AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **35** (2006), Pages 181–192

Sierpiński gasket graphs and some of their properties

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Abstract

The *Sierpiński fractal* or *Sierpiński gasket* Σ is a familiar object studied by specialists in dynamical systems and probability. In this paper, we consider a graph S_n derived from the first n iterations of the process that leads to Σ, and study some of its properties, including its cycle structure, domination number and pebbling number. Various open questions are posed.

1 Introduction and Basic Properties

The structure known as the *Sierpiński fractal* or *Sierpiński gasket* Σ is a familiar object studied by schoolchildren and specialists in dynamical systems and probability alike. It is known to be a self-similar object with fractal dimension $d = \log 3/\log 2 \approx 1.585$ ([4]). In this paper, however, we consider the finite structure obtained by iterating, a finite number of times, the process that defines Σ, leading to (i) the finite Sierpiński gasket σ_n, and (ii) the associated *Sierpiński gasket graph* S_n, defined as one with vertex set V_n equal to the intersection points of the line segments in σ_n, and edge set E_n consisting of the line segments connecting two vertices; see Figure 1 for a portrayal of S_2, S_3 and S_4. It is immediate that S_{n+1} consists of three attached copies of S_n which we will refer to as the *top, bottom left* and *bottom right* components of S_{n+1} – and denote by $S_{n+1,T}, S_{n+1,L}$, and $S_{n+1,R}$ respectively (Figure 2).

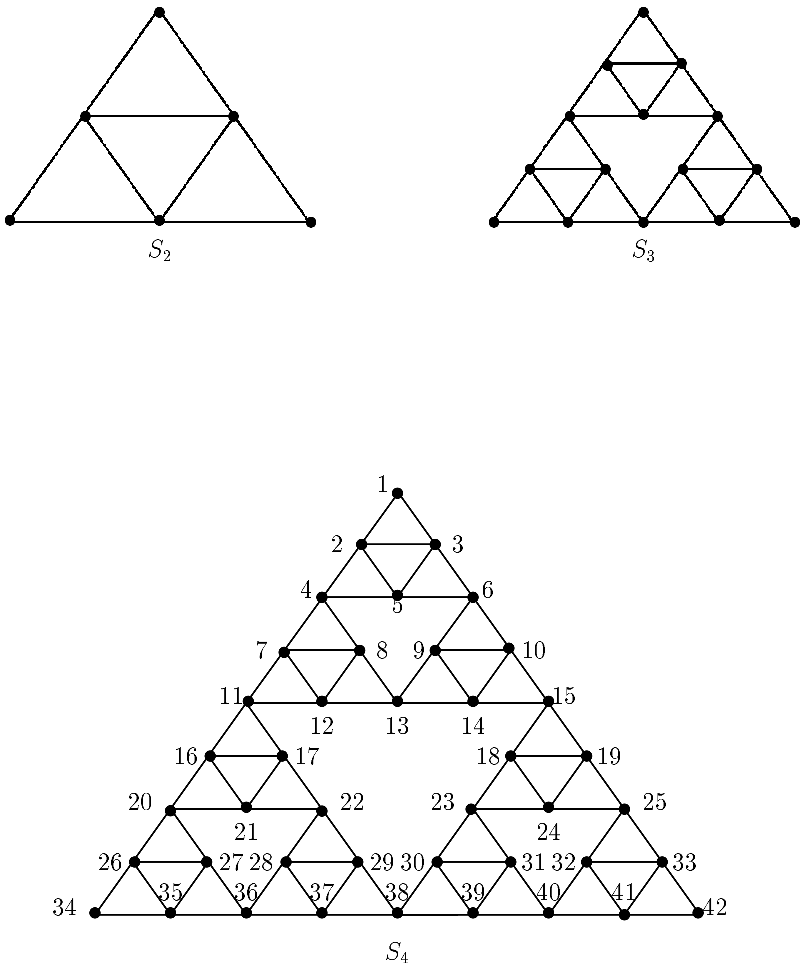
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Figure 1

