AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **34** (2006), Pages 331–342

Weighted Ramsey problem

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Abstract

A weighted graph is one in which every edge e is assigned a nonnegative number, called the weight of e. The weight of a graph is defined as the sum of the weights of its edges.

In 2-edge-colored complete graph, by using Ramsey-type theorems, we obtain the existence of monochromatic subgraph which have many edges compared with its order. In this paper, we extend the concept of Ramsey problem to the weighted graphs, and we show the existence of a heavy monochromatic subgraph in 2-edge-colored graph with small order.

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. We use [1] for basic terminology and notation not defined here. A weighted graph is one in which every edge e is assigned a nonnegative real number $w(e)$, called the *weight* of e. For a subgraph H of G, the *weight* of H is defined by the sum of the weight of the edges in H, denoted by $w_G(H)$. When there is no fear of confusion, we denote $w_G(H)$ by $w(H)$.

We say that a graph G can be *decomposed* into graphs $H_1, H_2, \ldots, H_l$ if and only if there is a set $\{G_1, G_2, \ldots, G_l\}$ of subgraphs of G such that each G_i is isomorphic to H_i and each edge of G is contained in exactly one of the graphs in $\{G_1, G_2, \ldots, G_l\}$. In this case we also say that $\{H_1, H_2, \ldots, H_l\}$ is a *decomposition* of G. Especially, we call a decomposition of a graph G into two weighted graphs R and B *a 2-edge-coloring* of G, so that the edges in R are colored red, and the edges in B are colored blue. For any subgraph H of a 2-edge-colored weighted graph G, we define

$$w_R(H) = \sum_{e \in E(R) \cap E(H)} w(e), \;\; w_B(H) = \sum_{e \in E(B) \cap E(H)} w(e).$$

In [4], some Turán-Ramsey theorems for weighted graphs in which every edge has weight 0, 1/2 or 1, are considered. And in [2] and [5], there are some results of Turán problems for weighted graphs, in which the weight of every edge is a rational number.

In this paper we deal with more general weighted graphs, i.e. every nonnegative real number is allowed for the weights of the edges. And, the aim is to introduce the Weighted Ramsey Problem, the extension of the Ramsey Problem to weighted graphs.

Definition 1. *Let n and s be two integers with $n > s \geq 3$. We define $WR(s;n)$ to be the supremum value c such that for any weighting function w of K_n, and for any 2-edge-coloring R and B of K_n, there exists a subgraph H of order s satisfying* $\max\{w_R(H), w_B(H)\} \geq c \cdot w(K_n)$.

The following proposition shows the relation between the Ramsey number $R(s,s)$ and the weighted Ramsey number $WR(s;n)$.

Proposition 1. *Let n and s be two integers with $n > s \geq 3$. Then $R(s,s) \leq n$ if*

$$WR(s;n) > \frac{s(s-1)-2}{n(n-1)}. \tag{1}$$

Proof. Consider a weighted complete graph G of order n such that $w(e) = 1$ for every edge e in G. By (1) and the fact $w(G) = n(n-1)/2$, we can find $H \simeq K_s$ such that

$$\max\{w_R(H), w_B(H)\} > \frac{s(s-1)-2}{n(n-1)} \cdot w(G) = \frac{s(s-1)-2}{2} = \frac{s(s-1)}{2} - 1.$$

Since $w(e) = 1$ for every edge in G, H is a monochromatic K_s, which implies $R(s) \leq n$. □

Since $\max\{w(R), w(B)\} \geq w(G)/2$ for any 2-edge-coloring of weighted complete graph G with order n, we easily obtain the following proposition from the straightforward averaging argument.

Proposition 2. *Let n and s be two integers with $n > s \geq 3$. Then*

$$WR(s;n) \geq \frac{1}{2} \cdot \frac{s(s-1)}{n(n-1)}.$$

□

On the other hand, the Turán graph and its complement give an upper bound of $WR(s,n)$.

