AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **35** (2006), Pages 211–220

On property $M(3)$ of some complete multipartite graphs

WENJIE HE

Applied Mathematics Institute
Hebei University of Technology
Tianjin 300130
P.R. China
he_wenjie@yahoo.com

YUFA SHEN* YONGQIANG ZHAO†

Hebei Normal University
Shijiazhuang 050016
P.R. China
syf030514@163.com

YANNING WANG XINMIAO MA

Applied Mathematics Institute
Hebei University of Technology
Tianjin 300130
P.R. China

Abstract

Let G be a graph and suppose that for each vertex v in G, there exists a list of k colors, $L(v)$, such that there is a unique proper L-coloring for G; then G is called a uniquely k-list colorable graph. M. Ghebleh and E. S. Mahmoodian have characterized almost all uniquely 3-list colorable complete multipartite graphs except for nine of them. In this paper, some graphs, which are exempt by Ghebleh and Mahmoodian, are studied, and it is proved that $K_{1*4,5}$, $K_{1*4,4}$, $K_{2,2,r}$ ($r = 4, 5, 6$) have property $M(3)$. Finally, we obtain an improvement of Ghebleh and Mahmoodian's theorem.

* Also at Hebei Normal University of Science and Technology, Qinhuangdao 066004, P.R. China.
† Also at Shijiazhuang College, Shijiazhuang 050801, P.R. China.

1 Introduction

In this section we mention some of the definitions and results which are referred to throughout this paper. For the necessary definitions and notation we refer the reader to standard texts, such as [9]. For each vertex v in a graph G, let $L(v)$ denote a list of colors available for v. *A list coloring* from the given collection of lists is a proper coloring c such that $c(v)$ is chosen from $L(v)$. We will refer to such a coloring as an *L-coloring*. The idea of list colorings of graphs is due independently to Vizing [8] and to Erdős, Rubin and Taylor [2]. For a survey on list coloring and some more information we refer the interested reader to Alon [1] and West [9].

Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors, $L(v)$, such that there is a unique proper L-coloring for G from this collection of lists. Then G is called a *uniquely k-list colorable graph* or a *UkLC* graph for short. We say that a graph G has the property $M(k)$ (M for Marshall Hall) if and only if it is not uniquely k-list colorable. So G has property $M(k)$ if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist at least two list colorings. *The m-number* of a graph G, denoted by $m(G)$, is defined to be the least integer k such that G has property $M(k)$.

On uniquely k-list colorable complete multipartite graphs, some results have been obtained as follows.

Theorem 1.1 (Mahdian and Mahmoodian [6]) *A connected graph G has property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

Lemma 1.2 (Ghebleh and Mahmoodian [5]) *If L is a k-list assignment to the vertices in the graph G, and G has a unique L-coloring, then $|\bigcup_v L(v)| \geq k+1$ and all these colors are used in the unique L-coloring of G.*

Lemma 1.3 (Ganjali et al. [4]) *If G is a complete tripartite $U3LC$ graph, then all vertices in each part cannot take the same color in any unique 3-list coloring of G.*

Lemma 1.4 (Ghebleh and Mahmoodian [5]) *If G is a complete multipartite graph which has an induced UkLC subgraph, then G is UkLC.*

In the following, we use the notation K_{s*r} for a complete r-partite graph in which each part is of size s. The notation $K_{s*r,t}$, etc. is used similarly. Ghebleh and Mahmoodian also show the following result.

Theorem 1.5 (Ghebleh and Mahmoodian [5]) *The graphs $K_{3,3,3}$, $K_{2,4,4}$, $K_{2,3,5}$, $K_{2,2,9}$, $K_{1,2,2,2}$, $K_{1,1,2,3}$, $K_{1,1,1,2,2}$, $K_{1*4,6}$, $K_{1*6,4}$ and $K_{1*5,5}$ are U3LC.*

Theorem 1.6 (Ghebleh and Mahmoodian [5]) *Let G be a complete multipartite graph that is not $K_{2,2,r}$, for $r = 4, 5, \ldots, 8$, $K_{2,3,4}$, $K_{1*4,4}$, $K_{1*4,5}$, or $K_{1*5,4}$. Then G is $U3LC$ if and only if it has one of the graphs in Theorem 1.5 as an induced subgraph.*

For the result in Theorem 1.6, there are nine graphs exempted. In order to perfect Theorem 1.6, it is clear that the remaining work is to determine, for the nine graphs above, whether they are $U3LC$ or not. So Ghebleh and Mahmoodian give the open problem as follows.

Problem [5] *Verify property $M(3)$ for the graphs exempted in Theorem 1.6, i.e. $K_{2,2,r}$, for $r = 4, 5, \cdots, 8$, $K_{2,3,4}$, $K_{1*4,4}$, $K_{1*4,5}$, and $K_{1*5,4}$.*

In this paper, we research some complete multipartite graphs for which property $M(3)$ is not decided in Theorem 1.6, and prove that $K_{1*4,5}$,$K_{1*4,4}$, $K_{2,2,r}$ $(r = 4, 5, 6)$ have property $M(3)$. Furthermore, an improved version of Theorem 1.6 is obtained.

