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Minimum cycle bases of direct products of bipartite graphs

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Abstract

This article describes a construction of a minimum cycle basis for the direct product of two connected bipartite graphs, in terms of minimum cycle bases of the factors.

1 Introduction

In the article *Minimum Cycle Bases of Product Graphs* [5], W. Imrich and P. Stadler construct minimum cycle bases for Cartesian and strong products of graphs, in terms of minimum cycle bases of the factors. In [1], F. Berger solves the same problem for the lexicographical product. The corresponding construction for the direct product appears to be elusive. The present article presents a step in this direction by addressing case in which the factors are bipartite. Though the general problem appears to be even more involved, our approach offers some possible insights.

This introduction reviews the preliminaries.

The *edge space* $\mathcal{E}(G)$ of a simple graph $G = (V(G), E(G))$ is the power set of its edges $E(G)$ endowed with the structure of a vector space over the two-element field $\mathbb{F}_2 = \{0, 1\}$. Addition in $\mathcal{E}(G)$ is symmetric difference of sets, and zero is the empty set. Similarly, the *vertex space* $\mathcal{V}(G)$ of G is the power set of the vertices $V(G)$ viewed as a vector space over $\mathbb{F}_2$. The set $E(G)$ is a basis for $\mathcal{E}(G)$. To avoid proliferation of notation, we blur the distinction between a subgraph K of G and its edge set $E(K) \in \mathcal{E}(G)$. Thus, if J and K are subgraphs, an expression such as $J + K$ means $E(J) + E(K)$, with the operation taking place in $\mathcal{E}(G)$.

For any graph G, there is a linear *incidence map* $B_G : \mathcal{E}(G) \to \mathcal{V}(G)$ whose effect on the basis $E(G)$ is $B_G(vw) = v + w$. The kernel of this map is denoted denoted $\mathcal{C}(G)$. It is called the *cycle space* of G, and consists exactly of the edge sets $E(K)$ of Eulerian subgraphs K of G, that is, subgraphs having no vertex of odd degree ([2], Proposition 1.9.2). Elements of $\mathcal{C}(G)$ are called *generalized cycles*, or just *cycles*. The dimension of $\mathcal{C}(G)$ is $\nu(G) = |E(G)| - |V(G)| + c$, where c is the number of components of G ([2], Theorem 1.9.6). A graph homomorphism $g : G \to H$ induces a linear map $g^* : \mathcal{E}(G) \to \mathcal{E}(H)$ defined on the basis $E(G)$ as $g^*(vw) = g(v)g(w)$. It is easy to check that g^* restricts to a linear map $g^* : \mathcal{C}(G) \to \mathcal{C}(H)$.

A basis $\mathcal{B}$ of $\mathcal{C}(G)$ is called a *cycle basis of* G, and its *length* is $l(\mathcal{B}) = \sum_{C \in \mathcal{B}} |C|$. Among all cycle bases of G, one with smallest possible length is called a *minimal cycle basis*, or *MCB*. A cycle in $\mathcal{C}(G)$ is called *relevant* if it belongs to some MCB of G. Our first proposition is proved in [6].

Proposition 1. *A cycle is relevant if and only if it is not the sum of shorter cycles.*

Because $\mathcal{C}(G)$ is a weighted matriod where any cycle C has weight $|C|$, the Greedy Algorithm [7] is guaranteed to terminate with an MCB. (I.e. begin with $\mathcal{M} = \emptyset$; then append shortest cycles to it, maintaining independence of $\mathcal{M}$, until no further shortest cycles can be appended; then append next-shortest cycles, maintaining independence, until no further next-shortest cycles can be appended; and so on, until $\mathcal{M}$ is a maximal independent set.) Although Horton's Algorithm [3] finds an MCB in polynomial time, our goal here is a deeper, structural, understanding of MCB's of a direct product, in which an MCB of the product is expressed in terms of MCB's of the factors.

The *direct product* of graphs G and H is the graph $G \times H$ whose vertex set is the cartesian product $V(G) \times V(H)$ and whose edges are $(u,x)(v,y)$ where $uv \in E(G)$ and $xy \in E(H)$. We quickly mention a few relevant facts; the reader requiring more background is referred to Imrich and Klavžar [4].