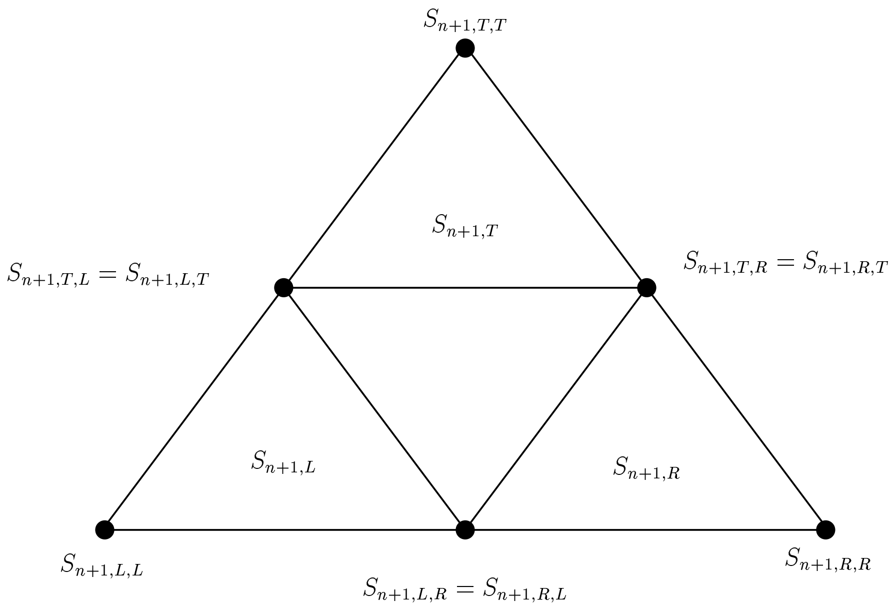

Figure 2, S_{n+1}

It turns out that structures similar to ours have been studied in two other contexts. It is unfortunate, but not surprising, that the phrase "Sierpiński graph" is used in each of these two situations, described below, as well. There will be little danger of confusion, however, since we will deal exclusively, in this paper, with the definition in the previous paragraph.

First, we mention the body of work that treats simple random walks and Brownian motion on the "infinite Sierpiński graph" and the Sierpiński gasket respectively. A typical publication in this genre is by Teufl [15], where a sharp average displacement result is proved for random walk on a Sierpiński graph that is an infinite version of the graph S_n defined above – but with each edge being of length one and with each vertex having degree four – in our case all but three vertices have degree four and the edges can be arbitrarily short. Several key references may be found in Teufl's paper.

Second, in [12], for example, the authors study crossing numbers for "Sierpiński-like graphs," which are extended versions of the "Sierpiński graphs" $S(n,k)$ first studied in [10]. These graphs were motivated by topological studies of the Lipscomb space that generalizes the Sierpiński gasket and are defined as follows: $S(n,k)$ has vertex set $\{1,2,\dots,k\}^n$, and there is an edge between two vertices $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_n)$ iff there is an $h\in[n]$ such that

- $u_j = v_j$ for $j=1,\dots,h-1$;

- $u_h \neq v_h$; and
- $u_j = v_h; v_j = u_h$ for $j = h+1, \ldots, n$.

An interesting connection is that the graph $S(n,3), n \geq 1$ is isomorphic to the Tower of Hanoi game graph with n disks: see [16] for a popular account and [9] for a list of scholarly references.

After discussing some baseline results in this section, we will prove in Section 2 that S_n is Hamiltonian and pancyclic (i.e., has cycles of all possible sizes). In Section 3, we determine the domination number of S_n, proving in the process that it has efficiency that is asymptotically 90%. This is in sharp contrast to the Sierpiński graphs of [10] – it has been exhibited by Klavžar, Milutinović, and Petr [11] that the graphs $S(n,k)$ have perfect dominating sets, i.e., are 100% efficient. Finally, in Section 4, we will show that the determination of the cover pebbling number $\pi(S_n)$ of S_n is not trivial, even given the so-called "stacking theorem" of Vuong and Wyckoff [17]. We conclude with a statement of some open questions.