Proposition 3. *Let n and s be two integers with $n > s \geq 3$. Then*

$$WR(s;n) < \frac{s^2-1}{s^2+1} \cdot \frac{s(s-1)}{n(n-1)}.$$

Proof. Let $T_r(n)$ be the *Turán graph*, the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. Consider the 2-edge-coloring of K_n where $R \simeq T_{s-1}(n)$ and B is the complement of R. Now we assign weight

$$\frac{1}{\binom{s}{2} - 1}$$

for every red edge and weight

$$\frac{1}{\binom{s}{2}}$$

for every blue edge. Then $\max\{w_R(H), w_B(H)\} \leq 1$ for every induced subgraph H of order s and there are

$$\left(1 - \frac{1}{s-1} + f(s,n)\right)\binom{n}{2}$$

red edges and

$$\left(\frac{1}{s-1} - f(s,n)\right)\binom{n}{2}$$

blue edges, where $f(s,n)$ is a function such that $f(s,n) > 0$ for every s, n and $f(s,n) \to 0$ as $n \to \infty$. Hence

$$\begin{aligned} w(G) &= \frac{2}{s(s-1)-2} \cdot \left(1 - \frac{1}{s-1} + f(s,n)\right)\binom{n}{2} \\ &\qquad + \frac{2}{s(s-1)} \cdot \left(\frac{1}{s-1} - f(s,n)\right)\binom{n}{2} \\ &> \left(\frac{s^2+1}{(s-1)^2 s(s+1)}\right) \cdot 2 \cdot \binom{n}{2} \\ &= \frac{s^2+1}{(s-1)(s+1)} \cdot \frac{n(n-1)}{s(s-1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} WR(s;n) &\leq \frac{1}{w(G)} \\ &< \frac{s^2-1}{s^2+1} \cdot \frac{s(s-1)}{n(n-1)}. \end{aligned}$$

□

In this paper, we determine the exact value of $WR(3;n)$ for $n = 5$ and 6.

Theorem 1. $WR(3;5) = 1/5$.

Theorem 2. $WR(3;6) = 1/7$.

We prove Theorems 1 and 2 in the later section. By Proposition 1, we obtain that Theorem 2 implies the fact $R(3,3) \leq 6$. Note that Theorem 1 implies that the equality

$$WR(s;n) = \frac{s(s-1)-2}{n(n-1)}$$

holds for $s = 3$ and $n = 5$. In this sense, we can say that the fact $R(3,3) > 5$ is optimal even for weighted graphs.

By using Theorem 2, we can improve the lower bound of $WR(3;n)$ in Proposition 2.

Proposition 4. *If* $n \geq 6$, *then*

$$WR(3;n) \geq \frac{30}{7} \cdot \frac{1}{n(n-1)}.$$

Proof. Let G be a weighted complete graph of order n with $w(G) = 1$. By the straightforward averaging argument, we obtain the existence of a subgraph $G' \simeq K_6$ in G such that

$$w(G') \geq \frac{30}{n(n-1)}.$$

Then, it follows from Theorem 2 that there exists an induced subgraph $H \simeq K_3$ satisfying

$$\max\{w_R(H), w_B(H)\} \geq w(G')/7 \geq \frac{30}{7} \cdot \frac{1}{n(n-1)},$$

which implies the assertion. □

We shall discuss the value $WR(3;n)$ further in Section 5.

2 Lemmas

Let B_1 and B_2 be two graphs. $B_1 + B_2$ denotes a graph obtained by joining every vertex in B_1 and every vertex in B_2. For a graph B and $v \in V(B)$, $d_B(v)$ is the number of neighbors of v in B. We say $E(B)$ is *connected* if $E(B)$ induces a connected graph. A path with r vertices is denoted by P_r, and the graph $K_{1,r}$ is called a *star*. In a star $K_{1,r}$, the vertex of degree r is called its *center*, and degree 1 its *leaf*. The star with the center u and the leaves $v_1, v_2, \ldots, v_r$ is denoted by $u\text{-}v_1v_2\ldots v_r$. A graph is called *claw-free* if it contains no $K_{1,3}$ as an induced subgraph.

To prove Theorems 1 and 2, for the technical reason, we consider the following weighting functions for a given graph B;

$$\mathcal{W}(B) = \{w : E(B) \to \mathbb{R}^+ \mid w(B') \leq 6 \text{ for any subgraph } B' \text{ of } B \text{ with } |V(B')| \leq 3\},$$

and investigate the following invariant.

$$W(B) = \sup\{w(B) \mid w \in \mathcal{W}(B)\}.$$

Now we prepare some facts and lemmas, which determine the values of $\mathcal{W}(B)$ for several graphs B. The following fact is obvious, so we omit the proof.