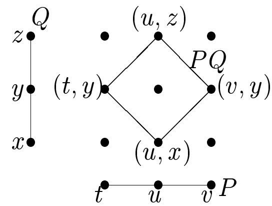
The projections $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ which project vertices onto the first and second components, respectively, are graph homomorphisms, and thus induce linear maps $\pi_G^* : \mathcal{C}(G \times H) \to \mathcal{C}(G)$ and $\pi_H^* : \mathcal{C}(G \times H) \to \mathcal{C}(H)$.

Suppose G and H are connected and bipartite. Then $G \times H$ has exactly two components, and, as it has $|V(G)||V(H)|$ vertices and $2|E(G)||E(H)|$ edges, $\nu(G \times H) = 2|E(G)||E(H)| - |V(G)||V(H)| + 2$. If $e \in E(G)$, then the subgraph $e \times H$ of $G \times H$ consists of two components, each isomorphic to H, one in each component of $G \times H$. Restriction of π_H to either component of $e \times H$ is an isomorphism. Likewise, if $f \in E(H)$, then the subgraph $G \times f$ consists of two components, each isomorphic to G, one in each component of $G \times H$. Restriction of π_G to either component is an isomorphism.

2 A Direct Sum Decomposition

This section introduces a direct sum decomposition of the cycle space of the direct product of two bipartite graphs, which, once established, leads readily to an MCB. We begin with a description of one of the terms of this sum.

If $P = tuv$ and $Q = xyz$ are paths of length 2 in G and H, respectively, then PQ denotes the 4-cycle in $G \times H$ induced on the edges $(u,x)(v,y)$, $(v,y)(u,z)$, $(u,z)(t,y)$ and $(t,y)(u,x)$, as illustrated in the figure below. Such a subgraph PQ is called a *diamond* in $G \times H$. The subspace $\mathcal{D}(G \times H)$ of $\mathcal{C}(G \times H)$ spanned by all diamonds is called the *diamond subspace* of $\mathcal{C}(G \times H)$. Since $\pi_G^*(PQ) = 0 = \pi_H^*(PQ)$, it follows that $\mathcal{D}(G \times H) \subseteq \text{Null}(\pi_G^*) \cap \text{Null}(\pi_H^*)$.



Proposition 2. *If G and H are connected bipartite graphs, with $e \in E(G)$ and $f \in E(H)$, then there is a direct sum $\mathcal{C}(G \times f) \oplus \mathcal{D}(G \times H) \oplus \mathcal{C}(e \times H)$ of subspaces of $\mathcal{C}(G \times H)$, and* $\dim(\mathcal{D}(G \times H)) \leq \nu(G \times H) - 2\nu(G) - 2\nu(H)$.

Proof. Suppose $C \in \mathcal{C}(G\times H)$ can be written as $C = A+D+B$ and $C = A'+D'+B'$, with $A, A' \in \mathcal{C}(G \times f)$, $D, D' \in \mathcal{D}(G \times H)$, and $B, B' \in \mathcal{C}(e \times H)$. It must be shown $A = A'$, $D = D'$ and $B = B'$. First assume C lies in a single component of $G \times H$. Applying π_G^* to both sides of $A + D + B = A' + D' + B'$ gives $\pi_G^*(A) = \pi_G^*(A')$. (Note $\pi_G^*(B) = 0 = \pi_G^*(B')$ since these are mappings of even cycles onto a single edge.) From this, $A = A'$, because π_G^* restricts to an isomorphism on the cycle space of a component of $G \times f$. Likewise, applying π_H^* to both sides yields $B = B'$, whence $D = D'$. If C lies in two components, write $C = C_1 + C_2$ as a sum of two cycles in different components and apply the above reasoning to each cycle in the sum. Since $\mathcal{C}(G \times f) \oplus \mathcal{D}(G \times H) \oplus \mathcal{C}(e \times H)$ is a subspace of $\mathcal{C}(G \times H)$, we have $\dim(\mathcal{D}(G \times H)) \leq \dim(\mathcal{C}(G \times H)) - \dim(\mathcal{C}(G \times f))- \dim(\mathcal{C}(e \times H))$. Since $G \times f$ and $e \times H$ consist of two isomorphic copies of G and H, respectively, $\dim(\mathcal{D}(G \times H))$ $\leq \nu(G \times H) - 2\nu(G) - 2\nu(H)$. ∎

Bipartiteness of factors is crucial for Proposition 2. Establishment of the direct sum relied on the fact that π_G^* and π_H^* restrict to isomorphisms on components of $G \times f$ and $e \times H$, and there are no such isomorphisms in the absence of bipartiteness. Consider the case $G = H = K_3$ where both $G \times f$ and $e \times H$ are hexagons on which π_G^* and π_H^* vanish. Indeed, in this case the direct sum of Proposition 2 fails to exist, since $(G \times f) + (e \times H) \in \mathcal{D}(G \times H)$, as the reader may check.

