AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **35** (2006), Pages 253–261

Super edge-antimagic labeling of a cycle with a chord

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Abstract

An (a, d)-edge-antimagic total labeling of G is a one-to-one mapping g taking the vertices and edges onto $1, 2, \dots, |V(G)| + |E(G)|$ so that the edge-weights $w(uv) = g(u)+g(v)+g(uv)$, $uv \in E(G)$, form an arithmetic progression with initial term a and common difference d. Such a labeling is called super if the smallest labels appear on the vertices. In this paper, we investigate the existence of super (a, d)-edge-antimagic total labelings of graphs derived from cycles by adding one chord.

1. Introduction and Definitions

We follow either Wallis [9] or West [10] for most of the graph theory terminology and notation used in this paper. In particular, we will consider a graph to be finite and without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, whereas the edge set of G is denoted by $E(G)$.

A *labeling* of a graph is any mapping that sends some set of graph elements to a set of positive integers. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. Moreover, in this paper we deal with the case where the domain is $V(G) \cup E(G)$, and these are called *total labelings*.

We define *edge-weight* of an edge uv under a vertex labeling to be the sum of the vertex labels corresponding to every vertex u and v. Under a total labeling, we also add the label of uv.

By an (a,d)*-edge-antimagic vertex labeling* we mean a one-to-one mapping from $V(G)$ onto $\{1,2,\dots,|V(G)|\}$ such that the set of edge-weights of all edges in G is $\{a, a+d, \dots, a+(|E(G)|-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

An (a,d)*-edge-antimagic total labeling* is defined as a one-to-one mapping from $V(G) \cup E(G)$ onto the set $\{1,2,\dots,|V(G)|+|E(G)|\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a+d, ..., a+(|E(G)|-1)d\}$, for two integers $a > 0$ and $d \geq 0$.

An (a,d)-edge-antimagic total labeling will be called *super* if it has the property that the vertex-labels are the integers $1,2,\dots,|V(G)|$, the smallest possible labels. A graph with an (a,d)-edge-antimagic total labeling or super (a,d)-edge-antimagic total labeling will be called (a,d)-edge-antimagic total or super (a,d)-edge-antimagic total, respectively.

These labelings are natural extensions of the notions of edge-magic labeling which was introduced by Kotzig and Rosa [5,6] and studied in [2,4,8], and the notion of super edge-magic labeling which was defined by Enomoto *et al.* in [3]. Note that MacDougall and Wallis in [7] called super edge-magic labeling "strongly edge-magic".

Additionally, Acharya and Hegde in [1] introduced the concept of a *strongly* (k,d)*-indexable labeling* which is equivalent to an (a,d)-edge-antimagic vertex labeling.

Let C_n be the cycle with $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. We shall write C_n^t to mean the graph constructed from a cycle C_n by joining two vertices whose distance in the cycle is t. For $n \geq 4$, $2 \leq t \leq n-2$, the graph C_n^t is of course also the graph C_n^{n-t}.

In the present article we investigate the values of t for which there exists a super (a,d)-edge-antimagic total labeling of C_n^t. If n is odd we can restrict our attention to t odd or even and if n is even we will pay attention to t at most $\frac{n}{2}$.

2. Basic Counting

The first result in this section finds an upper bound for the feasible values of the parameter d for a super (a,d)-edge-antimagic total labeling of C_n^t.

Theorem 1. *If C_n^t, $n \geq 4$, $2 \leq t \leq n-2$, is super (a,d)-edge-antimagic total then $d < 3$.*

Proof. Suppose C_n^t, $n \geq 4$, $2 \leq t \leq n-2$, has a super (a,d)-edge-antimagic total labeling $g : V(C_n^t) \cup E(C_n^t) \to \{1,2,\dots,n\} \cup \{n+1, n+2, \dots, 2n+1\}$. Consider the extreme values of vertices and edges. The maximum edge-weight is no more than $4n$. On the other hand, the minimum possible edge-weight is at least $n+4$.

Thus we have

$$a + nd \leq 4n$$

and

$$d \leq \frac{3n-4}{n} = 3 - \frac{4}{n} < 3$$

for every $n \geq 4$.

□

Suppose the endpoints of the chord receive labels x and y. The following result provides the values a and $x+y$ under a super (a,d)-edge-antimagic total labeling.