2 $K_{1*4,5}$ and $K_{1*4,4}$ have property $M(3)$

First, we introduce some lemmas.

Lemma 2.1. *For any five sets $S(i)$ $(i = 1, 2, \ldots, 5)$ where $|S(i)| = 2$ and $S(i) \subset \{1, 2, 3, 4\}$, there exist $S \subset \{1, 2, 3, 4\}$ of size 2 such that $S \cap S(i) \neq \emptyset$ for each $i = 1, 2, \ldots, 5$.*

Proof. By contradiction. Assume that there exist five sets $S^*(i)$ $(i = 1, 2, \ldots, 5)$ which do not satisfy the lemma. We conclude $\{1, 2\} \in \{S^*(1), S^*(2), \ldots, S^*(5)\}$, (otherwise, let $S = \{3, 4\}$, for any $i = 1, 2, \ldots, 5$, $S \cap S^*(i) \neq \emptyset$). By the same reason, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ also belong to $\{S^*(1), S^*(2), \ldots, S^*(5)\}$, which is impossible by the fact that there are only five sets. □

For the graph $K_{1*4,5}$, let $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$ and $\{v_5, v_6, v_7, v_8, v_9\}$ be the five parts of $K_{1*4,5}$. Assign the list $L(i) = \{c_{i1}, c_{i2}, c_{i3}\}$ to the vertex $v_i, i = 1, 2, \ldots, 9$. Now we assume that there exists an L-coloring c for $K_{1*4,5}$, say $c(v_i) = c_{i1}, i = 1, 2, \ldots, 9$. Let $2 \leq k \leq 4$, $i_1, i_2, \ldots, i_k \in \{1, 2, 3, 4\}$, and they are different pairwise. If $c_{i_j1} \in L(i_{j+1})$ for $j < k$, and $c_{i_k1} \in L(i_1)$, then $[i_1, i_2, \ldots, i_k]$ is called a *k-coloring rotation* in this paper.

Now we assume that c is the unique L-coloring for $K_{1*4,5}$. With this assumption, we obtain the following lemmas.

Lemma 2.2. *There is no k-coloring rotation in the graph $K_{1*4,5}$ for $2 \leq k \leq 4$.*

Proof. Otherwise, for a k-coloring rotation $[i_1, i_2, \ldots, i_k]$, let $c'(v_{i_1}) = c_{i_k}$, $c'(v_{i_j}) = c_{i_{j-1}}$ where $j > 1$, and for any $j \notin [i_1, i_2, \ldots, i_k]$, $c'(v_j) = c(v_j)$. Obviously, c' is another proper coloring of G from the given list assignment, which is a contradiction to the fact that c is a unique L-coloring for $K_{1*4,5}$. □

Lemma 2.3. *There is at least one $i \in \{1, 2, 3, 4\}$ such that $\{c_{i2}, c_{i3}\} \subseteq \{c_{51}, c_{61}, \ldots, c_{91}\}$.*

Proof. By contradiction. Suppose $|\{c_{i2}, c_{i3}\} \cap \{c_{51}, c_{61}, \ldots, c_{91}\}| \leq 1$ for each $i = 1, 2, 3, 4$. Let H denote the induced subgraph of $K_{1*4,5}$ by the vertex set $\{v_1, v_2, v_3, v_4\}$; then H is obviously a complete graph. We introduce a 2-list assignment L' of H as follows. For every vertex v_i, $i = 1, 2, 3, 4$, if there exists $s \in L(i) \cap \{c_{51}, c_{61}, \ldots, c_{91}\}$, then $L'(i) = L(i) \backslash \{s\}$, otherwise $L'(i) = L(i) \backslash \{t\}$, where $t \in L(i)$ and $t \neq c_{i1}$. Obviously $\{c_{11}, c_{21}, c_{31}, c_{41}\}$ is also a proper L'-coloring of H. We can obtain a new coloring c' of H from L' since every complete graph has the $M(2)$ property (by Theorem 1.1). From the construction of L', c' can be extended to an L-coloring of $K_{1*4,5}$—a contradiction. □

From above lemma, it is evident that $|\{c_{51}, c_{61}, \ldots, c_{91}\}| \geq 2$.

Lemma 2.4. $\{c_{i2}, c_{i3}\} \subseteq \{c_{11}, c_{21}, c_{31}, c_{41}\}$, *where* $i = 5, 6, \ldots, 9$.