Fact 1. *Let* B *be a graph with at most* 6 *vertices. If* $|E(B)| \geq 8$, *then* $E(B)$ *induces a connected graph.* □

Lemma 1. *Let* B' *be a subgraph of* B, *then* $W(B') \leq W(B)$.

Proof. Assume that $w'(B') > W(B)$ for some $w' \in \mathcal{W}(B')$. Consider the weighting function w such that $w(e) = w'(e)$ if $e \in B'$ and $w(e) = 0$ if $e \notin B'$. Then it is clear that $w(B) = w'(B') > W(B)$ and $w \in \mathcal{W}(B)$, which contradicts the definition of $W(B)$. $\square$

Lemma 2. *If B is an edge-disjoint union of the graphs B_1 and B_2, then $W(B) \leq W(B_1) + W(B_2)$.*

Proof. If $w \in \mathcal{W}(B)$ and $w(B) > W(B_1) + W(B_2)$, then $w(B_i) > W(B_i)$ for $i = 1$ or 2, which contradicts the definition of $W(B_i)$. $\square$

Lemma 3. *If B is a star $K_{1,r}$ with $r \geq 2$, then $W(B) = 3r$.*

Proof. Let u be the center of B, let $v_1, v_2, \dots, v_r$ be the leaves of B and let $v_{r+1} = v_1$. For any $w \in \mathcal{W}(B)$, we have $w(v_i u v_{i+1}) \leq 6$ for every i, where the index i is taken as modulo r. Hence

$$w(B) = \frac{1}{2}\sum_{i=1}^{r} w(v_i u v_{i+1}) \leq \frac{1}{2} \cdot 6r = 3r.$$

The constant weight with $w(e) \equiv 3$ shows that $W(B) = 3r$. $\square$

Lemma 4. *If B is a cycle with length at least 4, then $W(B) = 3|E(B)|$.*

Proof. Let $B = v_1 v_2 \dots v_r$. For any $w \in \mathcal{W}(B)$, we have $w(v_i v_{i+1} v_{i+2}) \leq 6$ for every i, where the index i is taken as modulo r. Hence

$$w(B) = \frac{1}{2}\sum_{i=1}^{r} w(v_i v_{i+1} v_{i+2}) \leq \frac{1}{2} \cdot 6r = 3|E(B)|.$$

The constant weight with $w(e) \equiv 3$ shows that $W(B) = 3|E(B)|$. $\square$

Lemma 5. *If B is a complete graph of order $n \geq 3$, then $W(B) = n(n-1)$.*

Proof. Let $\mathcal{T} = \{T \mid T \text{ is a triangle in } B\}$ and let $w \in \mathcal{W}(B)$. Then $w(T) \leq 6$ for all $T \in \mathcal{T}$. Hence

$$\begin{aligned} w(B) &= \frac{1}{n-2}\sum_{T \in \mathcal{T}} w(T) \\ &\leq \frac{1}{n-2} \cdot \binom{n}{3} \cdot 6 \\ &= n(n-1). \end{aligned}$$

The constant weight with $w(e) \equiv 2$ shows that $W(B) = n(n-1)$. $\square$

Lemma 6. *If C is a cycle of length $r \geq 4$ and $B = K_1 + C$, then $W(B) = 9r/2$.*

Proof. Let $C = v_1v_2\dots v_r$ and let u be the vertex of $V(B) \setminus V(C)$. Moreover, let T_i be the triangle uv_iv_{i+1}, where the indices i and j are taken as modulo r. If $w \in \mathcal{W}(B)$, then by Lemma 4, we have $w(C) \leq 3r$. Hence,

$$\begin{aligned} W(B) &\leq w(B) \\ &= \frac{1}{2}\Big(\sum_{i=1}^{r} w(T_i) + w(C)\Big) \\ &\leq \frac{1}{2}\cdot(6r+3r) \\ &\leq \frac{9}{2}r. \end{aligned}$$

On the other hand, there is $w \in \mathcal{W}(B)$ such that $w(e) = 3$ for every $e \in E(C)$ and $w(e) = 3/2$ for all the other edges. This shows $W(B) = 9r/2$. □