Theorem 2. *Let C_n^t, $n \geq 4$, $t \geq 2$, be super (a,d)-edge-antimagic total.*
If $d=0$ and $n=2k$, then $x+y=2k+1$ and $a=5k+2$.
If $d=0$ and $n=2k+1$, then either $x+y=k+1$ and $a=5k+4$, or
$x+y=3k+3$ and $a=5k+5$.
If $d=1$ then $x+y=n+1$ and $a=2n+2$.
If $d=2$ and $n=2k$, then $x+y=2k+1$ and $a=3k+2$.
If $d=2$ and $n=2k+1$, then either $x+y=k+1$ and $a=3k+3$, or
$x+y=3k+3$ and $a=3k+4$.

Proof. Assume that C_n^t, $n \geq 4$, $t \geq 2$, has a super (a,d)-edge-antimagic total labeling $g : V(C_n^t) \cup E(C_n^t) \to \{1,2,\dots,2n+1\}$ and $W = \{w(uv) : w(uv) = g(u)+g(v)+g(uv), uv \in E(C_n^t)\} = \{a, a+d, \dots, a+nd\}$ is the set of edge-weights.

The sum of edge-weights in the set W is

$$\sum_{uv \in E(C_n^t)} w(uv) = (n+1)a + \frac{(n+1)n}{2}d \ .$$

In the computation of the edge-weights of C_n^t the label of each edge is used once, the labels of endpoints of the chord are used three times each and the labels of all the other vertices are used two times each.

Thus

$$2\sum_{i=1}^{n} g(v_i) + x + y + \sum_{uv \in E(C_n^t)} g(uv) = \frac{5n^2+7n+2}{2} + x + y \ .$$

Combining these two equations gives us

$$a = \frac{5n+2}{2} + \frac{x+y}{n+1} - \frac{n}{2}d \ .$$

Let us consider three cases.

Case 1. $d = 0$.

If n is even, say $n = 2k$, then $a = 5k + 1 + \frac{x+y}{2k+1}$. The value a is an integer when $x + y \equiv 0 \pmod{2k+1}$. The condition $1 \le x, y \le 2k$ implies $x + y = 2k + 1$. Consequently $a = 5k + 2$.

If n is odd, $n = 2k + 1$, then $a = 5k + \frac{7}{2} + \frac{x+y}{2k+2}$. Applying the conditions that a is an integer and $1 \le x, y \le 2k + 1$ we get either $x + y = k + 1$ and $a = 5k + 4$, or $x + y = 3k + 3$ and $a = 5k + 5$.

Case 2. $d = 1$.

In this case $a = 2n + 1 + \frac{x+y}{n+1}$. Using the condition $1 \le x, y \le n$ we can see that a is an integer if and only if $x + y = n + 1$. Thus $a = 2n + 2$.

Case 3. $d = 2$.

If n is even, $n = 2k$, then $a = 3k + 1 + \frac{x+y}{2k+1}$ is an integer if $x + y \equiv 0 \pmod{2k+1}$. Applying the condition $1 \le x, y \le 2k$ implies $x + y = 2k + 1$ and $a = 3k + 2$.

When n is odd, $n = 2k + 1$, we get $a = 3k + \frac{5}{2} + \frac{x+y}{2k+2}$. Since $1 \le x, y \le 2k + 1$, it follows that either $x + y = k + 1$ and $a = 3k + 3$, or $x + y = 3k + 3$ and $a = 3k + 4$. □

3. Known Results

The following was proved in [2]:

Proposition A. *Let G be a graph which admits total labeling and whose edge labels constitute an arithmetic progression with difference d. Then the following are equivalent.*

(i) G has a $(k, 0)$-edge-antimagic total labeling.

(ii) G has a $(k - (|E(G)| - 1)d, 2d)$-edge-antimagic total labeling.

The following five results were obtained by MacDougall and Wallis in [7]. We rewrite them in the light of our terminology.

Proposition B. *C^t_{4m+3}, $m \ge 1$, has a super $(a, 0)$-edge-antimagic total labeling for all possible values t with $a = 10m + 9$ or $a = 10m + 10$.*

Proposition C. *C^t_{4m+1}, $m \ge 3$, has a super $(a, 0)$-edge-antimagic total labeling for every t other than $t = 5, 9, 4m - 4, 4m - 8$ with $a = 10m + 4$ or $a = 10m + 5$.*

Proposition D. *C^t_{4m+1}, $m \ge 1$, has a super $(10m + 4, 0)$-edge-antimagic total labeling for every $t \equiv 1 \pmod 4$ except $4m - 3$.*

Proposition E. *C^t_{4m}, $m \ge 1$, has a super $(10m + 2, 0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$.*

Proposition F. *C^t_{4m+2}, $m \geq 4$, has a super $(10m+7,0)$-edge-antimagic total labeling for all $t \equiv 3 \pmod 4$ and for $t = 2, 6$.*

4. Super (a,d)-edge-antimagic total labeling for C^t_n

In this section we present the values of t for which there exists a super (a,d)-edge-antimagic total labeling of C^t_n.

