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Contractible edges and removable edges in a graph with large minimum degree

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Abstract

For an edge e of a graph G, we denote by G/e the graph obtained from G by contraction of e. Let $k \geq 4$ be an integer. We prove that for each edge of a k-connected graph of minimum degree at least $\lfloor \frac{3}{2}k \rfloor - 1$, either G/e or $G - e$ is k-connected.

1 Introduction

An edge e of a graph G is said to be *k-contractible* if the graph obtained from G by contraction of e, denoted by G/e, is k-connected. When we investigate a property of k-connected graphs, a k-contractible edge gives us a basis for an inductive argument. For this reason, it is worthwhile to study the distribution of k-contractible edges in a k-connected graph. Tutte [7] proved that a 3-connected graph of order at least five has a 3-contractible edge. Since then numerous results on the distribution of 3-contractible edges in a 3-connected graph have been obtained. Those who are interested in this topic may consult Kriesell's survey [4].

An edge e of a graph G is said to be *3-removable* if $G - e$ is homeomorphic to a 3-connected graph. In other words, if the graph obtained from $G - e$ by suppressing vertices of degree two and replacing multiple edges with single edges is 3-connected, then we say that e is 3-removable. Since the above operation transforms a 3-connected graph to another 3-connected graph with fewer edges, a 3-removable edge can also act as a basis for an inductive argument for 3-connected graphs. The distribution of 3-removable edges in a 3-connected graph was studied in [3] and [6].

Though the previous researches have given insight into 3-contractible edges and 3-removable edges, few of them deal with both types of edges simultaneously. But Tutte [7] proved the following theorem.

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Theorem A ([7]) *Every edge in a 3-connected graph of order at least five is either 3-contractible or 3-removable.*

The purpose of this paper is to give an extension of the above theorem for graphs of higher connectivity.

One obstacle to this goal is the definition of a "k-removable" edge. Let e be an edge of a k-connected graph ($k \geq 3$). If e is incident with a vertex of degree k, then $G - e$ has a vertex of degree $k - 1$, and hence it is no longer k-connected. There exist infinitely many k-connected k-regular graphs, and for such graphs, we cannot simply delete an edge and expect the resulting graph is k-connected. For $k = 3$, in order to remove this obstacle, we suppress vertices of degree two. However, for $k \geq 4$, it is not easy to deal with vertices of degree $k - 1$ by a simple operation like suppression. Recently, Yin [9] defined a 4-removable edge in a 4-connected by introducing a relatively simple operation to handle vertices of degree three, and Wu, Li and Su [8] gave a lower bound to the number of 4-removable edges in a 4-connected graph. But for $k \geq 5$, no definition of a k-contractible edge has been proposed.

In this paper, we avoid this obstacle by restricting ourselves to k-connected graphs of minimum degree at least $k + 1$. For such graphs, no vertex of degree $k - 1$ arises by deleting an edge, and hence we can use the following simple definition.

Definition 1 *For $k \geq 4$, an edge e of a k-connected graph of minimum degree at least $k + 1$ is said to be k-removable if $G - e$ is k-connected.*

In this paper, we prove the following theorem. Note $\lfloor \frac{3}{2}k \rfloor - 1 \geq k + 1$ for $k \geq 4$.

Theorem 2 *Let $k \geq 4$ and let G be a k-connected graph of minimum degree at least $\lfloor \frac{3}{2}k \rfloor - 1$. Then every edge of G is either k-contractible or k-removable.*

For graph-theoretic terminology not defined in this paper, we refer the reader to [1]. Let G be a graph. For $x \in V(G)$, we denote by $N_G(x)$ the neighborhood of x in G, and for $A \subset V(G)$, we define $N_G(A)$ by $N_G(A) = \bigcup_{v \in A} N_G(v)$.

Let G be a non-complete graph, and let S be a minimum cutset of G. A union of at least one, but not all, components of $G - S$ is called a *fragment* associated with S. If $|S| = k$, it is also called a *k-fragment*. Note that if A is a k-fragment of a graph G, then $S = N_G(A) - A$ is a minimum cutset of G of order k, and both A and $V(G) - (A \cup N_G(A))$ are k-fragments associated with S.

2 Proof of the Main Theorem

In order to prove Theorem 2, we use the following lemma. Its proof is immediate from the definition of G/e and $G - e$.