Proposition 1 *S_n has $\frac{3}{2}(3^{n-1}+1)$ vertices and 3^n edges.*

Proof To construct S_{n+1} from S_n we add one downward facing triangle in each of the 3^{n-1} upward facing triangles of S_n. Thus we add 3^n points. In other words,

$$\begin{aligned} |V_{n+1}| &= |V_n| + 3^n \\ &= |V_1| + \sum_{i=1}^{n} 3^i \\ &= \frac{3}{2} \cdot (3^n + 1), \end{aligned}$$

as asserted. The number of edges in S_n may now be easily determined using the fact that the sum of the vertex degrees equals twice the number of edges:

$$\begin{aligned} |E_n| &= \frac{1}{2} \sum_{j=1}^{|V_n|} \deg(v_j) \\ &= \frac{1}{2} \cdot \left(4 \cdot \frac{3}{2}(3^{n-1} - 1) + 2 \cdot 3\right) \\ &= 3^n, \end{aligned}$$

completing the proof. □

Note that an alternative proof of Proposition 1 can be based on the facts that $|V_{n+1}| = 3|V_n| - 3$ and $|E_{n+1}| = 3|E_n|$.

Proposition 2 *S_n is properly three-colorable, i.e. $\chi(S_n) = 3$ for each n.*

Proof Clearly, $\chi(S_1) = 3$. Suppose $\chi(S_n) = 3$. Color S_n with three colors. We now properly 3-color S_{n+1} in the following fashion: After we insert one downward facing triangle in each upward facing triangle of S_n, each added vertex is assigned a previously used color different from the colors of the vertices of S_n adjacent to it. □

2 Hamiltonicity and Pancyclicity

We begin with an important lemma:

Lemma 3 *S_n has two Hamiltonian paths, say H_{n0} and H_{n1}, both starting at the same vertex of degree two and ending at different vertices of degree two.*

Proof The lemma is clearly true for S_1. Suppose it is true for S_n. Consider S_{n+1}, which is a three-fold "repetition" of S_n, as mentioned in Section 1, i.e., it consists of the three attached copies $S_{n+1,T}, S_{n+1,L}$, and $S_{n+1,R}$ of S_n. Consider a Hamiltonian path of $S_{n+1,T}$, moving from the top vertex (of degree 2) of $S_{n+1,T}$ to the top vertex of $S_{n+1,R}$. Using another Hamiltonian path (guaranteed by the induction hypothesis) we move from that vertex to left vertex of of $S_{n+1,R}$. Finally, we take the Hamiltonian path of $S_{n+1,L}$ starting at its right vertex and ending at its left vertex, but with a critical modification, namely avoiding the top vertex of $S_{n+1,L}$. In this fashion, we have constructed a Hamiltonian path of S_{n+1} from its top vertex to its left vertex. A similar argument is employed if the path is to end in the right vertex of S_{n+1}. □

We simplify our notation next. The top, left, and right vertices of $S_{n+1,T}$ will be denoted respectively by $S_{n+1,T,T}$, $S_{n+1,T,L}$, and $S_{n+1,T,R}$. Other critical vertices in S_{n+1} are analogously denoted by $S_{n+1,L,T}, S_{n+1,L,L}, S_{n+1,L,R}$, $S_{n+1,R,T}, S_{n+1,R,L}$, and $S_{n+1,R,R}$; of course we have $S_{n+1,T,R} = S_{n+1,R,T}$; $S_{n+1,R,L} = S_{n+1,L,R}$; and $S_{n+1,L,T} = S_{n+1,T,L}$. See Figure 2.

