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A note on decompositions of transitive tournaments II

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Abstract

For any positive integer n, we find all disconnected digraphs G of size at most four, such that a transitive tournament of order n is G-decomposable. It follows that the obvious necessary condition of divisibility is not always sufficient.

1 Introduction

This paper continues [3], where we investigated decompositions of transitive tournaments into small connected digraphs. In particular, we found graphs that decompose complete graphs, but no orientation of them decomposes transitive tournaments.

We start with some definitions and notation. Let G be a digraph of order n with the vertex set $V(G)$ and the arc set $E(G)$. The *outdegree* of a vertex $v \in V(G)$ is denoted by $d^+(v)$, and its *indegree* by $d^-(v)$. A *reverse of a digraph* G is the digraph $\overleftarrow{G}$ obtained from G by diverting each arc $(u,v) \in E(G)$ into $(v,u) \in E(\overleftarrow{G})$. An *oriented graph* is a digraph without directed cycles of length two. Replacing of every arc (u,v) in an oriented graph G by an edge uv yields the *underlying graph* Γ of G, and then G is called an *orientation* of Γ.

A *tournament* of order n is an orientation of a complete graph K_n. A transitive tournament of order n will be denoted by TT_n. Since TT_n is unique up to isomorphism, throughout the paper we will view it as shown in Figure 1. Namely, $V(TT_n) = \{1, ..., n\}$ and $E(TT_n) = \{(i, j) : 1 \leq i < j \leq n\}$. The vertices 1, 2 and n will be called the *first*, the *second* and the *last vertex* of TT_n, respectively.

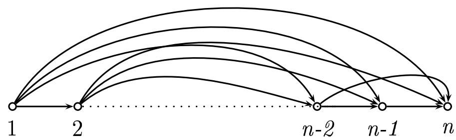

Figure 1: The transitive tournament TT_n

We say that a digraph H *can be decomposed into* a digraph G (or H *is* G*-decomposable*, for short), if there exists a partition of the arc set $E(H)$ into pairwise disjoint subsets each of which creates a subgraph isomorphic to G. An obvious necessary condition for the existence of a G-decomposition of H is the divisibility of $|E(H)|$ by $|E(G)|$.

In [3], we solved the following problem: for any positive integer n, determine all connected digraphs G of size at most four such that the transitive tournament TT_n is G-decomposable. In the present paper, we solve the same problem for disconnected digraphs.

If a digraph H can be decomposed into a digraph G then, clearly, the underlying graph of H can be decomposed into the underlying graph of G. In [3], we found examples of graphs Γ that decompose complete graphs K_n while the transitive tournament TT_n cannot be decomposed into any orientation of Γ. The smallest such example is a path P_3 of length two that decomposes K_n for any $n \equiv 0$ or 1 (mod 4). This shows that the decomposition problems for transitive tournaments are more delicate than those for complete graphs.

If TT_n is G-decomposable then G is an acyclic oriented graph since TT_n is acyclic.

We shall make use of some lemmas proved in [3].

Lemma 1 [3] *A transitive tournament TT_n is G-decomposable if and only if TT_n is $\overleftarrow{G}$-decomposable.* ■

Lemma 2 [3] *If for every arc $(u, v) \in E(G)$, at least one of its vertices u, v has both the outdegree and the indegree positive, then TT_n is not G-decomposable.* ■

Lemma 3 [3] *Assume that every subgraph of TT_n isomorphic to a digraph G has at most two arcs in the set*

$$F = \{(i, j) \in E(TT_n) : i \leq \frac{n}{2} < j\}.$$

If TT_n can be decomposed into G, then the number of copies of G in a decomposition cannot be smaller than $\lceil \frac{n^2-1}{8} \rceil$. ■

Another lemma is a modification of an observation proved in [3].

Lemma 4 *If a digraph G has two vertices x, y with $d^+(x) = d^+(y) = 2$ and $d^-(x) = d^-(y) = 0$, while all other vertices have outdegree zero, then no transitive tournament can be decomposed into G.*

Proof. If TT_n was decomposable into G, then in every copy of G the first vertex of TT_n would coincide with either x or y. It follows that the degree $n-1$ of every vertex in TT_n would be even. In the decomposition, there would be a unique copy of G containing the arc $(1, 2)$ of TT_n. In all other copies of G, the second vertex of TT_n could not be different than x or y. Thus $n-1$ would be odd. This yields a contradiction. ■

The following result of Alon is widely known in graph theory.

