AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **37** (2007), Pages 21–26

A note on potentially $K_{1,1,t}$-graphic sequences

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Abstract

For a given graph H, let $\sigma(H,n)$ be the smallest even integer such that every n-term non-increasing graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(H,n)$ has a realization G containing H as a subgraph. In this paper, we determine the values of $\sigma(K_{1,1,t}, n)$ for $t \geq 3$ and $n \geq 2[\frac{(t+5)^2}{4}] + 3$, where $K_{r,s,t}$ is the $r \times s \times t$ complete 3-partite graph.

1 Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers with $d_i \leq n-1$ for each i is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic non-increasing sequences in NS_n is denoted by GS_n. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For a given graph H, a graphic sequence π is *potentially H-graphic* if there exists a realization of π containing H as a subgraph. Gould et al. [3] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H,n)$ such that every $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H,n)$ is potentially H-graphic. If $H = K_r$, the complete graph on r vertices, this problem was considered by Erdős et al. [1] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_r, n) = (r-2)(2n-r+1)+2$ for sufficiently large n. Gould et al. [3] and Li and Song [6] independently proved that the conjecture holds for $r = 4$ and $n \geq 8$. Li et al. [7,8] showed that the conjecture is true for $r = 5$ and $n \geq 10$ and for $r \geq 6$ and $n \geq \binom{r-1}{2} + 3$. For $H = K_{r,s}$, the $r \times s$ complete bipartite graph, Gould et al. [3] determined $\sigma(K_{2,2}, n)$ for $n \geq 4$, Yin and Li [10] determined $\sigma(K_{3,3}, n)$ for $n \geq 6$ and $\sigma(K_{4,4}, n)$ for $n \geq 8$. Recently, Yin, Li and Chen [9,11,12] determined $\sigma(K_{r,s}, n)$ for $s \geq r \geq 1$ and sufficiently large n. We now consider the case of $H = K_{1,1,t}$. Erdős et al. in [1] determined $\sigma(K_{1,1,1}, n)$ for $n \geq 6$ since $K_{1,1,1} = K_3$, Lai in [5] determined $\sigma(K_{1,1,2}, n)$ for $n \geq 4$ since $K_{1,1,2} = K_4 - e$, a graph obtained from K_4 by deleting one edge. Moreover, Eschen and Niu [2] characterized the potentially $K_{1,1,2}$-graphic sequences. The purpose of this paper is to determine the values of $\sigma(K_{1,1,t}, n)$ for $t \geq 3$ and $n \geq 2[\frac{(t+5)^2}{4}] + 3$.

2 Preliminaries

In order to prove our main results, we need the following notations and results.

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence and $1 \leq k \leq n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n), & \text{if } d_k \geq k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n), & \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ is the rearrangement of the $n-1$ terms in π''_k. Then π'_k is called the *residual sequence* obtained by laying off d_k from π.

Theorem 2.1. [4] *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence and $1 \leq k \leq n$. Then π is graphic if and only if π'_k is graphic.*

Theorem 2.2. [9, 10] *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $\Delta = max\{d_1, d_2, \ldots, d_n\}$ and $\sigma(\pi)$ be even. The rearrangement sequence of π is denoted by $\pi^* = (d^*_1, d^*_2, \ldots, d^*_n)$, where $d^*_1 \geq d^*_2 \geq \cdots \geq d^*_n$ is the rearrangement of $d_1, d_2, \ldots, d_n$. If there exists an integer $n_1 \leq n$ such that $d^*_{n_1} \geq h \geq 1$ and $n_1 \geq \frac{1}{h}\left[\frac{(\Delta+h+1)^2}{4}\right]$, then π is graphic.*

Theorem 2.3. [9,10] *Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, where $d_r \geq r + s - 1$ and $d_n \geq r$. If $n \geq (r+2)(s-1)$, then π is potentially $K_{r,s}$-graphic.*

Theorem 2.4.

(1) [7] *If $r = 4$, then $\sigma(K_{r+1}, n) = 6n - 10 = (r-1)(2n-r) + 2$ for $n \geq 10$;*

(2) [8] *If $r \geq 5$, then $\sigma(K_{r+1}, n) = (r-1)(2n-r) + 2$ for $n \geq \binom{r}{2} + 3$;*

(3) [8] *If $r \geq 5$, then $\sigma(K_{r+1}, n) \leq 2n(r-2) + 8$ for $2r + 2 \leq n \leq \binom{r}{2} + 3$.*

Theorem 2.5. [3] *If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

Lemma 2.1. *Let $\pi \in GS_n$. If π is potentially $K_{2,t+2}$-graphic, then π is potentially $K_{1,1,t}$-graphic.*