Theorem 4 *S_n is Hamiltonian for each n.*

Proof By Lemma 3, we take a Hamiltonian path of $S_{n+1,T}$ that moves from $S_{n+1,T,L}$ to $S_{n+1,T,R}$. Next, take a Hamiltonian path from $S_{n+1,T,R}$ to $S_{n+1,R,L}$ and finally move from this vertex to our starting vertex using another Hamiltonian path of S_n. This gives us a Hamiltonian cycle for S_{n+1}. □

Lemma 5 *Each Hamiltonian path of S_{n+1} as constructed in Lemma 3 above, and moving, say, from $S_{n+1,T,T}$ to $S_{n+1,L,L}$, can be sequentially reduced in length by one at each step, while maintaining the starting and ending vertices, with the process ending in a path from $S_{n+1,T,T}$ to $S_{n+1,L,L}$ along the left side of S_{n+1}.*

Proof We proceed by induction, noting that the result is clearly true for $n = 1$. Assume that the result is true for S_n. There exists a Hamiltonian path from $S_{n+1,T,T}$ to $S_{n+1,L,L}$ via $S_{n+1,T,R}$ and $S_{n+1,R,L}$. Suppose we need a path from $S_{n+1,T,T}$ to $S_{n+1,L,L}$ but with a reduction in length of $r \geq 1$. If $r \leq |S_n| - s$, where s is the length of the side of S_n, then adjustments need only be made in the first of the three above-mentioned Hamiltonian paths. If $|S_n| - s \leq r \leq 2|S_n| - 2s$, then we make adjustments in the lengths of two Hamiltonian paths. Likewise, if $2|S_n| - 2s \leq r \leq 3|S_n| - 3s$, we adjust three paths. Finally, if $3|S_n| - 3s \leq r \leq 3|S_n| - 2s$, then we modify the reduced path $S_{n+1,T,T} \to S_{n+1,T,R} \to S_{n+1,R,L} \to S_{n+1,L,L}$ of length $3s$ as follows to achieve the further required reduction: (i) $S_{n+1,T,T} \to S_{n+1,T,R} \to S_{n+1,R,L}$ is replaced by a path from $S_{n+1,T,T}$ to $S_{n+1,T,L}$ of length $2s$; (ii) the path $S_{n+1,R,L} \to S_{n+1,L,L}$ of length s is replaced by the path $S_{n+1,T,L} \to S_{n+1,L,L}$, also of length s. This yields a path from $S_{n+1,T,T}$ to $S_{n+1,L,L}$ of length $3s$. Finally, the first component of this path is shrunk, by the induction hypothesis, to a path of length $s+\delta$, leading to a path of length $2s+\delta; 0 \le \delta \le s$ from $S_{n+1,T,T}$ to $S_{n+1,L,L}$. This completes the proof. □

Theorem 6 *S_n is pancyclic for each n.*

Proof Once again the proof is by induction. Assume the result is true for all $m \le n$. The Hamiltonian cycle $S_{n+1,T,L} \to S_{n+1,T,R} \to S_{n+1,R,L} \to S_{n+1,T,L}$ of S_{n+1} consists of three Hamiltonian paths in $S_{n+1,T}$, $S_{n+1,R}$ and $S_{n+1,L}$ respectively. By Lemma 5 applied to S_n, we reduce these as necessary to get cycles of all sizes $\ge 3s$. Cycles of smaller sizes are obtained by invoking the induction hypothesis on S_n, noting that $|S_n| \ge 3s$. □

3 Domination Numbers and Efficiency

The three degree two vertices $\{v_T, v_L, v_R\}$ of S_n will be called "corner vertices" and the three vertices $S_{n,T,R}, S_{n,R,L}$ and $S_{n,L,T}$ will be called the "middle vertices" of S_n. Let $S'_n = S_n \setminus \{v_T, v_L, v_R\}$. For $k = 0, 1, 2, 3$, let ${\gamma_n}^k$ be the minimum number of vertices needed to dominate S'_n *in addition to* k corner vertices that are to assist in the dominating of S'_n. Let γ_n denote the domination number of S_n.

Theorem 7 *For every $n \ge 4$ we have*

$$\gamma_n = 3 \cdot \gamma_{n-1}$$

and

$${\gamma_n}^k \ge \begin{cases} \gamma_n & \text{if } k = 0,\ 1 \\ \gamma_n - 1 & \text{if } k = 2,\ 3 \end{cases}$$

Proof The fact that $\gamma_n \le 3\gamma_{n-1}$ is a trivial consequence of the decomposition of S_{n+1} into its three components.