Theorem 3. *For n odd, $n = 2k+1 \geq 5$, and for all possible values t every graph C^t_n has*

(i) a super $(a,0)$-edge-antimagic total labeling with $a = 5k+4$ or $a = 5k+5$ and

(ii) a super $(a,2)$-edge-antimagic total labeling with $a = 3k+3$ or $a = 3k+4$.

Proof. It follows from the propositions B, C and D that every graph C^t_n, n odd, $n = 2k+1 \geq 13$, has a super $(a,0)$-edge-antimagic total labeling for all possible values t with $a = 5k+4$ or $5k+5$.

Now, for $i = 1,2,3,4$ we construct the vertex labeling $g_i : V(C^t_{2i+3}) \to \{1, 2, \ldots, 2i+3\}$ in the following way:

$$g_1(v_j) = \begin{cases} \frac{j+1}{2} & \text{if } j = 1,3,5 \\ \frac{j+6}{2} & \text{if } j = 2,4 \end{cases}$$

$g_2(v_1) = 1$, $g_2(v_2) = 4$, $g_2(v_3) = 2$, $g_2(v_4) = 5$, $g_2(v_5) = 6$, $g_2(v_6) = 3$, $g_2(v_7) = 7$.

$g_3(v_1) = 1$, $g_3(v_2) = 5$, $g_3(v_3) = 7$, $g_3(v_4) = 2$, $g_3(v_5) = 6$, $g_3(v_6) = 8$, $g_3(v_7) = 3$, $g_3(v_8) = 4$, $g_3(v_9) = 9$.

$g_4(v_1) = 6$, $g_4(v_2) = 7$, $g_4(v_3) = 1$, $g_4(v_4) = 8$, $g_4(v_5) = 2$, $g_4(v_6) = 9$, $g_4(v_7) = 3$, $g_4(v_8) = 4$, $g_4(v_9) = 10$, $g_4(v_{10}) = 5$, $g_4(v_{11}) = 11$.

It is a matter for routine checking to see that the labeling g_i is $(b,1)$-edge-antimagic vertex labeling of C^t_{2i+3} for $i \in \{1,2,3,4\}$ and for all possible t with $b = i+2$ or $b = i+3$. We are able to arrange the edge values $\{2i+4, 2i+5, \ldots, 4i+7\}$ to the edges of C^t_{2i+3} for $i \in \{1,2,3,4\}$ such that the resulting labelings are super $(c,0)$-edge-antimagic total for all possible values t with $c = 5i+9$ or $c = 5i+10$.

Note that, under every super (a,d)-edge-antimagic total labeling, the set of edge values of G consists of the consecutive integers $\{|V(G)|+1, |V(G)|+2, \ldots, |V(G)|+|E(G)|\}$. Combining the above described facts with Proposition A, it is easy to see that C^t_n for n odd, $n = 2k+1 \geq 5$, has a super $(a,2)$-edge-antimagic total labeling for all possible values t with $a = 3k+3$ or $a = 3k+4$.

□

Theorem 4. *For $n \equiv 0 \pmod 4$, $n \geq 4$, the graph C_n^t has*

(i) a super $(\frac{5n}{2}+2, 0)$-edge-antimagic total labeling and

(ii) a super $(\frac{3n}{2}+2, 2)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$.

Proof. The existence of super $(\frac{5n}{2}+2, 0)$-edge-antimagic total labeling of C_n^t, $n \equiv 0 \pmod 4$, $n \geq 4$, for all $t \equiv 2 \pmod 4$ follows from Proposition E [7]. Combining this with Proposition A we can see that C_n^t, $n \equiv 0 \pmod 4$, $n \geq 4$, has a super $(\frac{3n}{2}+2, 2)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$.

□

Theorem 5. *For $n = 10$ and for $n \equiv 2 \pmod 4$, $n \geq 18$, the graph C_n^t has*

(i) a super $(\frac{5n}{2}+2, 0)$-edge-antimagic total labeling and

(ii) a super $(\frac{3n}{2}+2, 2)$-edge-antimagic total labeling for all $t \equiv 3 \pmod 4$ and for $t = 2$ and $t = 6$.

