}$.

Notice that the above notation allows to deduce that

$$\downarrow (a-b) = \downarrow (-(b-a)) = - \uparrow (b-a) = -(b - \langle a \rangle) = \langle a \rangle - b\,.$$

For this reason, the equality $\downarrow (a - b) = \langle a \rangle - b$ could alternatively be postulated in place of the equality $- \uparrow z = \downarrow -z$, which would then be turned from an axiom into an easy consequence of the two postulates. We now proceed to adapt a well-known notion to the present context.

Definition 2.2. The *type* of a cycle $B = (b_0, b_1, ..., b_{k-1})$ is the cardinality of the stabiliser of B with respect to the action $\diamond$ of $\mathbf{Z}_{(v-1)/2}$ over K_v, defined by $z \diamond w = z + w$, $z \diamond \langle w \rangle = \langle z + w \rangle$, $z \diamond \infty = \infty$) for every $z \in \mathbf{Z}_{(v-1)/2}$.

We recall that a straightforward necessary condition for the existence of a (K_v, C_k)-design with k even is that $2k|v(v-1)$. In particular, $4|v-1$.

Lemma 2.3. *Let $\mathcal{D}$ be a 2-rotational (K_v, C_k)-design with k even, and let $B \in \mathcal{D}$ be the unique cycle containing the edge $\{a, a+(v-1)/4\}$ for some fixed $a \in \mathbf{Z}_{(v-1)/2}$. Then B has type 2 and, by regarding this cycle as a regular polygon, $(v-1)/4$ acts on B as the reflection whose axis passes through the above edge. Furthermore, the edge opposing $\{a, a+(v-1)/4\}$ is of the form $\{\langle b\rangle, \langle b+(v-1)/4\rangle\}$.*

Proof. Because the action of $(v-1)/4$ on $\{a, a+(v-1)/4\}$ leaves that edge unchanged, the stabiliser of B must contain $(v-1)/4$, and $(v-1)/4$ acts precisely as the claimed reflection. It easily follows that no other adjacent vertices of B, except for the two opposing the above edge, can be of the form $\{w, (v-1)/4 \diamond w\}$ with $w \neq \infty$. For that reason, any further (nontrivial) element of the stabiliser should interchange the above edge and its opposed. Anyway, this is not possible, for the unique candidate w for such an action should yield the identity if applied twice. That is, $w = (v-1)/4$, a contradiction. Finally, if the edge opposing $\{a, a+(v-1)/4\}$ was again defined in $\mathbf{Z}_{(v-1)/2}$, then it would be equal to $y \diamond \{a, a+(v-1)/4\}$ for some $y \in \mathbf{Z}_{(v-1)/2}$, with $y \neq (v-1)/4$, whence the stabiliser size would be greater than 2. ☐

Cycles satisfying the hypothesis of Lemma 2.3 will be termed *involution cycles*. Now we extend the concept of partial difference to the 2-rotational environment. To this end we essentially utilise the already existing definition, with the prescription that any difference involving ∞ be left as it formally appears, thus without using the absorption rule $\infty + a = a$.

Definition 2.4. If $B = (b_0, b_1, \ldots, b_{k-1})$ is a non-involution k-cycle of type d, the *list of partial differences from B* is the multiset $\partial B = \{\pm(b_{i+1} - b_i) \colon 0 \le i < k/d\}$, where $b_k = b_0$. If B is an involution cycle with $b_0 - b_{k-1} = (v-1)/4$ (equivalently, with $b_{k/2} - b_{k/2-1} = \langle (v-1)/4\rangle$) then $\partial B = \{\pm(b_{i+1} - b_i) \colon 0 \le i \le k/2 - 2\} \cup \{(v-1)/4, \langle (v-1)/4\rangle\}$. More generally, if $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ is a set of k-cycles, the *list of partial differences from $\mathcal{F}$* is the (multi)set $\partial\mathcal{F} = \bigcup_i \partial S_i$.