Lemma 7. *If $B \simeq K_6 - E(3K_2)$, then $W(B) = 24$.*

Proof. Let $E(\overline{B}) = \{a_1b_1, a_2b_2, a_3b_3\}$. Then B can be decomposed into four triangles $a_1a_2a_3$, $a_1b_2b_3$, $b_1a_2b_3$ and $b_1b_2a_3$. For any $w \in \mathcal{W}(B)$, each of them has weight at most 6, hence we have $w(B) \leq 24$. The constant weight with $w(e) \equiv 2$ shows that $W(B) = 24$. □

Sumner [7] and Las Vergnas [6] proved that every connected claw-free graph of even order has a 1-factor. Since the line graph of any graph is claw-free, we obtain that if B is a connected graph with $|E(B)|$ even, then its line graph $L(B)$ has a 1-factor M. If $e_1e_2 \in M$, then e_1 and e_2 is adjacent in B, hence this 1-factor corresponds to a partition of $E(B)$ to pairwise adjacent edges. This implies the following fact.

Fact 2. *Let B be a connected graph with $|E(B)|$ even. Then B can be partitioned into $|E(B)|/2$ pairs of adjacent edges.* □

And, the following fact is easily obtained from Fact 2.

Fact 3. *Let B be a connected graph with $|E(B)|$ odd. Then B can be partitioned into an edge and $(|E(B)|-1)/2$ pairs of adjacent edges.* □

Using these facts, we obtain the following lemma.

Lemma 8. *Suppose that B is a connected graph. If B is a tree with a perfect matching, then $W(B) = 3|E(B)| + 3$. Otherwise, $W(B) \leq 3|E(B)|$.*

Proof. If $|E(B)|$ is even, then B can be decomposed into $|E(B)|/2$ edge-disjoint P_3s. Hence, by Lemma 2, $W(B) \leq (|E(B)|/2)W(P_3) = 3|E(B)|$.

Suppose that $|E(B)|$ is odd. Since B can be decomposed into $(|E(B)|-1)/2$ edge-disjoint P_3s and one K_2, then by Lemma 2 again, we have $W(B) \leq ((|E(B)|-1)/2)W(P_3) + W(K_2) = 3(|E(B)|-1) + 6 = 3|E(B)| + 3$. In fact, when B is a tree with a perfect matching M, if we assign $w(e) = 6$ for $e \in M$ and $w(e) = 0$

for $e \notin M$, then $w \in \mathcal{W}(B)$ and $w(B) = 3|V(B)| = 3|E(B)| + 3$, which shows that $W(B) = 3|E(B)| + 3$.

Suppose next that B is a tree without perfect matchings. Recall that $|V(B)| = |E(B)| + 1$ is even. It follows from the fact B does not have a perfect matching that B is not a path, which implies that there exists a vertex v such that $d_B(v) \geq 3$. Since B is a tree, $B - v$ contains at least three odd components B_1, B_2 and B_3. Let v_i be the neighbor of v in B_i for $i = 1, 2, 3$. It is easy to see that each component of $B - \{vv_1, vv_2, vv_3\}$ has even number of edges. This implies that B can be decomposed into $(|E(B)| - 3)/2$ edge-disjoint P_3s and one $K_{1,3}$. By Lemmas 2 and 3, we have

$$W(B) \leq ((|E(B)| - 3)/2)W(P_3) + W(K_{1,3}) = 3(|E(B)| - 3) + 9 = 3|E(B)|.$$

Suppose that B contains a cycle. We use induction on $|E(B)|$ to prove that $W(B) \leq 3|E(B)|$. Let C be a cycle in B. If $B = C$ itself, then by Lemma 4, we have $W(B) \leq 3|E(B)|$. Let T be a unicyclic spanning subgraph of B such that $C \subseteq T$. Since $B \neq C$, we can take a leaf u of T which is farthest from C in T. If we can take a $P \simeq P_3$ containing u such that $E(P) \cap E(C) = \emptyset$ and $E(B) \setminus E(P)$ induces a connected subgraph, then by induction, we have $W(B) \leq W(B \setminus E(P)) + W(P) \leq 3(|E(B)| - 2) + 6 = 3|E(B)|$. This is the case unless $d_B(u) = 1$ and the unique neighbor v of u in B is in C and $d_T(v) = 3$. In this case, let v_1 and v_2 be the neighbor of v in C. It is easy to see that $B - \{vu, vv_1, vv_2\}$ is connected. This implies that B can be decomposed into $(|E(B)| - 3)/2$ edge-disjoint P_3s and one $K_{1,3}$. By Lemmas 2 and 3, we have