Theorem 5 [1] *A complete graph K_n can be decomposed into a matching of size p if and only if $\binom{n}{2} = pk$ for some integer $k \geq n-1$.*

By an *oriented matching* we mean an oriented graph, in which every weakly connected component is a single arc. The proposition below follows immediately from the above theorem of Alon.

Proposition 6 *The transitive tournament TT_n can be decomposed into an oriented matching of size m if and only if $m \leq \frac{n}{2}$ and $n \equiv 0$ or $1 \pmod{2m}$.* ■

This implies that every transitive tournament TT_n of even size, i.e. for $n \equiv 0$ or $1 \pmod 4$, can be decomposed into the unique disconnected digraph of size two, that is an oriented matching of size two.

2 Decomposition into disconnected digraphs of size three

Theorem 7 *Every disconnected oriented graph of size three decomposes the transitive tournament TT_n of order $n \equiv 0$ or $1 \pmod 3$, $\quad n \geq 6$.*

Proof. Clearly, $n \equiv 0$ or $1 \pmod 3$ is the necessary condition for decomposability of TT_n into any digraph of size three. Moreover, n cannot be smaller than five since each disconnected digraph of size three has at least five vertices.

Up to isomorphism, there are four disconnected oriented graphs of order three: an oriented matching and three digraphs $L1, L2$ and $L3$ depicted in Figure 2. As Proposition 6 settles the decomposability of TT_n into oriented matchings, and $L3 = \overleftarrow{L2}$, it suffices to consider only $L1$ and $L2$.

We label the vertices of $L1$ and $L2$ with x, y, z, t, u in such a way that $L1$ has arcs $(x, y), (y, z), (t, u)$, and $L2$ has arcs $(x, y), (z, y), (t, u)$. As in [3], we will use the following notation to represent a decomposition of TT_n into a digraph G with a vertex set $\{x, y, z, t, u\}$. Namely, every copy of G in TT_n will be described by a sequence

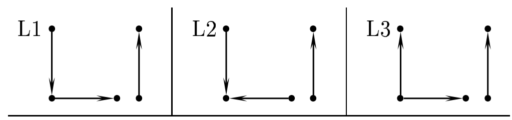

Figure 2: Disconnected digraphs of size three

$(x'y'z't'u')$ of five positive integers that indicate vertices of TT_n corresponding to x, y, z, t and u, respectively.

Using this convention, decompositions of the transitive tournaments TT_6, TT_7, TT_9 and TT_{10} into $L1$ can be described as follows.

TT_6 : (23516), (15624), (14625), (13645), (12634);

TT_7 : (12637), (13627), (14725), (16725), (23457), (24615), (35617);

TT_9 : (12458), (13859), (14528), (15734), (16829), (18946), (23947), (25619), (26917), (27935), (36748), (37849);

TT_{10} : (12930), (13640), (14720), (15878), (16089), (17869), (18067), (23890), (24610), (25037), (34928), (35619), (36829), (45927), (45927) — here 0 stands for the last vertex 10 of TT_{10}.

To show that TT_n is $L1$-decomposable for greater n, we distinguish four cases (mod 6). First, let $n = 6k$ with $k \geq 2$. Divide the set $V(TT_{6k}) = \{1, \ldots, 6k\}$ into $2k$ triples

$$V_i = \{i, 4k - i + 1, 6k - i + 1\}, \qquad i = 1, \ldots, 2k.$$

Further, define $W_l = V_{2l-1} \cup V_{2l}$, for $l = 1, \ldots, 3k$. In TT_n, each set W_l induces TT_6, and we have just proved its $L1$-decomposability. If (a, b) is an arc of TT_{6k}, then either both vertices a, b belong to the same set W_l for some l, or $a \in V_i$, $b \in V_j$ with V_i and V_j contained in different sets W_l. Hence, to show that $L1$ decomposes TT_{6k}, it suffices to show that, for each $i < j$, an oriented bipartite graph H_{ij} with the vertex set $V_i \cup V_j$ and the arc set $V_i \times V_j$ is $L1$-decomposable. To make the description more clear, denote the vertices of V_i and V_j by $a = i, b = 4k - i + 1, c = 6k - i + 1$, and $\breve{a} = j, \breve{b} = 4k - j + 1, \breve{c} = 6k - j + 1$. These integers are ordered $a < \breve{a} < \breve{b} < b < \breve{c} < c$, since $i < j$. Then an $L1$-decomposition of H_{ij} is given by a set of sequences: $(a\breve{c}c\breve{b}b), (a\breve{a}cc\breve{c}), (a\breve{b}c\breve{a}b)$.