Similarly to what happens with cyclic or 1-rotational cycle systems, a sufficient condition for obtaining a 2-rotational (K_v, C_k)-design is the existence of a set of k-cycles $\mathcal{F}$ such that $\partial\mathcal{F} = \{\pm x, \pm\langle x\rangle \colon 1 \le x \le (v-5)/4\} \cup \{(v-1)/4, \langle (v-1)/4\rangle\} \cup \{\pm\uparrow x, \pm\downarrow x \colon 1 \le x \le (v-5)/4\} \cup \{\pm\uparrow 0, \pm\uparrow (v-1)/4\} \cup \{\pm(x-\infty), \pm(\langle y\rangle - \infty)\ \exists! x, y\}$ without repetitions (we leave to the reader the easy proof of the claim). In simple terms, $\partial\mathcal{F}$ must not be a multiset (cycles satisfying even only this condition are termed *starter cycles*) and the cycles altogether must generate *all* possible partial differences. Whenever such a set of starter cycles is available, we call it a *2-rotational difference system* and denote respectively by S_∞, S_I the cycle passing through ∞ (of type 1 and necessarily unique) and the cycle generating the differences $(v-1)/4, \langle (v-1)/4\rangle$. Notice that the remaining starter cycles must generate as many partial differences as $(4\cdot(v-5)/4 + 2 + 4\cdot(v-5)/4 + 4 + 4) - 2k - k = 2v - 3k$. Since S_∞ and S_I do occur in any case, by the above calculation we instantly deduce the following.

Property 2.5. *In every 2-rotational (K_v, C_k)-design, $v \geq 3k/2$.*

It could be shown with little difficulty that any 2-rotational starter cycle $S = (s_0, s_1, ..., s_{k-1})$ of type $d > 2$ is characterised by the following properties.

1) ∂S is not a multiset.

2) $s_i \not\equiv s_j \pmod{(v-1)/(2d)}$ if $0 \leq i < j < k/d$. Instead, $s_{k/d} = (j(v-1)/2d) \diamond s_0$ for some j coprime to d.

3) $s_t = (qj(v-1)/2d) \diamond s_r$ for every t, where $t = (k/d)q + r$ according to the Euclidean division.

Any such starter cycle can be shortened to $[s_0, s_1, ..., s_{k/d}]_d$ with no loss of information. The case $d = 2$ requires some more care because there are two ways of realising a cycle of type 2, namely using either a rotation of 180 degrees or a reflection. The former case leads to the same properties as above, whereas the latter (namely, an involution cycle) has the following characterisation (cf. Definition 2.4).

S can be written as $(\sigma_0, ..., \sigma_{k-1})$ in such a way that

1') $\{\pm(\sigma_{i+1} - \sigma_i): 0 \leq i \leq k/2 - 2\}$ is not a multiset.

2') $\sigma_i \not\equiv \sigma_j \pmod{(v-1)/4}$ if $0 \leq i < j \leq k/2 - 1$.

3') $\sigma_t = (v-1)/4 \diamond \sigma_{k-t-1}$ if $k/2 \leq t \leq k-1$.

Any involution cycle will be shortly denoted by $[\sigma_0, \sigma_1, ..., \sigma_{k/2-1}]_I^v$. Although in the present paper we will utilise either type 1 or involution cycles only, it is worth remarking that the two characterisations hold for every 2-rotational k-cycle system with k even.

3 The recipe

Devising a 2-rotational (K_v, C_k)-design with k even is greatly facilitated by the assumption $\mathrm{GCD}((v-1)/2, k) = 2$. Anyway, the ad hoc construction we present relies on a general property, still true when the GCD constraint is removed.

Property 3.1. *For every 2-rotational (K_v, C_k)-design with k even, $2v - 3k$ is a multiple of $2k/\mathrm{GCD}((v-1)/2, k)$.*

Proof. Denoting such GCD by m, let us write k and $(v-1)/2$ as me and mf respectively. Since $k|vmf$ and $\mathrm{GCD}(e, f) = 1$, we obtain that $e|v$ and, in particular, that e is odd. Consequently $2e|k$, and the conclusion follows. □

We have now all the ingredients for establishing the main result.