Lemma 3 *Let G be a non-complete k-connected graph and let $e = xy$ be an edge of G.*

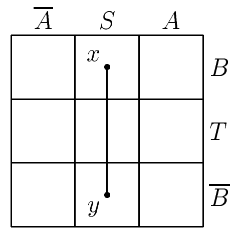

Figure 1: Fragments

(1) If G/e is not k-connected, then there exists a k-fragment A of G such that $\{x, y\} \subset N_G(A) - A$.

(2) If $G - e$ is not k-connected, then there exists a $(k-1)$-fragment B of $G - e$ such that $x \in B$ and $y \in \overline{B}$, where $\overline{B} = V(G) - (B \cup N_{G-e}(B))$.

Now we prove Theorem 2.

Proof of Theorem 2. The theorem is trivial if G is complete. Thus, we may assume that G is not a complete graph. Assume $e = xy \in E(G)$ is neither k-contractible nor k-removable. Since e is not k-contractible, there exists a k-fragment A of G such that $\{x, y\} \subset N_G(A) - A$ by Lemma 3 (1). Furthermore, since e is not k-removable, there exists a $(k-1)$-fragment B of $G - e$ such that $x \in B$ and $y \in \overline{B}$, where $\overline{B} = V(G) - (B \cup N_{G-e}(B))$, by Lemma 3 (2). Let $S = N_G(A) - A$, $\overline{A} = V(G) - (A \cup N_G(A))$ and $T = N_{G-e}(B) - B$. Let

$$\begin{aligned} X_1 &= (S \cap B) \cup (S \cap T) \cup (A \cap T), \\ X_2 &= (A \cap T) \cup (S \cap T) \cup (S \cap \overline{B}), \\ X_3 &= (S \cap \overline{B}) \cup (S \cap T) \cup (\overline{A} \cap T), \text{and} \\ X_4 &= (\overline{A} \cap T) \cup (S \cap T) \cup (S \cap B). \end{aligned}$$

Note $x \in B \cap S$ and $y \in \overline{B} \cap S$, $|S| = k$ and $|T| = k - 1$. (See Figure 1.)

Claim 1

(1) If $A \cap B \neq \emptyset$, then $|X_1| \geq k$.

(2) If $A \cap \overline{B} \neq \emptyset$, then $|X_2| \geq k$.

(3) If $\overline{A} \cap \overline{B} \neq \emptyset$, then $|X_3| \geq k$.

(4) If $\overline{A} \cap B \neq \emptyset$, then $|X_4| \geq k$.

Proof. We prove (1). Since S is a cutset of G and T is a cutset of $G-xy$, there does not exist an edge in G which joins a vertex in $A \cap B$ and a vertex in $\overline{A} \cup \overline{B}$. Therefore, if $A \cap B \neq \emptyset$, then X_1 is a cutset of G and hence $|X_1| \geq k$ since G is k-connected. We can prove (2), (3) and (4) in a similar way. $\square$

Claim 2

(1) Either $A \cap B = \emptyset$ *or* $\overline{A} \cap \overline{B} = \emptyset$.

(2) Either $\overline{A} \cap B = \emptyset$ *or* $A \cap \overline{B} = \emptyset$.

Proof. We prove (1). If $A \cap B \neq \emptyset$, then $|X_1| \geq k$ by Claim 1. Since $|X_1| + |X_3| = |S| + |T| = 2k-1$, we have $|X_3| \leq k-1$, which implies $\overline{A} \cap \overline{B} = \emptyset$, again by Claim 1. We can prove (2) in a similar way. $\square$

Claim 3

(1) $|A| \geq \left\lfloor \frac{1}{2}k \right\rfloor$ *and* $|\overline{A}| \geq \left\lfloor \frac{1}{2}k \right\rfloor$

(2) $|B| \geq \left\lfloor \frac{1}{2}k \right\rfloor + 1$ *and* $|\overline{B}| \geq \left\lfloor \frac{1}{2}k \right\rfloor + 1$