If $n = 6k + 3$, $k \geq 2$, we argue in the same way, with the only exception as we have $2k + 1$ triples V_i and the last set W_{3k} is defined as $W_{3k} = V_{3k-1} \cup V_{3k} \cup V_{3k+1}$, whence it induces TT_9.

For $n = 6k + 1$, $k \geq 2$, we consider the same partition of the set

$$V(TT_n) \setminus \{n\} = \{1, \ldots, 6k\}$$

into subsets V_i $(1 \leq i \leq 2k)$ and W_l $(1 \leq l \leq k)$ as in the case $n = 6k$. Observe that, for each $l = 1, \ldots, k$, the set $W_l \cup \{n\}$ induces TT_7 that is $L1$-decomposable.

The claim that TT_{6k+1} can be decomposed into $L1$ follows from $L1$-decomposability of the oriented bipartite graph H_{ij} with $i < j$.

In case $n = 6k + 4,\ k \geq 2$, we only modify the method of gluing sets V_i into W_l by putting $W_{3k} = V_{3k-1} \cup V_{3k} \cup V_{3k+1}$. Thus, $W_{3k} \cup \{6k + 4\}$ induces TT_{10}. The rest is the same as in the previous case.

To prove that $L2$ decomposes TT_n we proceed in an analogous way as for $L1$. First, we find $L2$-decompositions of transitive tournaments TT_6, TT_7, TT_9, and TT_{10}.

TT_6 : (14326), (15236), (16325), (13245), (46512);

TT_7 : (15246), (16327), (24137), (45312), (56213), (57423), (67134);

TT_9 : (14326), (15236), (16325), (17329), (18239), (19328), (47659), (48569), (49658), (13245), (46578), (79812);

TT_{10} : (13279), (14257), (15234), (16258), (17235), (18256), (19245), (10278), (36490), (37480), (39470), (30467), (48612), (59638), (50689) - here 0 indicates the vertex 10.

For greater n, we consider four cases. Let $n = 6k$. Partition the set $V(TT_{6k})$ into $2k$ triples $V_i = \{3i - 2, 3i - 1, 3i\}, i = 1, \dots, 2k$, and define $W_l = V_{2l-1} \cup V_{2l}$, for $l = 1, \dots, k$. Each set W_l induces TT_6 which is $L2$-decomposable. For $i < j$, consider the oriented bipartite graph H_{ij} with $V(H_{ij}) = V_i \cup V_j$ and $E(H_{ij}) = V_i \times V_j$. Denote $a = 3i - 2, b = 3i - 1, c = 3i$, and $\breve{a} = 3j - 2, \breve{b} = 3j - 1, \breve{c} = 3j$. Clearly, $a < b < c < \breve{a} < \breve{b} < \breve{c}$. The oriented graph H_{ij} has the following decomposition into $L2 : (a\breve{a}cb\breve{c}), (a\breve{b}bc\breve{c}), (a\breve{c}cb\breve{b})$.

In cases $n \equiv 1, 3$ and 4 (mod 6), we use the same partitions of sets $V(TT_n)$ or $V(TT_n) \setminus \{n\}$ into subsets V_i and W_l, as we did for the digraph $L1$. Then, analogously, we show that the arc set of TT_n is a union of pairwise disjoint $L2$-decomposable parts that are isomorphic either to TT_6, TT_7, TT_9, TT_{10}, or to an orientation H_{ij} of the complete bipartite graph $K_{3,3}$. ■

3 Decomposition into disconnected digraphs of size four

Theorem 8 *Let G be a disconnected digraph of size four. Then TT_n is G-decomposable if and only if $n \equiv 0$ or 1 (mod 8), $n \geq 8$, and either G or $\overleftarrow{G}$ is isomorphic to one of the following ten digraphs: $G1$, $G2$, $G3$, $G4$, $G5$, $G8$, $G9$, $G11$, $G12$, $G13$ shown in Figure 3.*

Proof. The congruence $n \equiv 0$ or 1 (mod 8) is equivalent to the obvious necessary condition for decomposability of TT_n into any digraph of size four.