Theorem 3.2. *Let k be an even integer greater than 2. There exists a 2-rotational (K_v, C_k)-design for all pairs (v, k) satisfying $3k/2 \leq v$, $2k|v(v-1)$, and such that $v < 3k$ and GCD$((v-1)/2, k) = 2$.*

Proof. Due to the GCD constraint, Property 3.1 yields either $v = 3k/2$ or $v = 5k/2$. In the former case we have that $v \equiv 1 \pmod 4$, whence $k/2 \equiv 3 \pmod 4$ and eventually $k = 8q + 6, v = 12q + 9$ for some non-negative integer q. As $2v - 3k = 0$, the related difference system consists of the only cycles S_I, S_∞. If $q > 0$ we define them as follows (the reader should pay more careful attention to bold numbers).

S_I : $[\ 3q+2, 1, 3q+1, 2, \ \dots\ , 2q+2, q+1, \langle q+1\rangle, \langle \mathbf{2q+3}\rangle, \langle q\rangle, \langle 2q+4\rangle, \ \dots\ , \langle 2\rangle, \langle 3q+2\rangle, \langle 1\rangle\ ]_I^v$.

If q is even,

S_∞ : $(\ \infty, 0, 1, -1, 2, -2, \ \dots\ , q/2, -q/2, \langle 5q/2+2\rangle, -q/2-1, \langle 5q/2+3\rangle, -q/2-2, \ \dots\ , \langle 4q+2\rangle, -2q-1, \langle \mathbf{4q+4}\rangle, -2q-2, \langle 4q+5\rangle, -2q-3, \ \dots\ , \langle 11q/2+3\rangle, -7q/2-1, \langle 11q/2+4\rangle, \langle \mathbf{q/2+1}\rangle, \langle 11q/2+5\rangle, \langle q/2\rangle, \ \dots\ , \langle 6q+3\rangle, \langle 2\rangle, \langle 0\rangle, \langle 1\rangle\ )$.

If q is odd,

S_∞ : $(\ \infty, 0, 1, -1, 2, -2, \ \dots\ , (-q+1)/2, (q+1)/2, \langle (7q+5)/2\rangle, (q+3)/2, \langle (7q+3)/2\rangle, (q+5)/2, \ \dots\ , 2q+1, \langle 2q+2\rangle, \mathbf{2q+3}, \langle 2q+1\rangle, 2q+4, \langle 2q\rangle, \ \dots\ , (7q+5)/2, \langle (q+3)/2\rangle, \langle (-q+1)/2\rangle, \langle (q+1)/2\rangle, \langle (-q+3)/2\rangle, \ \dots\ , \langle 2\rangle, \langle 0\rangle, \langle 1\rangle\ )$.

In both cases, $\partial S_I \bigcup \partial S_\infty = (\{\pm x, \pm\langle x\rangle : q+2 \leq x \leq 3q+1\} \cup \{\pm(q+1), 3q+2, \langle 3q+2\rangle, \pm\uparrow 0\}) \bigcup (\{\pm x, \pm\langle x\rangle : 1 \leq x \leq q\} \cup \{\pm\uparrow x, \pm\downarrow x : 1 \leq x \leq 3q+1\} \cup \{\pm(0-\infty), \pm(\langle 1\rangle - \infty), \pm\langle q+1\rangle, \pm\uparrow(3q+2)\})$. Thus, the above cycles make up a difference system for every choice of $q > 0$. Finally, it can be easily checked that $[2, 1, \langle 1\rangle]_I^9$ and $(\infty, 0, \langle 1\rangle, -1, \langle 2\rangle, \langle 3\rangle)$ are suitable starter cycles when $k = 6, v = 9$.