Proof. MacDougall and Wallis (Proposition F) proved the existence of super $(\frac{5n}{2}+2, 0)$-edge-antimagic total labeling of C_n^t for $n \equiv 2 \pmod 4$, $n \geq 18$ and for all $t \equiv 3 \pmod 4$ and for $t = 2, 6$. In [7] are described the 17 dual pairs of labelings for C_{10} and there is shown that C_{10}^t, for $t \in \{2, 3, 4, 5\}$, has super $(27, 0)$-edge-antimagic total labeling. Now, if we apply Proposition A then we arrive at the desired result.

□

In this notation, we remark that C_6^2 is not super (a, d)-edge-antimagic total for $d \in \{0, 2\}$ (see [7]). Moreover, for C_{14}^6 and $d \in \{0, 2\}$, we have not yet found a construction that will produce super (a, d)-edge-antimagic total labeling.

Theorem 6. *For n odd, $n \geq 5$, and for all possible values t every graph C_n^t has a super $(2n+2, 1)$-edge-antimagic total labeling.*

Proof. Let $V(C_n^t) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n^t) = \{v_i v_{i+1} : i = 1, 2, \dots, n-1\} \cup \{v_n v_1\} \cup \{chord\}$. Consider the bijection

$g_5 : V(C_n^t) \cup E(C_n^t) \to \{1, 2, \dots, n\} \cup \{n+1, n+2, \dots, 2n+1\}$ where

$$g_5(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq n \\ \frac{n+1+i}{2} & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

$$g_5(v_i v_{i+1}) = \begin{cases} \frac{4n+1-i}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ \frac{3n+1-i}{2} & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

$$g_5(v_n v_1) = \frac{3n+1}{2}$$

$$g_5(chord) = 2n+1.$$

It follows from Theorem 2 that the sum of labels of the endpoints of the chord must be $x+y=n+1$. We can see that for $1 \le i \le \frac{n-1}{2}-1$, $g_5(v_i)+g_5(v_{n-i})=n+1$. So the endpoints of the chords v_iv_{n-i} for $1 \le i \le \frac{n-1}{2}-1$ cover all even distances $\{2,4,6,\dots,n-3\}$ in the cycle and obviously all odd distances $\{n-2,n-4,n-6,\dots,3\}$. It is a routine procedure to verify that the set of edge-weights of all edges in the cycle consists of the consecutive integers $\{g_5(v_n)+g_5(v_1)+g_5(v_nv_1)\} \cup \{g_5(v_i)+g_5(v_{i+1})+g_5(v_iv_{i+1}) : 1 \le i \le n-1\} = \{2n+2, 2n+3, \dots, 3n+1\}$ and $g_5(v_i)+g_5(v_{n-i})+g_5(chord)=3n+2$ for $1 \le i \le \frac{n-1}{2}-1$.

This implies that g_5 is a super $(2n+2,1)$-edge-antimagic total labeling of C_n^t for n odd and for all possible values t.

□

Theorem 7. *For n even, $n \ge 6$, and for t odd, $t \ge 3$, the graph C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling.*

Proof. Name the vertices in C_n^t as $v_1, v_2, \dots, v_n$ and the set of edges is $E(C_n^t) = \{v_iv_{i+1} : i=1,2,\dots,n-1\} \cup \{v_nv_1\} \cup \{chord\}$. Then attach labels to all the vertices and edges as follows:

$$g_6(v_i) = i \qquad \text{for } i=1,2,\dots,n$$

$$g_6(v_iv_{i+1}) = 2n+1-i \qquad \text{for } i=1,2,\dots,n-1$$

$$g_6(v_nv_1) = n+1$$

$$g_6(chord) = 2n+1.$$

There is no problem in seeing that the labeling g_6 uses each integer $1,2,\dots,2n+1$ exactly once and this implies that the labeling g_6 is a bijection from the set $V(C_n^t) \cup E(C_n^t)$ onto the set $\{1,2,\dots,2n+1\}$.

Consider the following chords v_iv_{n+1-i} for $i=2,3,\dots,\frac{n}{2}-1$. The distances of endpoints of the chords cover all odd lengths $3,5,7,\dots,n-3$ in the cycle and the edge-weight for every chord is $g_6(v_i)+g_6(v_{n+1-i})+g_6(chord)=(n+1)+(2n+1)$. Moreover, it is easy to verify that the edge-weights of all edges in the cycle are $2n+2, 2n+3, 2n+4, \dots, 3n+1$. So g_6 is a super $(2n+2,1)$-edge-antimagic total labeling of C_n^t, n even, for all odd t.