The main result of this paper will follow from a sharpening Proposition 2. We will show $\dim(\mathcal{D}(G \times H)) = \nu(G \times H) - 2\nu(G) - 2\nu(H)$, whence it follows $\mathcal{C}(G \times H)$ $= \mathcal{C}(G \times f) \oplus \mathcal{D}(G \times H) \oplus \mathcal{C}(e \times H)$. Achieving this sharpening involves examining certain path spaces.

3 Space of Even Paths

We define the *P_2-space* of a graph G as the subspace $\mathcal{P}(G) \subseteq \mathcal{E}(G)$ spanned by the edge sets of the paths of length two in G. (We call such a path a *P_2 in G.*) Since any even path is the sum of P_2's, it follows that $\mathcal{P}(G)$ is spanned by the even paths of G. A basis for $\mathcal{P}(G)$ consisting entirely of P_2's is called a *P_2 basis* of $\mathcal{P}(G)$. Since any pair of edges in the same component of G is the sum of the P_2's on a path beginning and ending with an element of the pair, it follows that any element of $\mathcal{E}(G)$ that has an even number of edges in each component of G is in $\mathcal{P}(G)$. Conversely, if $P \in \mathcal{P}(G)$, and X is a component of G, then $P \cap E(X)$ is the sum of P_2's in X, so $|P \cap E(X)|$ is even. Therefore, $\mathcal{P}(G)$ is the set of all elements in $\mathcal{E}(G)$ having an even number of edges in each component of G.

Consequently, if the nontrivial components of G are $G_1, G_2, \cdots G_c$, then $\mathcal{P}(G)$ is the kernel of the surjective linear map $\lambda : \mathcal{E}(G) \to {\mathbb{F}_2}^c$ defined as

$$\lambda(P) = (|P \cap E(G_1)|, |P \cap E(G_2)|, \cdots, |P \cap E(G_c)|),$$

where the cardinalities are taken mod 2. From this, $\dim(\mathcal{P}(G)) = \dim(\mathcal{E}(G)) - \dim({\mathbb{F}_2}^c) = |E(G)| - c$.

Proposition 3. *If G has c nontrivial components, then* $\dim(\mathcal{P}(G)) = |E(G)| - c$.

If $x \in V(G)$, the *star at x*, denoted $S(x)$, is the subgraph of G induced on the edges adjacent to x. The subspace $\mathcal{L}_G(x) = \mathcal{P}(S(x))$ of $\mathcal{P}(G)$ is called the *local P_2 space at x*. If x is not isolated, Proposition 3 implies $\dim(\mathcal{L}_G(x)) = \deg_G(x) - 1$. If the neighbors of x are $v_1, v_2, \cdots, v_k$, then $\{v_1xv_2, v_1xv_3, v_1xv_4, \cdots, v_1xv_k\}$ is a basis for $\mathcal{L}_G(x)$.

Proposition 4. *Suppose G is bipartite, with partite sets A and B. Let $\mathcal{A}$ and $\mathcal{B}$ be the subspaces of $\mathcal{P}(G)$ spanned by P_2's whose middle vertices are in A or B, respectively. Then:*

1. $\mathcal{A} = \bigoplus_{a \in A} \mathcal{L}_G(a)$
2. $\mathcal{B} = \bigoplus_{b \in B} \mathcal{L}_G(b)$
3. $\mathcal{P}(G) = \mathcal{A} + \mathcal{B}$
4. $\mathcal{A} \cap \mathcal{B} = \mathcal{C}(G)$

Proof. By definition, $\mathcal{A}$ is the sum of the vector spaces $\mathcal{L}(a)$ with $a \in A$. If $A = \{a_1, a_2, \cdots, a_n\}$, then since the stars $S(a_i)$ for $1 \le i \le n$ are edge disjoint, it follows $(\mathcal{L}(a_1) + \mathcal{L}(a_2) + \cdots + \mathcal{L}(a_i)) \cap \mathcal{L}(a_{i+1}) = 0$ for $1 \le i < n$. Thus Statement 1 holds, and identical reasoning for B gives Statement 2.