Proof. If for some $k \in \{5, 6, 7, 8, 9\}$, $\{c_{k2}, c_{k3}\} \cap \{c_{11}, c_{21}, c_{31}, c_{41}\} \neq \{c_{k2}, c_{k3}\}$, say $c_{k3} \notin \{c_{11}, c_{21}, c_{31}, c_{41}\}$, then let $c'(v_k) = c_{k3}$ and $c'(v_i) = c(v_i)$ for $i \neq k$. Then c' is a new L-coloring, a contradiction. □

Theorem 2.5. $K_{1*4,5}$ *has the property M(3).*

Proof. By contradiction. Suppose that $K_{1*4,5}$ is $U3LC$. Let $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$ and $\{v_5, v_6, \ldots, v_9\}$ be the three parts of $K_{1*4,5}$. Assign the list $L(i) = \{c_{i1}, c_{i2}, c_{i3}\}$ to the vertex $v_i, i = 1, 2, \ldots, 9$, and assume that there exists a unique L-coloring c for $K_{1*4,5}$, say $c(v_i) = c_{i1}, i = 1, 2, \ldots, 9$. For clarity, it is assumed that $c_{i1} = i$ for $i = 1, 2, 3, 4$.

By Lemma 2.3, there exists $\{u, v\} \subseteq \{c_{51}, c_{61}, \ldots, c_{91}\}$ such that $\{c_{i2}, c_{i3}\} = \{u, v\}$ for some $i \in \{1, 2, 3, 4\}$. Without loss of generality, it is assumed that $i = 1$. For clear, let $\{c_{12}, c_{13}\} = \{5, 6\}$ and $M = \bigcup_{i=5}^{9} \{c_{i1}\}$, so $\{5, 6\} \subseteq M$. By Lemma 2.4, it is known that $\bigcup_{i=5}^{9} \{c_{i2}, c_{i3}\} \subseteq \{1, 2, 3, 4\}$, then by Lemma 2.1 there exist two colors $\{x, y\} \subseteq \{1, 2, 3, 4\}$ such that $\{x, y\} \cap \{c_{i2}, c_{i3}\} \neq \emptyset$ for each $i = 5, 6, \ldots, 9$. $\{x, y\}$ may be one of $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$ and $\{3, 4\}$. By whether 1 belongs to $\{x, y\}$, we just need to consider the following two cases: $\{x, y\} = \{1, 2\}$ and $\{x, y\} = \{3, 4\}$. In the following statements, it is shown case by case that there is another proper coloring from the given list assignment, which is contradicting with the assumption. In all the following new L-coloring c', except the vertices with special announce, we will let $c'(v_i) \in \{c_{i2}, c_{i3}\} \cap \{x, y\}$ for $i = 5, 6, \ldots, 9$, and other vertices were assigned the same color as in c.

Case 1. $\{x, y\} = \{1, 2\}$.

Subcase 1.1, $L(2) \cap M \neq \emptyset$.

Suppose $s \in L(2) \cap M$ and $z \in \{5, 6\} \backslash \{s\}$. Let $c'(v_2) = s, c'(v_1) = z, c'(v_i) = c(v_i)$ for $i = 3, 4$ and $c'(v_i) \in \{c_{i2}, c_{i3}\} \cap \{1, 2\}$ for $i = 5, 6, \ldots, 9$. Obviously, c' is a new L-coloring, which is a contradiction with $K_{1*4,5}$ is $U3LC$.

Subcase 1.2, $L(2) \cap M = \emptyset$.

Then $L(2) \subset \{2, 1, 3, 4\}$. So $L(2) \cap \{3, 4\} \neq \emptyset$. Without loss of generality, it is assumed that $3 \in L(2)$. By Lemma 2.2, $2 \notin L\{3\}$. If $L(3) \cap M \neq \emptyset$, let $s \in L(3) \cap M$, $z \in \{5, 6\} \backslash \{s\}$, and $c'(v_3) = s, c'(v_1) = z$ and $c'(v_2) = 3$, a new L-coloring then a contradiction. So $L(3) \cap M = \emptyset$ then $L(3) = \{3, 1, 4\}$. For $L(4)$, since $4 \in L(3)$, then $3 \notin L(4)$ (otherwise, [4,3] is a 2-coloring rotation, then a contradiction is obtained by Lemma 2.2; since $3 \in L(2), 4 \in L(3)$, then $2 \notin L(4)$ (otherwise, [2,4,3] is a 3-coloring rotation). In sum, $L(4) \subseteq \{4, 1\} \cup M$, then $L(4) \cap M \neq \emptyset$. Suppose $s \in L(4) \cap M$ and

$z \in \{5,6\}\backslash\{s\}$. Let $c'(v_4) = s, c'(v_3) = 4, c'(v_2) = 3, c'(v_1) = z$, a new L-coloring.

Case 2. $\{x, y\} = \{3, 4\}$.

Let $J = M \cup \{1\}$.

Subcase 2.1 $|J \cap (L(3) \cup L(4))| \geq 2$.