Up to isomorphism and reverse, there are 13 disconnected acyclic oriented graphs of size four (cf. Figure 3). We shall first show that three of them, namely $G6$, $G7$ and $G10$, do not decompose any transitive tournament.

The number c of copies of an oriented graph G of size four in any decomposition of TT_n into G equals $c = \frac{1}{8}(n^2 - n)$. The oriented graph **G6** satisfies the assumptions

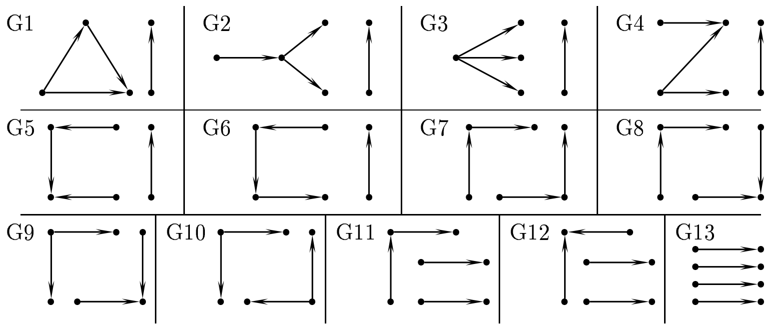

Figure 3: Disconnected acyclic digraphs of size four

of Lemma 3, hence $c \geq \lceil \frac{1}{8}(n^2-1) \rceil$. If $n = 8k$, for some $k \in I\!N$, then

$$\frac{1}{8}(n^2-n) = k(8k-1) < 8k^2 = \lceil \frac{1}{8}(n^2-1) \rceil.$$

Also for $n = 8k+1$ we have a contradiction:

$$\frac{1}{8}(n^2-n) = k(8k+1) < k(8k+2) = \lceil \frac{1}{8}(n^2-1) \rceil.$$

Any transitive tournament cannot be decomposed into **G7** by Lemma 2, and into **G10** by Lemma 4.

We have to show now that all ten remaining digraphs in Figure 3 decompose TT_{8k} and TT_{8k+1} for any $k \in I\!N$. This is immediate for the oriented matching **G13**, due to Proposition 6.

To show that TT_n can be decomposed into **G1** it is enough to decompose the complete graph K_n into the underlying graph $\Gamma = K_3 \cup K_2$ of $G1$. Indeed, if we orient the edges of K_n to obtain a transitive tournament TT_n, then each copy of Γ becomes $G1$. Consider first the case $n = 8k$. View K_{8k} in the following way: vertices $1, \ldots, 8k-1$ are situated equidistantly on a circle, and the last vertex n in a center. We shall describe every copy of Γ in K_{8k} by a sequence of five numbers (representing vertices of K_{8k}) such that the first three of them induce a triangle K_3, and the remaining two induce an edge K_2. At the beginning, we locate k copies of Γ in K_{8k}. For $i = 1, \ldots, k-1$, the i-th copy is the sequence

$$\Gamma^i = (1, 4k-i+1, 2k+i, i+1, 2k-i+1),$$

and the k-th copy is $\Gamma^k = (1, 3k+1, 3k, k+1, 8k)$. Exactly one edge of these copies is a radius of the circle, and all remaining ones are chords (cf. Figure 4). It is easy to observe that each cord is of different length. Hence, rotations of the circle on its center with the angle $\frac{2\pi}{8k-1}j$, $j = 1, \ldots, 8k-2$, produce the remaining $k(8k-2)$ copies of Γ in the decomposition. More precisely, consider a permutation $\sigma = (1 \quad \ldots \quad 8k-1)(n)$

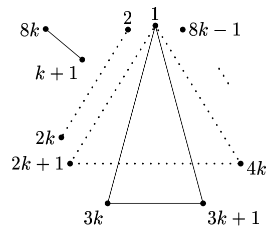

Figure 4: The k initial copies of $K_3 \cup K_2$ in K_{8k}

of the vertices of K_{8k}. Then the remaining copies can be defined as $\Gamma^i_j = \sigma^j(\Gamma^i)$ for $i = 1, \dots, k$ and $j = 1, \dots, 8k - 2$.