As $\mathcal{P}(G)$ is spanned by P_2's, and any such path is in $\mathcal{A}$ or $\mathcal{B}$, we have statement 3.

If $C \in \mathcal{C}(G)$, every vertex x of C has even degree, so $C \cap E(S(x)) \in \mathcal{L}_G(x)$. Since G is bipartite, all of its edges are incident with vertices of A (or B). We thus have $C = \sum_{a \in A} C \cap E(S(a)) \in \bigoplus_{a \in A} \mathcal{L}_G(a) = \mathcal{A}$ and $C = \sum_{b \in B} C \cap E(S(b)) \in \bigoplus_{b \in B} \mathcal{L}_G(b) = \mathcal{B}$, so $C \in \mathcal{A} \cap \mathcal{B}$. On the other hand, suppose $C \in \mathcal{A} \cap \mathcal{B}$. Apply the incidence map. Since $C \in \mathcal{A}$, $B_G(C) \subseteq B$, and since $C \in \mathcal{B}$, $B_G(C) \subseteq A$. As $A \cap B = \emptyset$, $B_G(C) = 0$, which means $C \in \mathcal{C}(G)$. ∎

Proposition 5. *If G is acyclic,* $\mathcal{P}(G) = \bigoplus_{x \in V(G)} \mathcal{L}_G(x)$.

Proof. Immediate from 1, 2, 3 and 4 of Proposition 4, combined with $\mathcal{C}(G) = 0$. ∎

Proposition 3 tells how to find a P_2 basis of $\mathcal{P}(G)$ when G is a forest. For each $x \in V(G)$, select a P_2 basis $\mathcal{B}_x$ of $\mathcal{L}_G(x)$. Then $\bigcup_{x\in V(G)} \mathcal{B}_x$ is a P_2 basis for $\mathcal{P}(G)$. The next proposition explains how to construct a P_2 basis for an arbitrary graph.

Construction 1. (For finding a P_2 basis of an arbitrary graph.)
Given a graph G with c nontrivial components, let T be a maximal spanning acyclic subgraph (i.e. each component of T is a spanning tree of a component of G). Let $X = E(G) - E(T) = \{t_1u_1, t_2u_2, \cdots, t_{\nu(G)}u_{\nu(G)}\}$. Further, let $\widetilde{X} = \{t_1u_1v_1, t_2u_2v_2, \cdots, t_{\nu(G)}u_{\nu(G)}v_{\nu(G)}\}$, with each u_iv_i in $E(T)$. For each $x \in V(T)$, let $\mathcal{B}_x$ be a P_2 basis for $\mathcal{L}_T(x)$. Then $\mathcal{B} = (\bigcup_{x\in V(T)} \mathcal{B}_x) \cup \widetilde{X}$ is a P_2 basis for $\mathcal{P}(G)$.

Proof. By Proposition 5, $\bigcup_{x\in V(T)} \mathcal{B}_x$ is an independent set. Clearly $\widetilde{X}$ is independent, for each of its elements contains an edge that is in no other of its elements. Any nonzero element of span$(\widetilde{X})$ must contain edges in $E(G)-E(T)$, while no nonzero element of span$(\bigcup_{x\in V(T)} \mathcal{B}_x)$ has such elements. Therefore span$(\bigcup_{x\in V(T)} \mathcal{B}_x) \cap$ span$(\widetilde{X})$ $= 0$, so $\mathcal{B}$ is an independent set. As $|\mathcal{B}| = \dim(\mathcal{P}(T)) + \nu(G) = (|E(T)| - c) + (|E(G)| - |V(G)| + c) = |E(G)| - (|V(G)| - |E(T)|) = |E(G)| - c = \dim(\mathcal{P}(G))$, it follows $\mathcal{B}$ is indeed a basis. ■

The following lemma is needed in the next section.