□

Theorem 8. *For $n \equiv 0 \pmod 4$, $n \ge 4$, and for $t \equiv 2 \pmod 4$, $t \ge 2$, the graph C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling.*

Proof. First, we consider $n=4$ and define the labeling $g_7 : V(C_4^2) \cup E(C_4^2) \to \{1,2,3,\dots,9\}$ where

$$g_7(v_i) = \begin{cases} \frac{3i-1}{2} & \text{if } i=1,3 \\ \frac{i+2}{2} & \text{if } i=2,4 \end{cases}$$

$$g_7(v_iv_{i+1}) = \begin{cases} n+i+2 & \text{if } i = 1,2 \\ 5 & \text{if } i = 3 \end{cases}$$

$$g_7(v_4v_1) = 9$$

$$g_7(chord) = g_7(v_1v_3) = 6.$$

We can see that the edge-weights of edges of C_4^2 are $10, 11, 12, 13, 14$. Thus the labeling g_7 is super $(10,1)$-edge-antimagic total labeling of C_4^2.

For $n \geq 8$, $n \equiv 0 \pmod 4$, define the bijective function

$g_8 : V(C_n^t) \cup E(C_n^t) \to \{1, 2, 3, \dots, 2n+1\}$ in the following way.

$$g_8(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq \frac{n}{2} - 1 \\ \frac{n+i+1}{2} & \text{if } i \text{ is odd, } \frac{n}{2} + 1 \leq i \leq n-1 \\ \frac{n+i}{2} + 1 & \text{if } i \text{ is even, } 2 \leq i \leq \frac{n}{2} - 2 \\ \frac{i}{2} + 1 & \text{if } i \text{ is even, } \frac{n}{2} \leq i \leq n \end{cases}$$

$$g_8(v_nv_1) = 2n+1$$

$$g_8(v_iv_{i+1}) = \begin{cases} 2n+1-\frac{i}{2} & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ \frac{3n-i+1}{2} & \text{if } i \text{ is odd except } i = \frac{n}{2} - 1 \\ \frac{3n}{2} + 1 & \text{if } i = \frac{n}{2} - 1 \end{cases}$$

$$g_8(chord) = \frac{5n}{4} + 1.$$

It can be seen that the weights of edges in the cycle, under the function g_8, clearly form two arithmetic progressions $2n+2, 2n+3, \dots, \frac{9n}{4}, \frac{9n}{4}+1$ and $\frac{9n}{4}+3, \frac{9n}{4}+4, \dots, 3n+2$. To prove that g_8 is a super $(2n+2,1)$-edge-antimagic total labeling of C_n^t it suffices to exhibit a chord with edge-weight $\frac{9n}{4}+2$ and distance $t \equiv 2 \pmod 4$.

When i is odd and $1 \leq i \leq \frac{n}{2} - 1$, the distances of endpoints of the chords v_iv_{n-i} cover all lengths $t \equiv 2 \pmod 4$ in the cycle and $g_8(v_i) + g_8(v_{n-i}) + g_8(chord) = n+1+\frac{5n}{4}+1 = \frac{9n}{4}+2$ is the edge-weight of every chord.

Thus g_8 is the required labeling.

□

5. Conclusion

In the foregoing sections we presented the values of t for which there exists a super (a,d)-edge-antimagic total labeling of C_n^t. We have shown a bound for the feasible values of the parameter d and have proved that graph C_n^t has a super $(2n+2,1)$-

edge-antimagic total labeling

(i) for n odd, $n \geq 5$, and for all possible values t,

(ii) for n even, $n \geq 6$, and for t odd, $t \geq 3$,

(iii) for $n \equiv 0 \pmod 4$, $n \geq 4$, and for $t \equiv 2 \pmod 4$, $t \geq 2$.

We have not yet found a construction that will produce super $(2n+2,1)$-edge-antimagic total labeling of C_n^t for the other values of n and t. However, we suggest the following:

Conjecture 1. *There is a super $(2n+2,1)$-edge-antimagic total labeling of C_n^t*

for $n \equiv 0$ (mod 4) and for $t \equiv 0$ (mod 4) and

for $n \equiv 2$ (mod 4) and for t even.

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(Received 19 Jan 2005)