For $n = 8k + 1$, all the vertices of K_{8k+1} are situated on a circle (cf. Figure 5). Define the first k copies of Γ as follows:

$$\Gamma^i = (1, 4k - i + 2, 2k + i + 1, 8k - i + 1, 6k + i - 1),$$

for $i = 1, ..., k$. The remaining copies are obtained by rotations, as for $n = 8k$. Thus, $\Gamma^i_j = \sigma^j(\Gamma^i)$, for $i = 1, \dots, k$, $j = 1, \dots, 8k$.

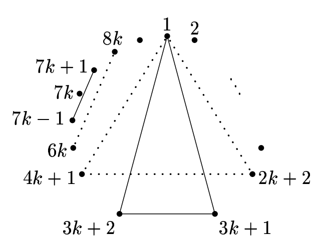

Figure 5: The k initial copies of $K_3 \cup K_2$ in K_{8k+1}

A scheme of the proof that each of the digraphs $G2$, $G3$, $G4$, $G5$, $G8$, $G9$, $G11$ i $G12$ decomposes TT_n, for $n \equiv 0$ or $1 \pmod 8$, is the same. First, we construct decompositions of TT_8 and TT_9. Then we partition the set $V(TT_{8k}) = \{1, \dots, 8k\}$ into 8-element sets V_i, $i = 1, \dots, k$. Each set V_i induces TT_8 in TT_{8k}, and the set $V_i \cup \{8k + 1\}$ induces TT_9 in TT_{8k+1}. Hence, it is enough to show decompositions of an oriented graph D_{ij} with the vertex set $V_i \cup V_j$, the arc set $E(TT_n) \cap ((V_i \times V_j) \cup (V_j \times V_i))$, and $i < j$.

Label the vertices of **G2** with x, y, w, z, s, t such that $(x, y), (y, w), (y, z)$, (s, t) are arcs of $G2$ (cf. Figure 6). Using the same convention for representing decompositions as in the proof of Theorem 7, decompositions of TT_8 and TT_9 into $G2$ can be given by the following sequences:

TT_8 : (234718), (245817), (146728), (167825), (126758), (156738), (135678);

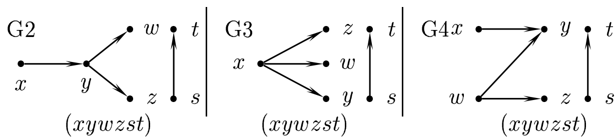

Figure 6: Labelling $G2$, $G3$and $G4$

TT_9 : (234519), (245618), (128936), (258917), (138927), (168937), (148967), (156789), (478926).

Define the sets

$$V_i = \{2i-1, 2i, 4i-1, 4i, 6i-1, 6i, 8i-1, 8i\ \}, \quad i = 1, \ldots, k.$$

For $i < j$, denote $V_i = \{a, b, c, d, e, f, g, h\}$ and $V_j = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}, \breve{e}, \breve{f}, \breve{g}, \breve{h}\}$, and assume that the elements in each ot these two sets are ordered increasingly. It is easy to verify that the sequence $(a\ b\ \breve{a}\ \breve{b}\ c\ d\ \breve{c}\ \breve{d}\ e\ f\ \breve{e}\ \breve{f}\ g\ h\ \breve{g}\ \breve{h}\ )$ is increasing. A decomposition of the oriented graph D_{ij} is given by the following set of sequences: $(\breve{b}h\breve{g}\breve{h}a\breve{e})$, $(\breve{b}g\breve{g}ha\breve{f})$, $(b\breve{f}ghd\breve{h})$, $(b\breve{e}gha\breve{a})$, $(\breve{b}f\breve{e}\breve{f}c\breve{h})$, $(\breve{a}f\breve{g}\breve{h}b\breve{b})$, $(\breve{b}e\breve{e}\breve{f}b\breve{a})$ $(\breve{a}e\breve{g}\breve{h}a\breve{b})$, $(b\breve{d}efa\breve{h})$, $(a\breve{d}ghb\breve{g})$, $(b\breve{c}efa\breve{g})$, $(a\breve{c}ghd\breve{g})$, $(\breve{b}d\breve{c}\breve{d}\breve{b}h)$, $(\breve{a}d\breve{e}\breve{f}c\breve{g})$, $(\breve{b}c\breve{c}\breve{d}\breve{a}g)$, $(\breve{a}c\breve{e}\breve{f}b\breve{h})$.