Lemma 1. *If $\mathcal{B}$ is an independent set of P_2's in $\mathcal{P}(G)$, then some edge of G belongs to just one element of $\mathcal{B}$.*

Proof. Say the subgraph K of G induced on the edges $\bigcup_{P\in\mathcal{B}} E(P)$ has c (nontrivial) components. Then $|\mathcal{B}| \leq \dim(\mathcal{P}(K)) = |E(K)| - c$. Since each element of $\mathcal{B}$ has two edges, then, on average, the number of elements of $\mathcal{B}$ that an edge of K intersects is $\frac{2|\mathcal{B}|}{|E(K)|} \leq \frac{2(|E(K)|-c)}{|E(K)|} < 2$. Thus some edge of K belongs to just one element of $\mathcal{B}$. ■

4 A Diamond-Space Basis

This section uses the P_2 spaces of the factors of a direct product to produce a basis for its diamond space. The following proposition shows how P_2 bases can be used to produce linearly independent sets of diamonds.

Proposition 6. *If $\mathcal{P}$ and $\mathcal{Q}$ are linearly independent sets of P_2's in $\mathcal{P}(G)$ and $\mathcal{P}(H)$, respectively, then $\mathcal{C} = \{PQ | P \in \mathcal{P}, Q \in \mathcal{Q}\}$ is an independent set of diamonds in $\mathcal{C}(G \times H)$.*

Proof. Induction on $|\mathcal{P}|+|\mathcal{Q}|$ is employed. If $|\mathcal{P}|+|\mathcal{Q}| = 2$, then $\mathcal{C}$ consists of a single (independent) diamond. Assume $|\mathcal{P}| + |\mathcal{Q}| > 2$. There is no loss of generality in also assuming $|\mathcal{P}| > 1$. By Lemma 1, G has an edge ab belonging to just one element abc of $\mathcal{P}$. Now, $\mathcal{C}$ is the disjoint union of the two sets $\mathcal{C}_1 = \{PQ | P \in \mathcal{P} - \{abc\}, Q \in \mathcal{Q}\}$ and $\mathcal{C}_2 = \{(abc)Q | Q \in \mathcal{Q}\}$, and, by the inductive hypothesis, each of these sets is independent. Suppose $C \in$ span$(\mathcal{C}_1) \cap$ span$(\mathcal{C}_2)$. Since C is in $\mathcal{C}_1$, it has no edge of the form $(a, u)(b, v)$, by choice of ab. Then it follows that since C is in $\mathcal{C}_2$, every one of its edges is of form $(b, u)(c, v)$. Therefore C is a cycle in $\mathcal{C}(bc \times H)$. Write $C = C_1 + C_2$ as the sum of two cycles in different components of $bc \times H$, hence in different components of $G \times H$. As C_1 and C_2 are each sums of diamonds, $\pi_H^*(C_1) = 0 = \pi_H^*(C_2)$. But π_H^* restricts to an isomorphism on each component of $bc \times H$, so $C_1 = 0 = C_2$. Thus $C = 0$, and the proof is complete. ∎

Despite Proposition 6, if $\mathcal{P}$ and $\mathcal{Q}$ are maximal independent sets of P_2's, the set $\mathcal{C}$ may still not be a maximal independent set in $\mathcal{D}(G \times H)$. As an illustration, the following construction gives a basis for $\mathcal{D}(G \times H)$ that includes, as a proper subset, all diamonds PQ where P and Q belong to certain bases of $\mathcal{P}(G)$ and $\mathcal{P}(H)$, respectively.

Construction 2. (A basis of diamonds for $\mathcal{D}(G \times H)$, where G and H are bipartite.) Suppose G and H are connected graphs. Let T and U be spanning trees of G and H, respectively. Let $\mathcal{B}_T$ be a basis for $\mathcal{P}(T)$ (Construction 1) and let $\mathcal{B}_U$ be a basis for $\mathcal{P}(U)$. Set $E(G) - E(T) = \{t_1u_1, t_2u_2, \cdots, t_{\nu(G)}u_{\nu(G)}\}$, and $E(H) - E(U) = \{x_1y_1, x_2y_2, \cdots, x_{\nu(H)}y_{\nu(H)}\}$. For each $1 \le i \le \nu(G)$, select P_2 paths $s_it_iu_i$ and $t_iu_iv_i$ with $s_it_i, u_iv_i \in E(T)$. For each $1 \le j \le \nu(H)$, select P_2 paths $w_jx_jy_j$ and $x_jy_jz_j$ with $w_jx_j, y_jz_j \in E(U)$. Form the following sets:

$$\begin{aligned}
\mathcal{D} &= \{PQ \mid P \in \mathcal{B}_T, Q \in \mathcal{B}_U\} \\
\mathcal{T}_j &= \{PQ \mid P \in \mathcal{B}_T, Q \in \{w_jx_jy_j, x_jy_jz_j\}\} && \text{for } 1 \le j \le \nu(H) \\
\mathcal{U}_i &= \{PQ \mid P \in \{s_it_iu_i, t_iu_iv_i\}, Q \in \mathcal{B}_U\} && \text{for } 1 \le i \le \nu(G) \\
\mathcal{S}_{ij} &= \{PQ \mid P = t_iu_iv_i, Q \in \{w_jx_jy_j, x_jy_jz_j\}\} && \text{for } 1 \le i \le \nu(G), 1 \le j \le \nu(H)
\end{aligned}$$

Then the following set is independent in $\mathcal{D}(G \times H)$:

$$\mathcal{B} = \mathcal{D} \cup \left(\bigcup_{1 \le j \le \nu(H)} \mathcal{T}_j \right) \cup \left(\bigcup_{1 \le i \le \nu(G)} \mathcal{U}_i \right) \cup \left(\bigcup_{1 \le i \le \nu(G)} \left(\bigcup_{1 \le j \le \nu(H)} \mathcal{S}_{ij} \right) \right)$$

Moreover, if G and H are bipartite, then $\mathcal{B}$ is a basis for $\mathcal{D}(G \times H)$, and $\dim(\mathcal{D}(G \times H)) = \nu(G \times H) - 2\nu(G) - 2\nu(H)$.

Proof. The sets $\mathcal{D}$, $\mathcal{T}_j$, $\mathcal{U}_i$ and $\mathcal{S}_{ij}$ (for $1 \le i \le \nu(G)$ and $1 \le j \le \nu(H)$) are pairwise disjoint, and each is independent by Proposition 6.

If $C \in \text{span}(\mathcal{D}) \cap \text{span}(\mathcal{T}_1)$, then since no element of $\mathcal{D}$ contains an edge in $\pi_H^{-1}(x_1y_1)$, it follows that C has no such edge either. Then, as $C \in \text{span}(\mathcal{T}_1)$, the definition of $\mathcal{T}_1$ implies the cycle C is in the forest $T \times (w_1x_1 \cup y_1z_1)$, so $C = 0$. Thus $\mathcal{D} \cup \mathcal{T}_1$ is an independent set.

By exactly the same reasoning, if $C \in \text{span}(\mathcal{D} \cup \mathcal{T}_1) \cap \mathcal{T}_2$, then $C = 0$ and $\mathcal{D} \cup \mathcal{T}_1 \cup \mathcal{T}_2$ is an independent set. Continuing in this fashion, $\mathcal{W} = \mathcal{D} \cup \left(\bigcup_{1 \le j \le \nu(H)} \mathcal{T}_j \right)$ is independent.

Observe that no element of $\mathcal{W}$ contains an edge t_iu_i for $1 \le i \le \nu(G)$, so no cycle in $\text{span}(\mathcal{W})$ contains such an edge. Hence, by definition of $\mathcal{U}_1$, if $C \in \text{span}(\mathcal{W}) \cap \mathcal{U}_1$, then C is in the forest $(s_1t_1 \cup u_1v_1) \times U$, so $C = 0$, and $\mathcal{W} \cup \mathcal{U}_1$ is an independent set. Continuing with this reasoning, we may append the remaining sets $\mathcal{U}_i$ to obtain the independent set $\mathcal{X} = \mathcal{D} \cup \left(\bigcup_{1\le j\le \nu(H)} \mathcal{T}_j\right) \cup \left(\bigcup_{1\le i\le \nu(G)} \mathcal{U}_i\right)$.