Now we handle the case $v = 5k/2$. As $k/2 \equiv 1 \pmod 4$, there exists some positive integer q such that $k = 8q+2$ and $v = 20q+5$. Noting that $2v - 3k = 2k$, it suffices to construct one starter cycle of type 1 besides S_I and S_∞. By defining the required cycle as

S : $(\ 0, \langle 1\rangle, -1, \langle 2\rangle, \ \dots\ , -2q, \langle 2q+1\rangle, \mathbf{6q+1}, \langle 2q+2\rangle, 6q, \langle 2q+3\rangle, 6q-1, \ \dots\ , \langle 4q\rangle, 4q+2, \mathbf{4q+1}\ )$,

the related set of partial differences is $\partial S = \{\pm\uparrow x, \pm\downarrow x : 2 \leq x \leq 4q\} \cup \{\pm 1, \pm(4q+1), \pm\downarrow 1, \pm\downarrow(4q+1)\}$. We proceed to construct the remaining two cycles.

If q is even,

S_I : $[\ 5q+1, 1, 5q, 2, \ \dots\ , 9q/2+3, q/2-1, 9q/2+2, q/2, \mathbf{9q/2}, q/2+1, 9q/2-1, \ \dots\ , 4q+2, q-1, 4q+1, q, \langle q\rangle, \langle 4q+1\rangle, \langle q-1\rangle, \langle 4q+2\rangle, \ \dots\ , \langle 1\rangle, \langle 5q\rangle, \langle 0\rangle\ ]_I^v$,

S_∞ : $(\ \infty, 0, 2, -1, 3, \ \dots\ , -3q/2+1, 3q/2+1, \langle 13q/2+2\rangle, 3q/2+2, \langle 13q/2+1\rangle, 3q/2+3, \ \dots\ , \langle 6q+3\rangle, 2q+1, \langle \mathbf{-2q}\rangle, 2q+2, \langle -2q-1\rangle, 2q+3, \ \dots\ , 5q/2, \langle -5q/2+1\rangle, 5q/2+1, \langle \mathbf{5q/2}\rangle, \langle -q/2\rangle, \langle 5q/2-1\rangle, \langle -q/2+1\rangle, \ \dots\ , \langle q-1\rangle, \langle q\rangle\ )$.

If q is odd,

S_I : $[\ 5q+1, 1, 5q, 2, \ \dots\ , (q-3)/2, (9q+5)/2, (q-1)/2, (9q+3)/2, (\mathbf{q+3})/\mathbf{2}, (9q+1)/2, (q+5)/2, \ \dots\ , 4q+3, q, 4q+2, q+1, \langle q+1\rangle, \langle 4q+2\rangle, \langle q\rangle, \langle 4q+3\rangle, \ \dots\ , \langle 2\rangle, \langle 5q+1\rangle, \langle 1\rangle\ ]_I^v$,

S_∞ : $(\ \infty, 0, 2, -1, 3, \ \dots\ , (3q+1)/2, -(3q-1)/2, \langle (7q+3)/2\rangle, -(3q+1)/2, \langle (7q/2+5)/2\rangle, -(3q+3)/2, \ \dots\ , \langle 4q+1\rangle, -2q, \langle \mathbf{2q+2}\rangle, -2q-1, \langle 2q+3\rangle, -2q-2, \ \dots\ , \langle (5q+1)/2\rangle, -(5q-1)/2, \langle -(\mathbf{5q+1})/\mathbf{2}\rangle, \langle -(11q+1)/2\rangle, \langle -(5q+3)/2\rangle, \langle -(11q-1)/2\rangle, \dots, \langle -4q\rangle, \langle -4q-1\rangle)$.

In both cases we have that $\partial S_I = \{\pm x, \pm\langle x\rangle : 3q+1 \leq x \leq 5q\} \setminus \{\pm(4q+1)\} \cup \{5q+1, \langle 5q+1\rangle, \pm\uparrow 0\}$, whereas $\partial S_\infty = \{\pm x, \pm\langle x\rangle : 2 \leq x \leq 3q\} \cup \{\pm\uparrow x, \pm\downarrow x : 4q+2 \leq x \leq 5q\} \cup \{\pm(0-\infty), \pm(\langle z\rangle - \infty), \pm\langle 1\rangle, \pm\uparrow 1, \pm\uparrow(4q+1), \pm\uparrow(5q+1)\}$, with $z = \langle q+1\rangle$ if q is even, $z = \langle -4q-1\rangle$ otherwise. Therefore, we have again obtained a difference system. □

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