The vertices of **G3** can be labelled such that its arcs are $(x, y), (x, w), (x, z)$, (s, t) (cf. Figure 6). Decompositions of the transitive tournaments of order 8 and 9 can be described as follows:

TT_8 : (356718), (145768), (234867), (123678), (256748), (456738), (567834);

TT_9 : (156978), (345618), (467829), (134759), (234879), (378912), (256789), (567849), (678945).

The sets V_i are defined as

$$V_i = \{8i-7, 8i-6, 8i-5, 8i-4, 8i-3, 8i-2, 8i-1, 8i\}.$$

Denote $V_i = \{a, b, c, d, e, f, g, h\}$ and $V_j = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}, \breve{e}, \breve{f}, \breve{g}, \breve{h}\}$, assuming that the elements are listed increasingly. If $i < j$ then $a < b < c < d < e < f < g < h < \breve{a} < \breve{b} < \breve{c} < \breve{d} < \breve{e} < \breve{f} < \breve{g} < \breve{h}$. It is easy to see that to show that the digraph D_{ij} can be decomposed into $G3$, it is enough to decompose its subgraph $D'ij$ induced by the set $V_i' \cup V_j'$, where $V_i' = \{a, b\}$ and $V_j' = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}\}$. This decomposition is given by two sequences $(a\breve{a}\breve{b}\breve{c}b\breve{d})$ and $(b\breve{a}\breve{b}\breve{c}a\breve{d})$.

Label the vertices of **G4** as in Figure 6, such that $G4$ has the arcs $(x, y), (w, y)$, $(w, z), (s, t)$. Decompositions into TT_8 and TT_9 follow:

TT_8 : (182345), (172534), (162435), (384612), (473615), (576814), (785613);

TT_9 : (192345), (182534), (172435), (461239), (263814), (574913), (896715), (685937), (487956).

Take the same sets V_i as considered for $G3$. Instead of decomposing D_{ij} it suffices to decompose a subgraph D'_{ij} of it induced by $V_i' \cup V_j'$, where $V_i' = \{a, b, c, d\}$ and

$V_j' = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}\}$. The set of sequences $(a\breve{a}b\breve{b}c\breve{c})$, $(a\breve{b}cd\breve{d}\breve{c})$, $(b\breve{c}ad\breve{d}\breve{a})$ and $(b\breve{d}d\breve{b}c\breve{a})$ represents a decomposition of D_{ij}' into $G4$.

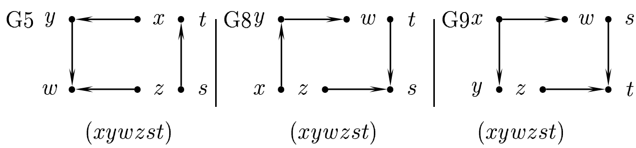

Figure 7: Labelling $G5$, $G8$, $G9$

The oriented graph **G5** has vertices labelled with x, y, w, z, s, t, so that its arcs are $(x, y), (y, w), (z, w), (s, t)$ as shown in Figure 7. Decompositions of TT_8 and TT_9 are given by the following sequences:

TT_8 : (238145), (257134), (128435), (246137), (156327), (268713), (147658);

TT_9 : (239145), (258134), (129435), (247136), (138246), (168427), (159678), (267514), (379856).

The sets V_i are the same as for $G2$. Analogously, it is enough to indicate a decomposition of a subgraph D_{ij}' of D_{ij} induced by $V_i' \cup V_j'$, where $V_i' = \{a, c, e, g\}$ and $V_j' = \{\breve{a}, \breve{c}, \breve{e}, \breve{g}\}$. This is given by the sequences: $(\breve{c}g\breve{g}ea\breve{a})$, $(a\breve{c}e\breve{a}c\breve{e})$, $(a\breve{e}g\breve{a}c\breve{c})$, $(\breve{a}c\breve{g}ae\breve{e})$.

The next oriented graph we have to consider is **G8**. Label the vertices with x, y, w, z, s, t, so that $(x, y), (y, w), (z, s), (t, s)$ are the arcs (cf. Figure 7). TT_8 and TT_9 can be decomposed into $G8$ as follows:

TT_8 : (345281), (235174), (248163), (146275), (138265), (158376), (125687);

TT_9 : (345192), (235184), (246173), (147283), (125396), (168275), (156497), (267598), (136587).