It is immediate that $\gamma_1 = 1; \gamma_2 = 2$; and $\gamma_3 = 3$. We next verify that both parts of the result are true for $n = 4$. First note that $\gamma_4 \le 9$. Also if $\gamma_4 = 8$, we contradict the fact that for any graph G with maximum degree Δ, $|\gamma(G)|(\Delta+1) \ge |V(G)|$. The fact that $\gamma_4^0 = \gamma_4^1 = 9$ may be checked by hand. Consider γ_4^2. Since a total of four vertices of S'_4 are dominated by the two external vertices, we need to dominate 35 others in S'_4. Assuming that the two aiding vertices are v_1 and v_{34} (see Figure 1), we must have v_4 or v_6 and v_{20} or v_{36} in the dominating set for S'_4. Supposing without loss of generality that v_4 and v_{20} are in the dominating set, we must now dominate 27 additional vertices and thus need at least 6 other vertices in the dominating set. Thus $\gamma_4^2 \ge 8$ as required. The fact that $\gamma_4^3 \ge 8$ is checked similarly.

Assume then that the statements of the theorem are both true for each m with $4 \leq m \leq n$. Let us start by proving the first part of the theorem. Since any dominating set of S_{n+1} contains either 0, 1, 2, or 3 "middle vertices," we have

$$\begin{aligned} \gamma_{n+1} &\geq \min\{3{\gamma_n}^0, 2{\gamma_n}^1 + {\gamma_n}^0 + 1, 2{\gamma_n}^1 + {\gamma_n}^2 + 2, 3{\gamma_n}^2 + 3\} \\ &\geq \min\{3\gamma_n, 3\gamma_n + 1, 3\gamma_n + 1, 3\gamma_n\} \\ &= 3\gamma_n, \end{aligned}$$

as required. A word of explanation might be in order: In the above calculation, the first quantity, namely $3\gamma_n^0$, is a lower bound on γ_{n+1} assuming that no middle vertices are in the dominating set of S_{n+1}. It is obtained as follows. There might be 0, 1, 2, or 3 *corner* vertices in the dominating set and we thus have, in this case,

$$\gamma_{n+1} \geq \min\{3\gamma_n^0, 2\gamma_n^0 + \gamma_n^1 + 1, \gamma_n^0 + 2\gamma_n^1 + 2, 3\gamma_n^1 + 3\} = 3\gamma_n^0.$$

Actually, it is evident that the minimum in each case corresponds to there being *no* corner vertices in the dominating set.

For the second part of the proposition, we note that $\gamma_n^0 \geq \gamma_n^1$ and $\gamma_n^2 \geq \gamma_n^3$, so that

$$\begin{aligned} \gamma_{n+1}^0 \geq {\gamma_{n+1}}^1 &\geq \min\{2{\gamma_n}^0 + {\gamma_n}^1, 3{\gamma_n}^1 + 1, {\gamma_n}^2 + {\gamma_n}^1 + {\gamma_n}^0 + 1, \\ &\qquad {\gamma_n}^3 + 2{\gamma_n}^1 + 2, {\gamma_n}^1 + 2{\gamma_n}^2 + 2, {\gamma_n}^3 + 2{\gamma_n}^2 + 3\} \\ &= 3\gamma_n \\ &= \gamma_{n+1}, \end{aligned}$$

and

$$\begin{aligned} \gamma_{n+1}^2 \geq {\gamma_{n+1}}^3 &\geq \min\{3{\gamma_n}^1, 2{\gamma_n}^2 + {\gamma_n}^1 + 1, 2{\gamma_n}^2 + {\gamma_n}^3 + 2, 3{\gamma_n}^3 + 3\} \\ &\geq 3\gamma_n - 1 \\ &= \gamma_{n+1} - 1, \end{aligned}$$

completing the proof. □

Remarks Note that by Theorem 7, $\gamma_n = 3^{n-2}$ for $n \geq 3$, and thus, since none of the outer vertices are in the minimum dominating set, it follows that this set "covers," with duplication, a total of $5 \cdot 3^{n-2}$ vertices. Now S_n has $\frac{3}{2}\left(3^{n-1} + 1\right)$ vertices, so that the "efficiency" of the domination is asymptotically 90%. This is in contrast to the fact, exhibited by Klavžar, Milutinović, and Petr [11] that the graphs $S(n, k)$ have perfect dominating sets, i.e., are 100% efficient. After we had completed this research, our colleague Teresa Haynes pointed out that domination numbers of the so-called E-graphs of [7] provide generalizations of Theorem 7. For this reason, we have given only an abbreviated proof of Theorem 7.