$$W(B) \leq ((|E(B)| - 3)/2)W(P_3) + W(K_{1,3}) = 3(|E(B)| - 3) + 9 = 3|E(B)|.$$

□

3 Proof of Theorem 1

Let G be a 2-edge-colored complete graph with 5 vertices. If each edge of G has weight 3 and $R \simeq B \simeq C_5$, then $w(G) = 30$ and $\max\{w_R(H), w_B(H)\} \leq 6 = w(G)/5$ for every triangle H in G, hence we have $WR(3; 5) \leq 1/5$. To prove the lower bound, we assume $\max\{w_R(T), w_B(T)\} \leq 6$ for every triangle T in G. Then it suffices to prove that $w(G) \leq 30$. If G has no monochromatic K_3, it is easy to see that $R \simeq B \simeq C_5$. Then Lemma 4 implies that $w(R) \leq 15$ and $w(B) \leq 15$, hence we obtain $w(G) = w(R) + w(B) \leq 30$. So we may assume that there exists a monochromatic triangle in G.

Now we consider the case where G has a monochromatic $2K_2 + K_1$. Without loss of generality we may assume that $2K_2 + K_1 \subseteq B$. Note that $R \subseteq C_4$. If $R \simeq K_2$ or P_3, we have $w(R) \leq 6$, and Lemmas 1 and 5 imply $w(B) \leq 24$. Hence $w(G) = w(R) + w(B) \leq 30$. Otherwise, $2K_2 \subseteq R$. Then $B \subseteq C_4 + K_1$, so Lemmas 1 and 6 imply that $w(B) \leq 18$. On the other hand, since $R \subseteq C_4$, we have $w(R) \leq 12$ by Lemmas 1 and 4. Thus $w(G) = w(R) + w(B) \leq 30$.

Therefore, we may assume that G has a monochromatic K_3 but no monochromatic $2K_2 + K_1$. Without loss of generality we may assume that $K_3 \subseteq B$. Since $2K_2 + K_1 \nsubseteq B$, we have $|E(R)| \geq 3$.

In case of $|E(R)| = 3$, $R \simeq P_4, K_{1,3}, K_3$ or $P_3 \cup K_2$. By Lemmas 2 and 8, we obtain $w(R) \leq 12$ in each case. Let B' be a graph obtained from B by deleting the edges of a triangle in B. Then $|E(B')| = 4$ and $E(B')$ must be connected, hence Lemma 8 shows $w(B') \leq 12$, which implies $w(B) \leq 18$. Thus $w(G) = w(R) + w(B) \leq 30$. In case of $|E(R)| = 4$, $E(R)$ must be connected, hence Lemma 8 implies that $w(R) \leq 12$. Since $|E(B)| = 6$, $E(B)$ is also connected, hence it follows from Lemma 8 that $w(B) \leq 18$. Therefore, we have $w(G) = w(R) + w(B) \leq 30$.

If $|E(R)| = 5$, then $3K_2 \not\subseteq R$ and $3K_2 \not\subseteq B$. Hence Lemma 8 implies $w(R), w(B) \leq 15$, thus we have $w(G) = w(R)+w(B) \leq 30$. If $|E(R)| = 6$, since $E(R)$ is connected, we have $w(R) \leq 18$ by Lemma 8. If $E(B)$ is connected, Lemma 8 implies $w(B) \leq 12$, and otherwise $B \simeq K_2 \cup K_3$, so $w(B) \leq 12$. Thus we have $w(G) \leq 30$. And if $|E(R)| = 7$, Lemma 8 implies $w(R) \leq 21$. Now the fact $B \simeq K_3$ shows $w(B) \leq 6$, hence we have $w(G) \leq 30$. This completes the proof of Theorem 1. □

4 Proof of Theorem 2

Let G be a 2-edge-colored complete graph with 6 vertices and $R \simeq 3K_2$. If each edge of R has weight 6 and each edge of B has weight 2, then $w(G) = 42$ and $\max\{w_R(H), w_B(H)\} \leq 6 = w(G)/7$ for every triangle H in G. Hence we have $WR(3;5) \leq 1/7$. To prove the lower bound, as in the proof of Theorem 1, we assume $\max\{w_R(T), w_B(T)\} \leq 6$ for every triangle T in G, and then it suffices to prove $w(G) \leq 42$. Without loss of generality, we may assume that $|E(R)| < 8 \leq |E(B)|$.