Proof. Let G be a realization of π that contains $K_{2,t+2}$ as a subgraph, where $X = \{u_1, u_2\}$, $Y = \{u_3, u_4, \ldots, u_{t+4}\}$ is the bipartite partition of the vertex set of $K_{2,t+2}$. If $u_1u_2 \in E(G)$, then G contains $K_{1,1,t}$ as a subgraph, i.e., π is potentially $K_{1,1,t}$-graphic. Assume $u_1u_2 \notin E(G)$. If there exist u_i, u_j, $3 \leq i < j \leq t+4$ such that $u_iu_j \notin E(G)$, then $G' = G + \{u_1u_2, u_iu_j\} - \{u_1u_i, u_2u_j\}$ is a realization of π, and clearly contains $K_{1,1,t}$ as a subgraph. Hence π is potentially $K_{1,1,t}$-graphic. If $u_iu_j \in E(G)$ for any u_i, u_j, $3 \le i < j \le t+4$, then $G[Y] = K_{t+2}$ contains $K_{1,1,t}$ as a subgraph, and hence π is also potentially $K_{1,1,t}$-graphic. □

Let $n \ge t+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$, where $d_2 \ge t+1$ and $d_{t+2} \ge 2$. Let

$$\rho_1'(\pi) = (d_2 - 1, d_3 - 1, \ldots, d_{t+2} - 1, d_{t+3} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n),$$

and denote $\rho_1(\pi) = (d_2 - 1, d_3 - 1, \ldots, d_{t+2} - 1, d_{t+3}^{(1)}, d_{t+4}^{(1)}, \ldots, d_n^{(1)})$, where $d_{t+3}^{(1)} \ge d_{t+4}^{(1)} \ge \cdots \ge d_n^{(1)}$ is the rearrangement of $d_{t+3} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. Let

$$\rho_2'(\pi) = (d_3 - 2, \ldots, d_{t+2} - 2, d_{t+3}^{(1)} - 1, \ldots, d_{d_2+1}^{(1)} - 1, d_{d_2+2}^{(1)}, \ldots, d_n^{(1)}),$$

and denote $\rho_2(\pi) = (d_3 - 2, \ldots, d_{t+2} - 2, d_{t+3}^{(2)}, d_{t+4}^{(2)}, \ldots, d_n^{(2)})$, where $d_{t+3}^{(2)} \ge d_{t+4}^{(2)} \ge \cdots \ge d_n^{(2)}$ is the rearrangement of $d_{t+3}^{(1)} - 1, \ldots, d_{d_2+1}^{(1)} - 1, d_{d_2+2}^{(1)}, \ldots, d_n^{(1)}$.

Lemma 2.2. *Let $n \ge t+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$, where $d_2 \ge t+1$ and $d_{t+2} \ge 2$. If $\rho_2(\pi)$ is graphic, then π is potentially $K_{1,1,t}$-graphic.*

Proof. It follows easily from the definition of $\rho_2(\pi)$ that π is potentially $K_{1,1,t}$-graphic. □

3 Main Result

Theorem 3.1. *Let $t \ge 3$ and $n \ge t+2$. Then*

$$\sigma(K_{1,1,t}, n) \ge \begin{cases} (n-1)(t+1) + 2 & \text{if } n \text{ is odd or } t \text{ is odd,} \\ (n-1)(t+1) + 1 & \text{if } n \text{ and } t \text{ are even.} \end{cases}$$

Proof. If n is odd or t is odd, let $\pi = (n-1, t^{n-1})$, where the symbol x^y in a sequence stands for y consecutive terms, each equal to x, then $\pi_1' = (d_1', \ldots, d_{n-1}') = ((t-1)^{n-1})$. It is easy to see that π_1' is graphic and not potentially $K_{1,t}$ or $K_{1,1,t-1}$-graphic. Hence π is graphic and not potentially $K_{1,1,t}$-graphic. Thus $\sigma(K_{1,1,t}, n) \ge \sigma(\pi) + 2 = (n-1)(t+1) + 2$.

Now assume that n and t are even. Take $\pi = (n-1, t^{n-2}, t-1)$. Also, it is easy to see that $\pi_1' = ((t-1)^{n-2}, t-2)$ is graphic and not potentially $K_{1,t}$ or $K_{1,1,t-1}$-graphic, and hence π is graphic and not potentially $K_{1,1,t}$-graphic. Thus $\sigma(K_{1,1,t}, n) \ge \sigma(\pi) + 2 = (n-1)(t+1) + 1$. □

In order to determine the exact values of $\sigma(K_{1,1,t}, n)$ for $t \ge 3$ and $n \ge 2[\frac{(t+5)^2}{4}] + 3$, we denote $m = [\frac{(t+5)^2}{4}]$ and also need the following Lemmas.