If there are two elements s, and z, such that $s \neq z$, $s \in L(3) \cap J$ and $z \in L(4) \cap J$, let $c'(v_3) = s, c'(v_4) = z$, $c'(v_1) \in L(1)\backslash\{s, z\}$, $c'(v_2) = 2$, and $c'(v_i) \in \{c_{i2}, c_{i3}\} \cap \{3, 4\}$ for $i = 5, 6, \ldots, 9$. Then c' is a new L-coloring, a contradiction. Otherwise, either $|L(3) \cap J| = 2$ and $|L(4) \cap J| = 0$, or $|L(3) \cap J| = 0$ and $|L(4) \cap J| = 2$. Without loss of generality, we may assume that $|L(3) \cap J| = 2$ and $|L(4) \cap J| = 0$. Then $L(4) = \{4, 2, 3\}$. Since $2 \in L(4)$, then $4 \notin L(2)$ by Lemma 2.2. So $L(2) \subset \{2, 3\} \cup J$, then $L(2) \cap J \neq \emptyset$. Suppose $s \in L(2) \cap J$ and $z \in L(3)\backslash\{s\}$. Let $c'(v_3) = z, c'(v_4) = 2, c'(v_2) = s, c'(v_1) \in L(1)\backslash\{s, z\}$. Then a new L-coloring is obtained.

Subcase 2.2 $|J \cap (L(3) \cup L(4))| = 1$.

Subcase 2.2.1 either $|L(4) \cap J| = 0$ or $|L(3) \cap J| = 0$.

Without loss of generality, it is assumed that $|L(4) \cap J| = 0$, then $|L(3) \cap J| = 1$. By $|L(4) \cap J| = 0$, it is concluded that $L(4) = \{4, 2, 3\}$. Since $3 \in L(4)$, then $4 \notin L(3)$ by Lemma 2.2. Together with the assumption $|L(3) \cap J| = 1$, it is affirmed that $L(3) = \{3, s, 2\}$, where $s \in J$. For $L(2)$, since $2 \in L(4)$ and $2 \in L(3)$, then $4 \notin L(2)$ and $3 \notin L(2)$ by Lemma 2.2. Then $|L(2) \cap J| = 2$. Let $c'(v_3) = s, c'(v_4) = 2, c'(v_2) = z$, where $z \in L(2)\backslash\{s\}$, $c'(v_1) \in L(1)\backslash\{s, z\}$. Then a new L-coloring.

Subcase 2.2.2 $|L(3) \cap J| = |L(4) \cap J| = 1$.

Let $\{s\} = L(3) \cap J = L(4) \cap J$. It is obvious that $\{2\} \cap (L(3) \cup L(4)) \neq \emptyset$ (otherwise, $3 \in L(4)$ and $4 \in L(3)$, then [3,4] is a 2-coloring rotation, which contradicts with Lemma 2.2. Suppose that $L(3) = \{3, s, 2\}$. For $L(2)$, since $2 \in L(3)$, then $3 \notin L(2)$. With the above assumptions, it is concluded that either $L(4) = \{4, s, 2\}$ or $L(4) = \{4, s, 3\}$, then $4 \notin L(2)$ (otherwise, [2,4] or [2,3,4] is a coloring rotation). So $|L(2) \cap J| = 2$. Let $c'(v_3) = 2, c'(v_4) = s, c'(v_2) = z$, where $z \in L(2)\backslash\{s\}$, $c'(v_1) \in L(1)\backslash\{s, z\}$. Then a new L-coloring is constructed.

Subcase 2.3 $J \cap (L(3) \cup L(4)) = 0$.

With the assumption, it is obvious that $L(3) = \{3, 4, 2\}$ and $L(4) = \{4, 3, 2\}$, which is impossible by Lemma 2.2.

Hence, in all cases there exist a new L-coloring, which contradicts with the assumption that $K_{1*4,4}$ is U3LC. □

Corollary 2.6. $K_{1*4,4}$ *has the property M(3).*

Proof. By contradiction. If $K_{1*4,4}$ is $U3LC$, we can obtain that $K_{1*4,5}$ is $UkLC$ by Lemma 1.4 since $K_{1*4,4}$ is a induced subgraph of $K_{1*4,5}$. A contradiction with Theorem 2.5. □

Corollary 2.7. $m(K_{1*4,4}) = m(K_{1*4,5}) = 3$.

Proof. By Theorem 1.1, Theorem 2.5 and above corollary, it is obvious. □

3 Some structural properties of the graph $K_{2,2,6}$

For the graph $K_{2,2,6}$, let $X_1 = \{v_1, v_2\}$, $X_2 = \{v_3, v_4\}$ and $X_3 = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ be the three parts of $K_{2,2,6}$ respectively. Assign the list $\{c_{i1}, c_{i2}, c_{i3}\}$ to the vertex $v_i, i = 1, 2, \ldots, 10$. Suppose there exists a unique L-coloring c for $K_{2,2,6}$, say $c(v_i) = c_{i1}, i = 1, 2, \ldots, 10$. Let $S = \{c_{51}, c_{61}, \ldots, c_{10,1}\}$. We have the following propositions.