4 Pebbling Numbers

Given a connected graph G, distribute t indistinguishable pebbles on its vertices in some configuration. Specifically, a configuration of weight t on a graph G is a function C from the vertex set $V(G)$ to $\mathbf{N} \cup \{0\}$ such that $\sum_{v \in V(G)} C(v) = t$. A *pebbling move* is defined as the removal of two pebbles from some vertex and the placement of one of these on an adjacent vertex. Given an initial configuration, a vertex v is called *reachable* if it is possible to place a pebble on it in finitely many pebbling moves. Given a configuration, the graph G is said to be *pebbleable* if any of its vertices can be thus reached. Define the pebbling number $\pi(G)$ to be the minimum number of pebbles that are sufficient to pebble the graph regardless of the initial configuration.

SPECIAL CASES: The pebbling number $\pi(P_n)$ of the path is 2^{n-1} ([5]). Chung [2] proved that $\pi(Q^d) = 2^d$ and $\pi(P_n^m) = 2^{(n-1)m}$, where Q^d is the d-dimensional binary cube and P_n^m is the cartesian product of m copies of P_n. An easy pigeonhole principle argument yields $\pi(K_n) = n$. The pebbling number of trees has been determined (see [5]). One of the key conjectures in pebbling, now proved in several special cases, is due to Graham; its resolution would clearly generalize Chung's result for m-dimensional grids:

GRAHAM'S CONJECTURE. *The pebbling number of the cartesian product of two graphs is no more that the product of the pebbling numbers of the two graphs, i.e.*

$$\pi(G \square H) \leq \pi(G)\pi(H).$$

A detailed survey of graph pebbling has been presented by Hurlbert [5], and a survey of open problems in graph pebbling may be found at [6].

Consider also the following variant of pebbling called cover pebbling, first discussed by Crull et al ([3]): The *cover pebbling number* $\lambda(G)$ is defined as the minimum number of pebbles required such that it is possible, given any initial configuration of at least $\lambda(G)$ pebbles on G, to make a series of pebbling moves that *simultaneously* reaches *each* vertex of G. A configuration is said to be *cover solvable* if it is possible to place a pebble on every vertex of G starting with that configuration. Various results on cover pebbling have been determined. For instance, we now know ([3]) that $\lambda(K_n) = 2n - 1$; $\lambda(P_n) = 2^n - 1$; and that for trees T_n,

$$\lambda(T_n) = \max_{v \in V(T_n)} \sum_{u \in V(T_n)} 2^{d(u,v)}, \tag{1}$$

where $d(u, v)$ denotes the distance between vertices u and v. Likewise, it was shown in [8] that $\lambda(Q^d) = 3^d$ and in [18] that $\lambda(K_{r_1,\ldots,r_m}) = 4r_1 + 2r_2 + \ldots + 2r_m - 3$, where $r_1 \geq 3$ and $r_1 \geq r_2 \geq \ldots \geq r_m$. The above examples reveal that for these special classes of graphs at any rate, the cover pebbling number equals the "stacking number", or, put another way, the worst possible distribution of pebbles consists of placing all the pebbles on a single vertex. The intuition built by computing the value of the cover pebbling number for the families K_n, P_n, and T_n in [3] led to Open

Question No. 10 in [3], and which was proved by Vuong and Wyckoff [17] and later, independently, by Sjöstrand [14]:

STACKING THEOREM: *For any connected graph G,*

$$\lambda(G) = \max_{v \in V(G)} \sum_{u \in V(G)} 2^{d(u,v)},$$

thereby proving that (1) holds for all graphs.