The sets V_i are the same as in the case of $G2$ (and $G5$), and the subgraph D_{ij}' is the same as for $G5$. A decomposition of D_{ij}' into $G8$ follows: $(\breve{a}e\breve{g}c\breve{c}a)$, $(a\breve{a}c\breve{e}g\breve{c})$, $(\breve{c}e\breve{e}c\breve{g}a)$, $(\breve{a}g\breve{g}c\breve{e}a)$.

Label the vertices of **G9** as depicted in Figure 7, so that the arcs of $G9$ are $(x, y), (x, w), (z, t), (s, t)$. Decompositions of the transitive tournaments of order 8 and 9 can be chosen in the following way:

TT_8 : (345128), (234157), (136247), (378456), (678125), (458236), (124578);

TT_9 : (345129), (234158), (137245), (145238), (126389), (689247), (489367), (579236), (789456).

Partition the set $V(TT_{8k})$ into the same sets V_i as for $G3$, and consider the same subgraph D_{ij}' of D_{ij}, as for $G5$. The following set of sequences represents a decomposition of D_{ij}' into $G9$: $(c\breve{b}\breve{c}ab\breve{a})$, $(d\breve{a}\breve{b}abd\breve{d})$, $(c\breve{a}\breve{d}ab\breve{c})$, $(d\breve{c}\breve{d}ab\breve{b})$.

In the digraph **G11**, label the vertices with x, y, w, z, s, t, u in such a way that $G11$ has the following arcs: $(x, y), (y, w), (z, s), (t, u)$ as depicted in (t, u) Figure 8. We decompose TT_8 and TT_9 as follows:

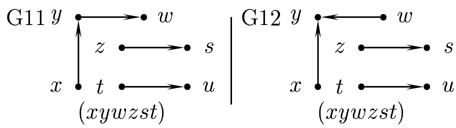

Figure 8: Labelling $G11$ and $G12$

TT_8 : (4561823), (1782534), (1283547), (1672438), (1482736), (1582637), (4681357);

TT_9 : (3561924), (3461825), (1452936), (4891723), (4791328), (5781239), (1692738), (1584967), (2683759).

Define

$$V_i = \{4i-3, 4i-2, 4i-1, 4i, 8i-3, 8i-2, 8i-1, 8i\}, \quad i = 1, \ldots, k.$$

For $i < j$, denote $V_i = \{a,b,c,d,e,f,g,h\}$ and $V_j = \{\breve{a},\breve{b},\breve{c},\breve{d},\breve{e},\breve{f},\breve{g},\breve{h}\}$, and assume that the elements in each of these two sets are listed in the increasing order. Then the sequence $(a,b,c,d,\breve{a},\breve{b},\breve{c},\breve{d},e,f,g,h,\breve{e},\breve{f},\breve{g},\breve{h})$ is also increasing. Analogously to the most of the previous cases, it is enough to give a decomposition of a subgraph D'_{ij} induced in D_{ij} by the set of vertices $V'_i \cup V'_j$ with $V'_i = \{a,b,c,d\}$ and $V'_j = \{\breve{a},\breve{b},\breve{c},\breve{d}\}$. This is represented by $(b\breve{a}ca\breve{c}d\breve{d})$, $(b\breve{b}da\breve{d}c\breve{c})$, $(\breve{a}c\breve{d}a\breve{b}b\breve{c})$, $(\breve{b}d\breve{c}a\breve{a}b\breve{d})$.

The last digraph to be considered is **G12**. Label the vertices to obtain the arcs $(x,y),(y,w),(z,s),(t,u)$, as in Figure 8. Then decompositions of the transitive tournaments of orders 8 and 9 can be represented as:

TT_8 : (1823456), (1723645), (1623547), (1523846), (4851237), (5761423), (6871324);

TT_9 : (1923456), (1823645), (1723546), (3941526), (4751623), (3862514), (3761289), (5971348), (5872469).

Consider the same sets V_i, $i = 1,\ldots,k$, as for $G3$, and the same subgraph D'_{ij} as for $G4$. A decomposition of D'_{ij} into $G12$ is represented by the following set of sequences: $(a\breve{a}bc\breve{c}d\breve{d})$, $(a\breve{b}cb\breve{d}d\breve{a})$, $(a\breve{c}db\breve{b}c\breve{a})$, $(a\breve{d}cd\breve{b}b\breve{c})$. ■

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