Proof. (1) Take a vertex v in A. Then $\deg_G v \geq \delta(G) \geq \left\lfloor \frac{3}{2}k \right\rfloor - 1$ and $\{v\} \cup N_G(v) \subset A \cup S$. Since $|S| = k$, we have $k + |A| = |A \cup S| \geq \deg_G v + 1 \geq \left\lfloor \frac{3}{2}k \right\rfloor$, which implies $|A| \geq \left\lfloor \frac{1}{2}k \right\rfloor$. We can prove $|\overline{A}| \geq \left\lfloor \frac{1}{2}k \right\rfloor$ in a similar way.
(2) Applying the same argument as above to x and observing $\{x\} \cup N_G(x) \subset B \cup T \cup \{y\}$, we have $\left\lfloor \frac{3}{2}k \right\rfloor \leq |B \cup T \cup \{y\}| = |B| + k$, which implies $|B| \geq \left\lfloor \frac{1}{2}k \right\rfloor$. Since $k \geq 4$, we have $|B| \geq 2$. Then we can take a vertex u in $B - \{x\}$, and apply the same argument to u, observing $\{u\} \cup N_G(u) \subset B \cup T$. This implies $|B| \geq \left\lfloor \frac{1}{2}k \right\rfloor + 1$. We can prove $|\overline{B}| \geq \left\lfloor \frac{1}{2}k \right\rfloor + 1$ in a similar way. $\square$

Assume $A \cap B \neq \emptyset$ and $A \cap \overline{B} \neq \emptyset$. Then $|X_1| \geq k$, $|X_2| \geq k$, $\overline{A} \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$ by Claims 1 and 2. These imply $\overline{A} \subset T$ and $|\overline{A} \cap T| = |\overline{A}| \geq \left\lfloor \frac{1}{2}k \right\rfloor$ by Claim 3. Since $|T| = k-1$, $|S \cap T| + |A \cap T| \leq k - 1 - \left\lfloor \frac{1}{2}k \right\rfloor = \left\lceil \frac{1}{2}k \right\rceil - 1$. Since $|X_1| \geq k$ and $|X_2| \geq k$, $|S \cap B| \geq k - \left(\left\lceil \frac{1}{2}k \right\rceil - 1\right) = \left\lfloor \frac{1}{2}k \right\rfloor + 1$ and $|S \cap \overline{B}| \geq \left\lfloor \frac{1}{2}k \right\rfloor + 1$. However, these imply $|S| \geq 2\left(\left\lfloor \frac{1}{2}k \right\rfloor + 1\right) \geq k+1$, a contradiction. Therefore, we have either $A \cap B = \emptyset$ or $A \cap \overline{B} = \emptyset$.

Assume $A \cap B \neq \emptyset$. Then $A \cap \overline{B} = \emptyset$. By Claims 1 and 2, $|X_1| \geq k$ and $\overline{A} \cap \overline{B} = \emptyset$. Therefore, $\overline{B} \subset S$. By Claim 3, $|S \cap \overline{B}| = |\overline{B}| \geq \left\lfloor \frac{1}{2}k \right\rfloor + 1$. This implies $|S \cap B| + |S \cap T| \leq k - \left(\left\lfloor \frac{1}{2}k \right\rfloor + 1\right) = \left\lceil \frac{1}{2}k \right\rceil - 1$. Since $|X_1| \geq k$, we have $|A \cap T| \geq k - \left(\left\lceil \frac{1}{2}k \right\rceil - 1\right) = \left\lfloor \frac{1}{2}k \right\rfloor + 1$, and since $|T| = k-1$, $|\overline{A} \cap T| + |S \cap T| \leq (k-1) - \left(\left\lfloor \frac{1}{2}k \right\rfloor + 1\right) = \left\lceil \frac{1}{2}k \right\rceil - 2$. Now, we have

$$|X_4| \leq (|S \cap B| + |S \cap T|) + (|S \cap T| + |\overline{A} \cap T|) \leq \left\lceil \frac{1}{2}k \right\rceil - 1 + \left\lceil \frac{1}{2}k \right\rceil - 2 \leq k - 2$$

and hence $\overline{A} \cap B = \emptyset$ by Claim 1. Therefore, we have $\overline{A} \subset T$. By Claim 3, we have $|\overline{A} \cap T| \geq \lfloor \frac{1}{2}k \rfloor$. However, this contradicts $|\overline{A} \cap T| + |S \cap T| \leq \lceil \frac{1}{2}k \rceil - 2$. Hence we have $A \cap B = \emptyset$. By symmetry, we also have $A \cap \overline{B} = \emptyset$, and hence $A \subset T$.