Proposition 3.1. *c_{11}, c_{21}, c_{31} and c_{41} are all different.*

Proof. It is obvious from Lemma 1.3. □

Proposition 3.2. *If v_i and v_j are in same part of $K_{2,2,6}$, then $c_{i1} \notin \{c_{j2}, c_{j3}\}$.*

Proof. If $c_{i1} = c_{j1}$, then the result is obvious. Else if $c_{i1} = c_{jk}$, where $k = 2$ or 3, let $c^{'}(v_j) = c_{jk} = c_{i1}$ and $c^{'}(v_i) = c(v_i)$ for $i \neq j$, then $c^{'}$ is another L-coloring of $K_{2,2,6}$. □

Proposition 3.3. *Let $i \in \{1, 2\}$ and $j \in \{3, 4\}$. If $c_{i1} \in \{c_{j2}, c_{j3}\}$, then $c_{j1} \notin \{c_{i2}, c_{i3}\}$.*

Proof. Otherwise, let $c^{'}(v_i) = c_{j1}$, $c^{'}(v_j) = c_{i1}$ and $c^{'}(v_k) = c(v_k)$ for $k \neq i, j$. Then $c^{'}$ is another L-coloring of $K_{2,2,6}$. □

Proposition 3.4. There exists at least one $i \in \{1, 2, 3, 4\}$ such that $\{c_{i2}, c_{i3}\} \subseteq S$.

Proof. By contradiction. Suppose $|\{c_{i2}, c_{i3}\} \cap S| \leq 1$ for each $i = 1, 2, 3, 4$. Let H denote the induced subgraph of $K_{2,2,6}$ by the vertices set $\{v_1, v_2, v_3, v_4\}$, then $H = K_{2,2}$ is obviously a complete bipartite graph. We introduce a 2-list assignment $L^{'}$ of H with $L^{'}(v_i) = L(v_i) \backslash S$. By the property $M(2)$ of complete graphs (Theorem 1.1), we can obtain another L-coloring c' for $K_{2,2}$ which is extendible to vertices of $K_{2,2,6}$. This is a contradiction to c being a uniquely 3-list coloring. □

Proposition 3.5. *$\{c_{i2}, c_{i3}\} \subseteq \{c_{11}, c_{21}, c_{31}, c_{41}\}$, where $i = 5, 6, 7, 8, 9, 10$.*

Proof. By contradiction. For some $k \in \{5, 6, 7, 8, 9, 10\}$, by Lemma 1.2, at least one of c_{k2} and c_{k3} is in $S \setminus \{c_{k1}\}$ (note $|S| \geq 2$, by Lemma 1.3). Suppose that $c_{k2} \in S \setminus \{c_{k1}\}$. Let $c^{'}(v_k) = c_{k2}$ and $c^{'}(v_i) = c(v_i)$ for $i \neq k$. Then $c^{'}$ is another L-coloring of $K_{2,2,6}$. □

Remark. By Proposition 3.1, without loss of generality, we assume $c_{i1} = i, i = 1, 2, 3, 4$. By Lemma 1.3, $S = \{c_{51}, c_{61}, \ldots, c_{10,1}\} = \{5, 6, \ldots\}$. We declare that by Proposition 3.4, without loss of generality, we assume that $L(v_1) = \{1, 5, 6\}$. We also declare that in the following proof, we use lowercase x, y, z, t to denote some color in S.

Proposition 3.6. *For $K_{2,2,6}$, suppose c is a uniquely 3-list coloring with $L(v_i) = \{c_{i1}, c_{i2}, c_{i3}\}$ and $c(v_i) = c_{i1}$, for $i = 1, 2, \ldots, 10$, then any three colors in $\{1, 2, 3, 4\}$ can be used to L-color the part X_3 (not considering part X_1 and X_2).*

Proof. Consider the 2×6 array $\begin{pmatrix} c_{52} & c_{62} & \cdots & c_{10,2} \\ c_{53} & c_{63} & \cdots & c_{10,3} \end{pmatrix}$. By Proposition 3.5, $c_{ik} \in \{1, 2, 3, 4\}$, for $i = 5, 6, \cdots, 10$, and $k = 2, 3$. Assume the statement of Proposition 3.6 is not true. There must be three colors, say $1, 2, 3$ which do not appear in some column of the 2×6 array. Notice $c_{i2} \neq c_{i3}$, for $i = 5, 6, \ldots, 10$; this is impossible. $\square$

Proposition 3.7. *Suppose* $L(v_1) = \{1, 5, 6\}$, $\{c_{22}, c_{23}\} \subseteq S$, *say* $c_{22} = x, c_{23} = y$, *then there exists another* L*-coloring* c' *for* $K_{2,2,6}$.

Proof. If $\{c_{32}, c_{33}, c_{42}, c_{43}\} \cap S \neq \emptyset$, say $c_{32} = z \in S$. Let $c'(v_i) = l_i, l_i \in \{c_{i2}, c_{i3}\} \backslash \{z\} (i = 1, 2)$, $c'(v_3) = z, c'(v_4) = 4$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1, 2, 3\} \cap L(v_i)$, for $i = 5, 6, \ldots, 10$. Then c' is another L-coloring of $K_{2,2,6}$.