The Sierpiński graph will now be revealed to be one for which the use of the Stacking Theorem does not reduce the computation of the cover pebbling number to a trivial exercise. We first prove that the diameter of S_n is 2^{n-1}, and, using this fact, that the worst vertex on which to stack pebbles is a corner vertex a of degree two:

Lemma 8 $\mathrm{diam}(S_n) = 2^{n-1}$.

Proof The fact that $\mathrm{diam}(S_n) \geq 2^{n-1}$ is obvious. We use induction for the reverse inequality. The result is clearly true for $n = 1$. Assume it to be true for n. Let x, y be any two points in (without loss of generality) $S_{n+1,T}$ and $S_{n+1,L}$ respectively. We will denote for brevity $S_{n+1,L,T}$ by z. We thus have

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) \\ &\leq 2^{n-1} + 2^{n-1} = 2^n, \end{aligned}$$

as required. □

Lemma 9 *$ST(a) \geq ST(v)$ for each $v \in S_n$, where $ST(v) = \sum_{u \in S_n} 2^{d(u,v)}$.*

Proof. We proceed by induction. The result is easy to verify for S_1 and S_2. Assume that it is true for S_n. In S_{n+1}, denote the vertices $S_{n+1,T,T}$, $S_{n+1,L,T}$, and $S_{n+1,R,T}$ by a, b and c respectively. Let d and e be arbitrary vertices in $S_{n+1,T}$ and $S_{n+1,L} \cup S_{n+1,R}$ respectively. By Lemma 8, we have

$$d(a,e) = 2^{n-1} + \min\{d(b,e), d(c,e)\},$$

and

$$d(d,e) = \min\{d(d,b) + d(b,e), d(d,c) + d(c,e)\}.$$

Since, however, $\min\{d(b,e), d(c,e)\} = d(\alpha, e)$, where $\alpha = b$ or $\alpha = c$, it follows that

$$d(d,e) \leq d(d,\alpha) + d(\alpha,e) \leq 2^{n-1} + d(\alpha,e) = d(a,e).$$

We thus have

$$\begin{aligned}
ST(d) &= \sum_{u\in S_{n+1}} 2^{d(d,u)} \\
&= \sum_{u\in S_{n+1,T}} 2^{d(d,u)} + \sum_{u\in[S_{n+1,L}\cup S_{n+1,R}]\setminus\{b,c\}} 2^{d(d,u)} \\
&\leq \sum_{u\in S_{n+1,T}} 2^{d(d,u)} + \sum_{u\in[S_{n+1,L}\cup S_{n+1,R}]\setminus\{b,c\}} 2^{d(a,u)} \\
&\leq \sum_{u\in S_{n+1,T}} 2^{d(a,u)} + \sum_{u\in[S_{n+1,L}\cup S_{n+1,R}]\setminus\{b,c\}} 2^{d(a,u)} \\
&= ST(a),
\end{aligned}$$

where the next to last line above follows due to the induction hypothesis. □

Theorem 10 *The cover pebbling number $\lambda(S_n)$ of the Sierpiński graph satisfies the recursion*

$$\lambda(S_{n+1}) = \left(1+2^{2^{n-1}+1}\right)\lambda(S_n) - \left(2^{2^n}+2^{2^{n-1}+1}\right).$$

Proof Notice that, with $\beta(i)$ denoting (in S_{n+1}) the number of points at distance i from the corner vertex a, the following two conditions hold:

- For any $j \in \{1, 2, \ldots, 2^{n-1}-1\}$, $\beta(j+2^{n-1}) = 2\beta(j)$;
- $\beta(2^n) = 2\beta(2^{n-1}) - 1$.

Thus,

$$\begin{aligned}
\lambda(S_{n+1}) &= ST(a) \\
&= \sum_{i=0}^{2^{n-1}} \beta(i)2^i + 2\sum_{i=1}^{2^{n-1}} \beta(i)2^{i+2^{n-1}} - 2^{2^n} \\
&= \lambda(S_n) + 2\cdot 2^{2^{n-1}}(\lambda(S_n)-1) - 2^{2^n} \\
&= \left(1+2^{2^{n-1}+1}\right)\lambda(S_n) - \left(2^{2^n}+2^{2^{n-1}+1}\right),
\end{aligned}$$

as asserted. □

5 Open Problems

Here are some open problems for readers of this paper to consider:

- What is the edge chromatic number (chromatic index) and total chromatic number of S_n? By Vizing's Theorem, the former is either 4 or 5, and if Behzad's total chromatic conjecture is true, then the latter is either 5 or 6.