By applying the same arguments as above to the fragment $\overline{A}$, we have $\overline{A} \cap B = \overline{A} \cap \overline{B} = \emptyset$. These imply $B \subset S$ and $\overline{B} \subset S$. Then Claim 3 (2) forces $|S \cap B| \geq \lfloor \frac{1}{2}k \rfloor + 1$ and $|S \cap \overline{B}| \geq \lfloor \frac{1}{2}k \rfloor + 1$. However, these imply $|S| \geq 2\left(\lfloor \frac{1}{2}k \rfloor + 1\right) \geq k+1$. This is a final contradiction, and the theorem follows. □

3 Concluding Remarks

Theorem 2 is best-possible in the sense that there exist infinitely many k-connected graphs G of minimum degree $\lfloor \frac{3}{2}k \rfloor - 2$ such that G has an edge which is neither k-contractible nor k-removable ($k \geq 4$). If k is an even integer, put $k = 2l$. Let m be an integer with $m \geq l-1$, and let H_1, H_2, H_3, H_4 and K be five complete graphs on mutually disjoint sets of vertices with $|H_1| = |H_2| = |H_4| = l$, $|H_3| = l-1$ and $|K| = m$. Take a vertex x in H_2 and a vertex y in H_4. Let G_m be the graph defined by

$$V(G) = \bigcup_{i=1}^{4} V(H_i) \cup V(K)$$

$$E(G) = \bigcup_{i=1}^{4} E(H_i) \cup E(K) \cup \{uv \colon u \in V(H_i), v \in V(H_{i+1}), 1 \leq i \leq 4\}$$
$$\cup \{uv \colon u \in V(K), v \in V(H_1) \cup V(H_4)\} \cup \{xy\},$$

where we consider $H_5 = H_1$. Then G_m is a $2l$-connected graph of minimum degree $3l - 2 = \lfloor \frac{3}{2}k \rfloor - 2$, and the edge xy is neither k-contractible nor k-removable. If k is an odd integer, then put $k = 2l+1$, where $l \geq 2$, and the follow the same construction as above, but with $|H_1| = |H_2| = |H_3| = l$, $|H_4| = l+1$ and $|K| = m$, where $m \geq l-1$. Then the resulting graph is a $(2l+1)$-connected graph of minimum degree $3l - 1 = \lfloor \frac{3}{2}k \rfloor - 2$ and the edge xy is neither k-contractible nor k-removable.

For $k \geq 4$, it is not difficult to construct a k-connected $\left(\lfloor \frac{3}{2}k \rfloor - 1\right)$-regular graph in which every edge lies in a triangle. (One example is the Cartesian product of $K_{\lfloor \frac{3}{2}k \rfloor - 2}$ and K_3.) For each edge e in such a graph, both contraction and removal of e yields a vertex of degree $\lfloor \frac{2}{3}k \rfloor - 2$. Thus, in Theorem 2, we cannot hope to find an edge e such that either removal or contraction of e results in a k-connected graph which also has minimum degree at least $\lfloor \frac{3}{2}k \rfloor - 1$.

By extending the definition of a k-removable edge in Definition 1, we can unite Theorem A and Theorem 2.

Definition 4 *For $k \geq 1$, an edge e of a k-connected graph G is said to be k-removable if $G - e$ is homeomorphic to a k-connected graph.*

For $k \geq 4$, $G-e$ is homeomorphic to a k-connected graph if and only if $G-e$ itself is a k-connected graph. Therefore, this definition is a common extension of Definition 1 and the definition of a 3-removable edge in [3] and [6].

Under Definition 4, Theorem A says that Theorem 2 also holds for 3-connected graphs of order at least five. Dirac [2] and Plummer [5] characterized the edges e in a 2-connected graph G such that $G-e$ is not 2-connected. From their results, it is easy to see that every edge in a 2-connected graph of order at least five is either 2-removable or 2-contractible. This implies that Theorem 2 holds for $k=2$. Trivially, every edge in a connected graph is 1-contractible. By combining these observations, we have the following.

Theorem 5 *Let k be a positive integer. Then every edge in a k-connected graph of minimum degree at least $\lfloor \frac{3}{2}k \rfloor - 1$ and order at least $k+2$ is either k-removable or k-contractible.*

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