If $\{c_{32}, c_{33}, c_{42}, c_{43}\} \cap S = \emptyset$, then $L(v_3) = \{3, 1, 2\}$ and $L(v_4) = \{4, 1, 2\}$, by Proposition 3.2. Let $c'(v_1) = 5, c'(v_2) = x, c'(v_3) = 1, c'(v_4) = 1$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{2, 3, 4\} \cap L(v_i)$, for $i = 5, 6, \ldots, 10$. Then c' is another L-coloring of $K_{2,2,6}$. $\square$

Proposition 3.8. *Suppose* $L(v_1) = \{1, 5, 6\}$. *If there exists* $i \in \{3, 4\}$ *such that* $\{c_{i2}, c_{i3}\} \subseteq S$, *then there exists another* L*-coloring* c' *for* $K_{2,2,6}$.

Proof. Without loss of generality, we suppose that $\{c_{32}, c_{33}\} \subseteq S$; say $c_{32} = x, c_{33} = y$.

If $\{c_{22}, c_{23}\} \cap S \neq \emptyset$, by Proposition 3.7, $\{c_{22}, c_{23}\} \cap S = \{z\}$. Then we put $c'(v_2) = z$, $c'(v_3) = l_3, l_3 \in \{x, y\} \backslash \{z\}$, $c'(v_1) = l_1, l_1 \in \{5, 6\} \backslash \{l_3\}$, $c'(v_4) = 4$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1, 2, 3\} \cap L(v_i)$, for $i = 5, 6, \ldots, 10$. Then c' is another L-coloring of $K_{2,2,6}$.

If $\{c_{22}, c_{23}\} \cap S = \emptyset$, by Proposition 3.2, $\{c_{22}, c_{23}\} = \{3, 4\}$; then by Proposition 3.3 $\{c_{42}, c_{43}\} \cap S \neq \emptyset$, say $\{c_{42}, c_{43}\} \cap S = \{z\}$. Then we put $c'(v_4) = z$, $c'(v_1) = l_1, l_1 \in \{5, 6\} \backslash \{z\}$, $c'(v_3) = l_3, l_3 \in \{x, y\} \backslash \{l_1\}$, $c'(v_2) = 2$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1, 3, 4\} \cap L(v_i)$, for $i = 5, 6, \ldots, 10$. Then c' is another L-coloring of $K_{2,2,6}$. $\square$

Proposition 3.9. *Suppose* $L(v_1) = \{1, 5, 6\}$ *and* $\{c_{22}, c_{23}\} \cap S = \{x\}$. *If there exists* $i \in \{3, 4\}$ *such that* $\{c_{i2}, c_{i3}\} = \{1, 2\}$, *then we can obtain another* L*-coloring of* $K_{2,2,6}$.

Proof. Without loss of generality, we suppose that $\{c_{32}, c_{33}\} = \{1, 2\}$. By Proposition 3.2 and Proposition 3.3, $\{c_{22}, c_{23}\} = \{4, x\}$. By Proposition 3.8, $1 \in \{c_{42}, c_{43}\}$, we put $c'(v_3) = c'(v_4) = 1, c'(v_1) = 5, c'(v_2) = x$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{2, 3, 4\} \cap L(v_i)$, for $i = 5, 6, \ldots, 10$. Then c' is another L-coloring of $K_{2,2,6}$. $\square$

Proposition 3.10. *Suppose* $L(v_1) = \{1, 5, 6\}$ *and* $\{c_{22}, c_{23}\} \cap S = \{x\}$. *Then if there exists* $i \in \{3, 4\}$, *such that* $\{c_{i2}, c_{i3}\} \cap S = \{y\}$, *we can obtain another* L*-coloring of* $K_{2,2,6}$.

Proof. Without loss of generality, we suppose that $\{c_{32}, c_{33}\} \cap S = \{y\}$. By Propositions 3.8 and 3.9, $\{c_{42}, c_{43}\} \cap S = \{z\}$.

Case 1. $x \neq y$.
We let $c'(v_2) = x, c'(v_3) = y, c'(v_4) = 4$. $c'(v_1) = l_1, l_1 \in \{5, 6\} \backslash \{y\}$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1, 2, 3\} \cap L(v_i)$, for $i = 5, 6, \ldots, 10$. Then c' is another L-coloring of $K_{2,2,6}$.

Case 2. $x = y$.

Subcase 2.1. $y = z$
We let $c'(v_3) = c'(v_4) = y, c'(v_2) = 2$, $c'(v_1) = l_1, l_1 \in \{5,6\}\backslash\{y\}$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1,3,4\} \cap L(v_i)$, for $i = 5,6,\ldots,10$. Then c' is another L-coloring of $K_{2,2,6}$.