- What is the pebbling number of S_n? Various bounds as in [1] may be used to estimate this quantity, but we consider the determination of $\pi(S_n)$ to be quite hard.
- What baseline structural properties similar to the ones we have studied in this paper can be established for the Sierpiński graphs of Klavžar and Milutinović [10]?
- In a similar vein, what can be said of the domination number, cycle structure, etc. of Sierpiński-like graphs generated by considering Pascal's triangle mod $p; p \geq 3$? (See [13] for details on this structure and recall that the Siepiński gasket graph is related to Pascal's triangle mod 2.)

Acknowledgment

Teguia was a graduate student at ETSU when this research was conducted — but not as part of his M.S. thesis, which was in operator theory. Godbole's research was supported by NSF Grant DMS-0139291. This paper has benefited greatly from the suggestions for improvement made by the anonymous referee.

References

[1] M. Chan and A. Godbole, Improved pebbling bounds, submitted to *Discrete Math.*, 2005.

[2] F. Chung, Pebbling in hypercubes, *SIAM J. Discrete Mathematics* **2** (1989), 467–472.

[3] B. Crull, T. Cundiff, P. Feltman, G. Hurlbert, L. Pudwell, Z. Szaniszlo and Z. Tuza, The cover pebbling number of graphs, *Discrete Math.* **296** (2005), 15–23.

[4] G. Edgar, *Measure, Topology and Fractal Geometry*, Springer Verlag, New York, 1990.

[5] G. Hurlbert, A survey of graph pebbling, *Congr. Numer.* **139** (1999), 41–64.

[6] G. Hurlbert (2005), `http://math.la.asu.edu/~hurlbert/HurlPebb.ppt`.

[7] T. Haynes and L. Lawson, Applications of E-graphs in network design, *Networks* **23** (1993), 473–479.

[8] G. Hurlbert and B. Munyan, The cover pebbling number of hypercubes, Preprint, to appear in *Bull. Inst. Comb. Appl.*, (2005) `http://arxiv.org/abs/math.CO/0409368`.

[9] S. Klavžar (2005), `http://www-mat.pfmb.uni-mb.si/personal/klavzar/hanoi.htm`

[10] S. Klavžar and U. Milutinović, Graphs $S(n,k)$ and a variant of the Tower of Hanoi problem, *Czechoslovak Math. J.* **122** (1997), 95–104

[11] S. Klavžar, U. Milutinović and C. Petr, 1-perfect codes in Sierpiński graphs,*Bull. Austral. Math. Soc.* **66** (2002), 369–384.

[12] S. Klavžar and B. Mohar, Crossing numbers of Sierpiński-like graphs,submitted to *J. Graph Theory.* (2004).

[13] K. Shannon and M. Bardzwell, Patterns in Pascal's triangle — with a twist, *J. Online Math.* (2003), `http://joma.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=493`.

[14] J. Sjöstrand, The cover pebbling theorem, Preprint. (2005) `http://arxiv.org/abs/math/0410129`

[15] E. Teufl, The average displacement of the simple random walk on the Sierpiński graph, *Comb. Prob. Computing* **12** (2003), 203–222.

[16] K. Touhey (2005), `http://www.math.ubc.ca/~cass/courses/m308-02b/projects/touhey/`.

[17] A. Vuong and I. Wyckoff, Conditions for Weighted Cover Pebbling of Graphs, submitted to *Graphs Comb.* (2004) `http://arxiv.org/abs/math/0410410`.

[18] N. Watson and C. Yerger, Cover pebbling numbers and bounds for certain families of graphs, Preprint, To appear in *Bull. Inst. Comb. Appl.* (2005) `http://arxiv.org/abs/math.CO/0409321`.

(Received 31 Dec 2004)