Lemma 3.1. *Let $t \ge 3$ and $n = m$. Then $\sigma(K_{1,1,t}, n) \le (n-1)(t+1) + 2 + (t-1)(m+3)$.*

Proof. If $t = 3$, then $n = m = 16 \geq 10$. If $4 \leq t \leq 9$, then $n = m \geq \binom{t+1}{2} + 3$. By Theorem 2.4(1) and (2),

$$\begin{aligned}\sigma(K_{1,1,t}, n) &\leq \sigma(K_{t+2}, n) = t(2n - t - 1) + 2\\ &= tn + n - t + 1 + (t-1)(n - t - 1)\\ &= (n-1)(t+1) + 2 + (t-1)(m - t - 1)\\ &\leq (n-1)(t+1) + 2 + (t-1)(m-4)\\ &\leq (n-1)(t+1) + 2 + (t-1)(m+3).\end{aligned}$$

If $t \geq 10$, then $n = m \leq \binom{t+1}{2} + 3$. By Theorem 2.4(3),

$$\begin{aligned}\sigma(K_{1,1,t}, n) &\leq \sigma(K_{t+2}, n) \leq 2n(t-1) + 8\\ &= tn + n - t + 1 + n(t-3) + t + 7\\ &\leq (n-1)(t+1) + 2 + m(t-3) + 3(t-3)\\ &\leq (n-1)(t+1) + 2 + (t-1)(m+3).\end{aligned}$$

□

Lemma 3.2. *Let $t \geq 3$, $n \geq m$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$, where $d_n \geq 2$. If $\sigma(\pi) \geq (n-1)(t+1) + 2$, then π is potentially $K_{1,1,t}$-graphic.*

Proof. Since $\sigma(\pi) \geq (n-1)(t+1) + 2$, $d_2 \geq t + 1$. If $d_2 \geq t + 3 = 2 + (t+2) - 1$, then by $n \geq m \geq 4(t+1)$ and Theorem 2.3, π is potentially $K_{2,t+2}$-graphic. Hence π is potentially $K_{1,1,t}$-graphic by Lemma 2.1. Assume $t + 1 \leq d_2 \leq t + 2$. Since $\pi_1' = (d_1', d_2', \ldots, d_{n-1}')$ satisfies $d_1' \geq t$, π_1' is potentially $K_{1,t}$-graphic. If $d_1 = n - 1$ or there exists an integer k, $t+2 \leq k \leq d_1+1$ such that $d_k > d_{k+1}$, then $d_1' = d_2 - 1, d_2' = d_3 - 1, \ldots, d_{t+1}' = d_{t+2} - 1$, by Theorem 2.5, π is potentially $K_{1,1,t}$-graphic. We now further assume that

$$n - 2 \geq d_1 \geq \cdots \geq d_{t+1} \geq d_{t+2} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq 2.$$

Denote $\ell = \max\{i : \ d_{t+2} = d_{d_1+2+i}\}$. We consider the following cases:

Case 1. $d_2 = t + 1$. By the definition of $\rho_2(\pi)$, it is easy to see that

$$\rho_2(\pi) = (d_3 - 2, \ldots, d_{t+2} - 2, d_{d_1+2}, \ldots, d_{d_1+2+\ell}, d_{t+3} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+3+\ell}, \ldots, d_n).$$

If $d_{t+2} \leq t - 1$, then by $d_2 = t + 1$, $\sigma(\pi) = d_1 + d_2 + \cdots + d_{t+1} + d_{t+2} + \cdots + d_n \leq n - 1 + t(t+1) + (n-t-1)(t-1) = nt + t$. Obviously, if $nt + t \geq \sigma(\pi) \geq (n-1)(t+1) + 2$, then $n \leq 2t - 1$. In fact, it is impossible that $n \leq 2t - 1$ for $n \geq m$, where $m = [\frac{(t+5)^2}{4}]$ and $t \geq 3$. So $nt + t < (n-1)(t+1) + 2$ is a contradiction. Hence $d_{t+2} \geq t$. So we have

$$\max\{d_3 - 2, \ldots, d_{t+2} - 2, d_{d_1+2}, \ldots, d_{d_1+2+\ell}, d_{t+3} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+3+\ell}, \ldots, d_n\} \leq t + 1,$$
$$\min\{d_3 - 2, \ldots, d_{t+2} - 2, d_{d_1+2}, \ldots, d_{d_1+2+\ell}, d_{t+3} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+3+\ell}, \ldots, d_n\} \geq 1.$$

It follows from $\sigma(\rho_2(\pi))$ is even, $[\frac{(t+1+1+1)^2}{4}] \leq [\frac{(t+5)^2}{4}] - 2 \leq n - 2$ and Theorem 2.2 that $\rho_2(\pi)$ is graphic. Hence π is potentially $K_{1,1,t}$-graphic by Lemma 2.2.