Finally, note that any set $\mathcal{S}_{ij}$ contains exactly two diamonds, call them D^1_{ij} and D^2_{ij}, the first containing the edge $e^1_{ij} = (t_i, x_j)(u_i, y_j)$, and the second containing the edge $e^2_{ij} = (t_i, y_j)(u_i, x_j)$. Observe that D^1_{ij} is the only element of $\mathcal{B}$ containing e^1_{ij}, and D^2_{ij} is the only element of $\mathcal{B}$ containing e^2_{ij}, so neither can be expressed as a linear combination of the other elements in $\mathcal{B}$. Therefore all the sets $\mathcal{S}_{ij}$ can be appended to $\mathcal{X}$, resulting in $\mathcal{B}$, with independence preserved. This completes the demonstration that $\mathcal{B}$ is independent.

Say G has p vertices and q edges, and H has r vertices and s edges. Computing cardinality, $|\mathcal{B}| = (p-2)(r-2) + 2\nu(H)(p-2) + 2\nu(G)(r-2) + 2\nu(G)\nu(H) = 2(p-2+\nu(G))(r-2+\nu(H)) - (p-2)(r-2) = 2(q-1)(s-1) - (p-2)(r-2) = (2qs - pr + 2) - 2(q-p+1) - 2(s-r+1) = \nu(G\times H) - 2\nu(G) - 2\nu(H)$. This, combined with Proposition 2, implies that $\mathcal{B}$ is a basis for $\mathcal{D}(G\times H)$ if G and H are bipartite. ■

Construction 2 implies $\dim(\mathcal{D}(G\times H)) = \nu(G\times H) - 2\nu(G) - 2\nu(H)$, and combining this with Proposition 2 produces

Corollary 1. *If G and H are connected bipartite graphs, $e \in E(G)$ and $f \in E(H)$, then $\mathcal{C}(G\times H) = \mathcal{C}(G\times f) \oplus \mathcal{D}(G\times H) \oplus \mathcal{C}(e\times H)$.*

5 A Minimum Cycle Basis

For brevity, set $\Lambda = \mathcal{C}(G\times f)$, $\Delta = \mathcal{D}(G\times H)$, and $\Phi = \mathcal{C}(e\times H)$, so Corollary 1 says $\mathcal{C}(G\times H) = \Lambda \oplus \Delta \oplus \Phi$.

Proposition 7. *If C is a relevant cycle in $\mathcal{C}(G\times H)$, then one of the following holds.*

1. $C \in \Lambda$, or $C \in \Delta$, or $C \in \Phi$
2. $C = L + D$ with $L \in \Lambda$, $D \in \Delta$, and $|C| = |L|$
3. $C = D + F$ with $D \in \Delta$, $F \in \Phi$, and $|C| = |F|$
4. $C = L + D + F$ with $L \in \Lambda$, $D \in \Delta$, $F \in \Phi$ and $|C| = \max\{|L|, |F|\}$

Proof. Since $C \in \Lambda \oplus \Delta \oplus \Phi$, then either $C = L$, $C = D$, $C = F$, $C = L + D$, $C = L + F$, $C = D + F$, or $C = L + D + F$, for nonzero elements $L \in \Lambda$, $D \in \Delta$, and $F \in \Phi$.

Note that if C is relevant, then it is impossible that $C = L + F$. Otherwise, by definition of Γ and Φ, $L \cap F \subseteq E(e\times f)$. Since $e\times f$ consists of just two edges, one in each component of $G\times H$, and the relevant cycle C must be in a single component, it follows that $|L \cap F| \le 1$. Then $C = L + F$ is the sum of two cycles shorter than itself, contradicting relevancy.

Suppose $C = L + D$ is relevant. Regard π_G as the projection of $G\times H$ onto the component of $G\times f$ that contains C. Then $\pi^*_G(C) = \pi^*_G(L) + \pi^*_G(D) = L$, from which $|C| \ge |L|$ is deduced. If $|C| > |L| \ge 4$, then as D is a sum of 4-cycles, $C = L + D$ is the sum of cycles shorter than C. Hence $|C| = |L|$, by Proposition 1. In an analogous manner, if $C = D + F$ is relevant, then $|C| = |F|$.

Finally, if $C = L + D + F$ is relevant, then, as above, $\pi_G^*(C) = L$ and $\pi_H^*(C) = F$, from which $|C| \geq \max\{|L|, |F|\}$. But if the inequality were strict, then C would be a sum of shorter cycles, so $|C| = \max\{|L|, |F|\}$. ■

Construction 3. (An MCB for $G \times H$ where G and H are connected bipartite graphs.)