Case 1. $|E(R)| \leq 2$.

In this case it is obvious that $w(R) \leq 12$, and Lemmas 1 and 5 imply that $w(B) \leq 30$, hence $w(G) \leq 42$.

Case 2. $|E(R)| = 3$.

In this case, $R \simeq P_4$, $K_{1,3}$, K_3, $P_3 \cup K_2$ or $3K_2$. If $R \not\simeq 3K_2$, then we obtain $w(R) \leq 12$ by Lemmas 2 and 8. On the other hand, Lemmas 1 and 5 imply that $w(B) \leq 30$, thus we have $w(G) \leq 42$. If $R \simeq 3K_2$, then $w(R) \leq 18$. Since Lemma 7 implies $w(B) \leq 24$, we obtain $w(G) \leq 42$.

Case 3. $|E(R)| = 4$.

Since $|E(B)| = 11$, there exists a triangle in B, say T. Let $B' = B - E(T)$, then it follows from Fact 1 that $E(B')$ is connected. Hence Lemma 8 implies $w(B') \leq 24$. Thus we have $w(B) = w(B') + w(T) \leq 30$.

Now suppose that $E(R)$ is connected. Then by Lemma 8, we have $w(R) \leq 12$, which implies $w(G) \leq 42$. Hence we may assume that $E(R)$ is not connected, then $R \simeq 2P_3$, $K_2 \cup K_3$, $K_2 \cup K_{1,3}$, or $K_2 \cup P_4$. If $R \simeq 2P_3$ or $K_2 \cup K_3$, then Lemmas 2 and 8 imply $w(R) \leq 12$, hence $w(G) \leq 42$. If $R \simeq K_2 \cup K_{1,3}$, we have $w(R) \leq 15$ by Lemma 3. Let v_1 and v_2 be the vertices of K_2, let v_3 be the center of $K_{1,3}$, and let v_4, v_5, v_6 be leaves of $K_{1,3}$ in R. Then B can be decomposed into two triangles $v_1v_4v_6$, $v_2v_5v_6$ and a cycle $v_1v_3v_2v_4v_5$. Hence by Lemma 4, we obtain $w(B) \leq 27$, which implies $w(G) \leq 42$. If $R \simeq K_2 \cup P_4$, by Lemma 8, we have $w(R) \leq 18$. Since

$B \subseteq K_6 - E(3K_2)$, we have $w(B) \leq 24$ by Lemmas 1 and 7, therefore $w(G) \leq 42$. □

Case 4. $|E(R)| = 5$.

Since $|E(B)| = 10$, there exists a triangle in B, say T. Let $B' = B - E(T)$. Since $|E(B')| = 7$ and B' is a graph obtained by deleting a triangle from B, B' is connected. Hence we have $w(B') \leq 21$ by Lemma 8. Thus $w(B) = w(B') + w(T) \leq 27$. If $w(R) \leq 15$, we are done, so we assume that $w(R) > 15$. Now Lemma 8 implies that one of the component of R is a tree with a perfect matching. Considering $|E(R)| = 5$, we have $w(R) \leq 18$ and $3K_2 \subseteq R$ by Lemma 8. Then $B \subseteq K_6 - E(3K_2)$, hence by Lemmas 1 and 7 we have $w(B) \leq 24$, which implies $w(G) \leq 42$. □

Case 5. $|E(R)| = 6$.