Subcase 2.2. $y \neq z$.
If $x \in \{5,6\}$, we let $c'(v_1) = c'(v_2) = x, c'(v_3) = 3, c'(v_4) = z$. If $x \notin \{5,6\}$, we let $c'(v_2) = x, c'(v_4) = z, c'(v_3) = 3$. $c'(v_1) = l_1, l_1 \in \{5,6\}\backslash\{z\}$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1,2,4\} \cap L(v_i)$, for $i = 5,6,\ldots,10$. Then c' is another L-coloring of $K_{2,2,6}$. □

Proposition 3.11. *Suppose $L(v_1) = \{1,5,6\}$ and $L(v_2) = \{2,3,4\}$. Then if there exists $i \in \{3,4\}$ such that $\{c_{i2}, c_{i3}\} \cap S = \{x\}$, $x \notin \{5,6\}$, we can obtain another L-coloring of $K_{2,2,6}$.*

Proof. without loss of generality, we suppose that $\{c_{32}, c_{33}\} \cap S = \{x\}, x \notin \{5,6\}$. By Proposition 3.2, Proposition 3.3 and Proposition 3.8, we only need verify that $L(v_3) = \{1,3,x\}, L(v_4) = \{1,4,y\}$. In fact, we can let $c'(v_2) = 2, c'(v_3) = x, c'(v_4) = y$, $c'(v_1) = l_1, l_1 \in \{5,6\}\backslash\{y\}$. By Proposition 3.6, we can let $c'(v_i) = m_i$, where $m_i \in \{1,3,4\} \cap L(v_i)$, for $i = 5,6,\ldots,10$. Then c' is another L-coloring of $K_{2,2,6}$. □

Proposition 3.12. *If there are no two colors in $\{1,2,3,4\}$ which can be used to L-color $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ (not considering part X_1 and X_2), then $\{c_{i2}, c_{i3}\} \in \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, and $\{c_{i2}, c_{i3}\} \neq \{c_{j2}, c_{j3}\}$, for $i,j = 5,6,$ $7,8,9,10$, and $i \neq j$.*

Proof. Above all, by Proposition 3.5, $\{c_{i2}, c_{i3}\} \subseteq \{c_{11}, c_{21}, c_{31}, c_{41}\}$ for $v_i, i = 5,6,7,8,9,10$. Denote $\mathcal{A} = \{\{c_{52}, c_{53}\}, \{c_{62}, c_{63}\}, \{c_{72}, c_{73}\}, \{c_{82}, c_{83}\}, \{c_{92}, c_{93}\}, \{c_{10,2}, c_{10,3}\}\}$. If one cannot use $\{1,2\}$ to L-color $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$, then $\{3,4\} \in \mathcal{A}$. By the same case, $\{2,4\} \in \mathcal{A}$. $\{2,3\} \in \mathcal{A}$. $\{1,2\} \in \mathcal{A}$. $\{1,3\} \in \mathcal{A}$. $\{1,4\} \in \mathcal{A}$. Then we are done. □

Proposition 3.13. *If there are no two colors in $\{1,2,3,4\}$ which can be used to L-color $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ (not considering part X_1 and X_2), then we can use any two colors in $\{1,2,3,4\}$ to L-color some five vertices in $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$.*

Proof. By Proposition 3.12, it is obvious. □

Proposition 3.14. *If there are no two colors in $\{1,2,3,4\}$ which can be used to L-color $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ (not considering part X_1 and X_2), then we can obtain another L-coloring of $K_{2,2,6}$.*

Proof. By Propositions 3.7–3.12 we only need verify the case that $L(v_1) = \{1,5,6\}$, $L(v_2) = \{2,3,4\}, L(v_3) = \{1,3,5\}, L(v_4) = \{1,4,6\}$. $\{c_{i2}, c_{i3}\} \in \{\{1,2\},\{1,3\},$ $\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, and $\{c_{i2}, c_{i3}\} \neq \{c_{j2}, c_{j3}\}$, for $i,j = 5,6,7,8,9,10$, and $i \neq j$.

In fact, we let $c'(v_3) = c'(v_4) = 1, c'(v_2) = 2$. By Proposition 3.13, we can use colors 3 and 4 to L-color some five vertices in $\{v_5, v_6, v_7, v_8, v_{10}\}$, say $\{v_5, v_6, v_7, v_8, v_9\}$. Then we let $c'(v_{10}) = c(v_{10})$, and we use 5 or 6 that is different from $c(v_{10})$ to L-color v_1. It is obvious that c' is another L-coloring of $K_{2,2,6}$. □

Proposition 3.15. *If there are two colors in* $\{1,2,3,4\}$ *which can be used to L-color* $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ *(not considering part* X_1 *and* X_2*), then we can obtain another L-coloring of* $K_{2,2,6}$.