Let e and f be edges of G and H, respectively, so $G \times f$ consists of two isomorphic copies of G, and $e \times H$ consists of two isomorphic copies of H. Let $\mathcal{L}$, $\mathcal{L}' \subseteq \Lambda$ be MCB's for the two components of $G \times f$, respectively. Let $\mathcal{F}$, $\mathcal{F}' \subseteq \Phi$ be MCB's for the two components of $e \times H$, respectively. Let $\mathcal{B}$ be the basis of diamonds for $\Delta = \mathcal{D}(G \times H)$, as described in Construction 2. Then the set $\mathcal{M} = \mathcal{B} \cup \mathcal{L} \cup \mathcal{L}' \cup \mathcal{F} \cup \mathcal{F}'$ is a MCB for $G \times H$. Moreover, the length of this basis is $l(G \times H) = 4\nu(G \times H) - 8\nu(G) - 8\nu(H) + 2l(G) + 2l(H)$.

Proof. By Corollary 1, $\mathcal{C}(G \times H) = \Lambda \oplus \Delta \oplus \Phi$. Using the Greedy Algorithm to extract an MCB, we begin by finding a maximal independent set of 4-cycles. Start with $\mathcal{M} = \mathcal{B} \subseteq \Delta$. Next, append to $\mathcal{M}$ a maximal independent set of (relevant) 4-cycles in $\Lambda \cup \Phi$, and for this it suffices to first take all the 4-cycles in $\mathcal{L} \cup \mathcal{L}' \cup \mathcal{F} \cup \mathcal{F}'$.

We claim that, at this point, all other relevant 4-cycles are linear combinations of those already in $\mathcal{M}$. For, using the result and notation of Proposition 7, any other relevant 4-cycle not in $\Lambda \cup \Delta \cup \Phi$ is either of form $C = L + D$ with $|C| = |L|$, or $C = D + F$ with $|C| = |F|$ or $C = L + D + F$ with $|C| = \max\{|L|, |F|\}$. If $C = L + D$, the 4-cycle $L \in \Lambda$ must be a linear combination of 4-cycles already in $\mathcal{M}$ (for otherwise it would have been previously appended to $\mathcal{M}$), and certainly D is a linear combination of elements of $\mathcal{B} \subseteq \mathcal{M}$. Similarly any $C = D + F$ is a combination of elements of $\mathcal{M}$. If $C = L + D + F$, with $|C| = \max\{|L|, |F|\} = 4$, then, again, 4-cycles $L \in \Lambda$ and $F \in \Phi$ must be linear combinations of those already in $\mathcal{M}$. Thus $\mathcal{M}$ contains a maximal independent set of relevant 4-cycles.

Next, we append relevant 6-cycles to $\mathcal{M}$. Begin by appending the 6-cycles in $\mathcal{L} \cup \mathcal{L}' \cup \mathcal{F} \cup \mathcal{F}' \subseteq \Gamma \cup \Phi$. We claim that, at this point, all other relevant 6-cycles are linear combinations of cycles already in $\mathcal{M}$. For any such 6-cycle is either of form $C = L + D$, or $C = D + F$ or $C = L + D + F$. If $C = L + D$, then $6 = |C| = |L|$, and the 6-cycle $L \in \Lambda$ must be a linear combination of 4- and 6-cycles already in $\mathcal{M}$, and certainly D is a linear combination of 4-cycles in $\mathcal{B} \subseteq \mathcal{M}$. Similarly any $C = D + F$ is a combination of elements of $\mathcal{M}$. If $C = L + D + F$, with $6 = |C| = \max\{|L|, |F|\}$, then since $L \in \Lambda$ and $F \in \Phi$ are cycles of length no greater than 6, they must be linear combinations of cycles already in $\mathcal{M}$. Thus no further relevant 6-cycles may be appended to $\mathcal{M}$ without destroying independence.

Applying this same reasoning to 8-cycles, then 10-cycles, and so on, $\mathcal{M}$ must eventually become an MCB, and, by construction, it is $\mathcal{M} = \mathcal{B} \cup \mathcal{L} \cup \mathcal{L}' \cup \mathcal{F} \cup \mathcal{F}'$. Computing length, $l(G \times H) = 4|\mathcal{B}| + 2l(G) + 2l(H) = 4\nu(G \times H) - 8\nu(G) - 8\nu(H) + 2l(G) + 2l(H)$. ■

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