Case 2. $d_2 = t+2$. By the definition of $\rho_2(\pi)$, $\rho_2(\pi) = (d_3-2,\dots,d_{t+2}-2, d_{t+3}-1,\dots,d_{d_1+2}-1, d_{d_1+3},\dots,d_n)$ for $\ell = 0$ or $\rho_2(\pi) = (d_3-2,\dots,d_{t+2}-2, d_{d_1+3},\dots,d_{d_1+2+\ell}, d_{t+3}-1,\dots,d_{d_1+2}-1, d_{d_1+3+\ell},\dots,d_n)$ for $\ell \ge 1$. Since the largest term in $\rho_2(\pi)$ is at most $t+2$ and the smallest term in $\rho_2(\pi)$ is at least 1, by $[\frac{(t+2+1+1)^2}{4}] \le [\frac{(t+5)^2}{4}] - 2 \le n-2$ and Theorem 2.2, $\rho_2(\pi)$ is graphic. Hence π is also potentially $K_{1,1,t}$-graphic by Lemma 2.2. □

Lemma 3.3. *Let $t \ge 3$ and $n = m+s$, where $0 \le s \le m+3$. Then*

$$\sigma(K_{1,1,t}, n) \le (n-1)(t+1) + 2 + (t-1)(m+3-s).$$

Proof. Use induction on s. By Lemma 3.1, Lemma 3.3 holds for $s = 0$. Now assume that Lemma 3.3 holds for $s-1$, $0 \le s-1 \le m+2$. Let $n = m+s$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (n-1)(t+1) + 2 + (t-1)(m+3-s)$. It is enough to prove that π is potentially $K_{1,1,t}$-graphic. Clearly, $\sigma(\pi) \ge (n-1)(t+1)+2$. If $d_n \ge 2$, then by Lemma 3.2, π is potentially $K_{1,1,t}$-graphic. If $d_n \le 1$, then π'_n satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n \ge (n-1)(t+1) + 2 + (t-1)(m+3-s) - 2 = (n-2)(t+1) + 2 + (t-1)(m+3-(s-1))$. By the induction hypothesis, π'_n is potentially $K_{1,1,t}$-graphic, and hence π is also potentially $K_{1,1,t}$-graphic. Thus, $\sigma(K_{1,1,t}, n) \le (n-1)(t+1) + 2 + (t-1)(m+3-s)$. □

Lemma 3.4. *If $t \ge 3$ and $n \ge 2m+3$, then $\sigma(K_{1,1,t}, n) \le (n-1)(t+1)+2$.*

Proof. Use induction on n. It follows from Lemma 3.3 that Lemma 3.4 holds for $n = 2m+3$. Now assume that Lemma 3.4 holds for $n-1$, $n-1 \ge 2m+3$. Let $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (n-1)(t+1)+2$. We only need to prove that π is potentially $K_{1,1,t}$-graphic. If $d_n \ge 2$, then by Lemma 3.2, π is potentially $K_{1,1,t}$-graphic. If $d_n \le 1$, then π'_n satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n \ge (n-1)(t+1)+2-2 \ge (n-2)(t+1)+2$. By the induction hypothesis, π'_n is potentially $K_{1,1,t}$-graphic, and hence π is also potentially $K_{1,1,t}$-graphic. □

Theorem 3.2. *Let $t \ge 3$ and $n \ge 2m+3$. Then*

$$\sigma(K_{1,1,t}, n) = \begin{cases} (n-1)(t+1)+2 & \text{if } n \text{ is odd or } t \text{ is odd,} \\ (n-1)(t+1)+1 & \text{if } n \text{ and } t \text{ are even.} \end{cases}$$

Proof. If n is odd or t is odd, then by Theorem 3.1 and Lemma 3.4, $\sigma(K_{1,1,t}, n) = (n-1)(t+1)+2$. If n and t are even, then $(n-1)(t+1)+1 \le \sigma(K_{1,1,t}, n) \le (n-1)(t+1)+2$ by Theorem 3.1 and Lemma 3.4. Since $\sigma(K_{1,1,t}, n)$ is even, we have $\sigma(K_{1,1,t}, n) = (n-1)(t+1)+1$. □

Acknowledgements

The author is grateful to the referee for his valuable comments and suggestions.

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(Received 20 Feb 2004; revised 21 July 2006)