Proof. By Propositions 3.7–3.11, we only need verify the case that $L(v_1) = \{1,5,6\}$, $L(v_2) = \{2,3,4\}, L(v_3) = \{1,3,5\}, L(v_4) = \{1,4,6\}$. It is obvious that if we can use some two colors in $\{1,2,3,4\}$ to L-coloring $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ (*Not considering part* X_1 *and* X_2), then we can use the remaining two colors in $\{1,2,3,4\}$ and colors $\{5,6\}$ to L-color $\{v_1, v_2, v_3, v_4\}$. For example, if we can use colors $\{1,2\}$ to L-coloring $\{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ (*Not considering part* X_1 *and* X_2), then we let $c'(v_1) = 5, c'(v_2) = 4, c'(v_3) = 3, c'(v_4) = 6$. Other cases can be similarly proved. So we are done. □

4 $K_{2,2,r}$ $(r = 4,5,6)$ has property M(3)

Theorem 4.1 $K_{2,2,6}$ ***has property M(3).***

Proof. By contradiction. For the graph $K_{2,2,6}$, let $X_1 = \{v_1, v_2\}$, $X_2 = \{v_3, v_4\}$ and $X_3 = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ be the three parts of $K_{2,2,6}$ respectively. Assign the list $\{c_{i1}, c_{i2}, c_{i3}\}$ to the vertex $v_i, i = 1,2,\ldots,10$. Suppose there exists a unique L-coloring c for $K_{2,2,6}$, say $c(v_i) = c_{i1}, i = 1,2,\ldots,10$. By the property of $L(v_i), i = 5,6,7,8,9,10$, there are two cases.

First case: there exist two colors $\{x,y\} \in \{1,2,3,4\}$ such that we may use them to L-color the vertices v_5, v_6, v_7, v_8, v_9 and v_{10} (*Not considering part* X_1 *and* X_2).

Second case: there are no two colors in $\{1,2,3,4\}$ such that we may use them to L-color the vertices v_5, v_6, v_7, v_8, v_9 and v_{10} (*Not considering part* X_1 *and* X_2).

By Propositions 3.14 and 3.15, we finish the proof. □

Corollary 4.2. $K_{2,2,4}$ *and* $K_{2,2,5}$ *have property M(3).*

Proof. By contradiction. For $r = 4,5$, it is clear that $K_{2,2,r}$ is an induced subgraph of $K_{2,2,6}$. If $K_{2,2,r}$ is $U3LC$, then $K_{2,2,6}$ is $U3LC$ by Lemma 1.4. This is a contradiction to Theorem 4.1. □

Corollary 4.3. $m(K_{2,2,4}) = m(K_{2,2,5}) = m(K_{2,2,6}) = 3$.

Proof. By Theorem 1.1, Theorem 4.1 and above corollary, it is obvious. □

5 On characterization of uniquely 3-list colorable complete multipartite graphs

According to our results in Sections 2 and 4, now we can restate Theorem 1.6 in an improved form, which is given by M. Ghebleh and E. S. Mahmoodian.

Theorem 5.1. *Let* G *be a complete multipartite graph that is not* $K_{2,2,r}, r = 7,8$. $K_{2,3,4}$, *or* $K_{1*5,4}$. *Then* G *is* $U3LC$ *if and only if it has one of the graphs in Theorem 1.5 as an induced subgraph.*

Acknowledgements

We would like to express our gratitude to the referee for their careful reading and helpful comments. In particular, we are indebted to the referee for pointing out the alternate proof for Lemma 2.4 and simpler proof for Proposition 3.4.

References

[1] N. Alon, Restricted colorings of graphs. In K. Walker (ed.), *Surveys in Combinatorics* 187, London Math. Soc. LNS (1993), pp. 1–33.

[2] P. Erdős, A.L. Rubin, and H. Taylor, Choosability in graphs. In Proc. West Coast Conf. Combinatorics, Graph Theory, and Computing, *Congr. Numer.* 26 (1979), 125–157.

[3] Ch. Eslahchi, M. Ghebleh and H. Hajiabolhassan, Some concepts in list coloring, *J. Combin. Math. Combin. Comput.* 41 (2002), 151–160.

[4] Y. G. Ganjali, M. Ghebleh, H. Hajiabohassan, M. Mirzadeh and B. S. Sadjad, Uniquely 2-list colorable graphs, *Discrete Appl. Math.* 119 (2002), 217–225.

[5] M. Ghebleh and E. S. Mahmoodian, On uniquely list colorable graphs, *Ars Combin.* 59 (2001), 307–318.

[6] M. Mahdian and E. S. Mahmoodian, A characterization of uniquely 2-list colorable graphs, *Ars Combin.* 51 (1999), 295–305.

[7] E. S. Mahmoodian and M. Mahdian, On the uniquely list colorable graphs. In Proc. 28th Annual Iranian Math. Conference, Part 1, no. 377 in Tabriz Univ. Ser., (1997), pp. 319–326.

[8] V. G. Vizing. Coloring the vertices of a graph in prescribed colors. In Diskret. Analiz, no. 29 in Metody Diskret. Anal. v Teorii Kodov i Shem, (1976), 3–10.

[9] D.B. West, *Introduction to graph theory*, Prentice Hall, Upper Saddle River, NJ, 1996.

(Received 7 Jan 2005)