First assume that $E(R)$ is not connected, then $R \simeq K_2 \cup K_4^-$ (K_4^- is the graph obtained from K_4 by deleting just one edge) or $2K_3$. If $R \simeq K_2 \cup K_4^-$, then the fact $K_4^- \subseteq K_4$ and Lemmas 1 and 5 imply that $w(R) \leq 6 + 12 = 18$. Let v_1 and v_2 be vertices of K_2, and v_3, v_4, v_5, v_6 be vertices of K_4^- in R such that $v_3v_4 \notin E(R)$. Then B has a triangle $T = v_1v_3v_4$. Let $B' = B - E(T)$, then $E(B')$ is connected and $|E(B')| = 6$, hence Lemma 8 implies $w(B') \leq 18$. Thus $w(B) \leq 24$, which implies $w(G) \leq 42$. In case of $R \simeq 2K_3$, we have $w(R) \leq 12$. Now $B \simeq K_{3,3}$, hence Lemma 8 implies $w(B) \leq 27$. Therefore we have $w(G) \leq 42$.

In the case where $E(R)$ is connected, by Lemma 8, we have $w(R) \leq 18$. Since $B \not\simeq K_{3,3}$ and $|E(B)| = 9$, there exists a triangle T in B. Let $B' = B - E(T)$, then $|E(B')| = 6$. So if we change B' into R and use the same argument as above, we obtain $w(B') \leq 18$. Hence $w(B) \leq 24$, this implies $w(G) \leq 42$. □

Case 6. $|E(R)| = 7$.

In case of R is not connected, $R \simeq K_2 \cup K_4$. Hence Lemmas 2 and 8 imply $w(R) \leq 18$. And if R is connected, Lemma 8 implies that $w(R) \leq 21$. Now suppose that B has a triangle T and let $B' = B - E(T)$. If $w(B') \leq 15$, we have $w(B) \leq 21$, this implies $w(G) \leq 42$. Hence we may assume that $w(B') > 15$, then Lemma 8 implies that $w(B') \leq 18$ and B' contains $3K_2$. Let F_1, F_2 and F_3 be the components of $3K_2$ in B', then each of them must contain just one vertex of T. Let $F_1 = a_1b_1$, $F_2 = a_2b_2$, $F_3 = a_3b_3$. Without loss of generality, we may assume that $T = a_1a_2a_3$. Let H be a graph such that $V(H) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $E(H) = E(T) \cup E(F_1) \cup E(F_2) \cup E(F_3)$, then $R \subseteq \overline{H}$. Since $\overline{H}$ can be decomposed into three triangles $a_1b_2b_3$, $b_1a_2b_3$ and $b_1b_2a_3$, we have $w(R) \leq 18$ by Lemma 1. Now $w(B) = w(B') + w(T) \leq 18 + 6 = 24$. Hence $w(G) \leq 42$, therefore we may assume that B is triangle-free.

Next, suppose that B has a C_5, say C. Since B is triangle-free, there is no chord in C. Hence the vertex which is not in C must adjacent to three vertices of the C, however this makes triangle in B, a contradiction.

Therefore, we may assume that B is bipartite. It follows from the fact $E(B)$ is connected and Lemma 8 that $w(B) \leq 24$. If $B \subseteq K_{3,3}$, then R can be decomposed into two triangles and a K_2. Hence $w(R) \leq 18$, which implies $w(G) \leq 42$. Otherwise, $B \simeq K_{2,4}$. Then R can be decomposed into a K_4 and a K_2. Hence by Lemma 5 we have $w(R) \leq 18$, which implies $w(G) \leq 42$. This completes the proof of Theorem 2. □

5 Weighted Ramsey number for large graphs

In this section, we observe the relation between the value $WR(3;n)$ and the number of edge-disjoint monochromatic triangles in 2-edge-colored graphs with n vertices, for sufficiently large n. For a 2-edge-coloring c of K_n, we define $N(n,k;c)$ as the maximum number of pairwise edge-disjoint monochromatic complete subgraphs K_k. And let

$$N(n,k) = \min\{N(n,k;c) \mid c \text{ is a 2-edge-coloring of } K_n\}.$$

Proposition 5.

$$WR(3;n) \geq \frac{4}{n^2 - 2N(n,3) + n}.$$

Proof of Proposition 5. Let G be a 2-colored graph with n vertices and set $m = N(n,3)$. As in the proofs of Theorems 1 and 2, we assume $\max\{w_R(T), w_B(T)\} \leq 6$ for every triangle T in G and prove that

$$w(G) \leq 3n^2/2 - 3m + 3n/2.$$

Let $\mathcal{T}$ be a set of edge-disjoint monochromatic triangles of cardinality m, $E(\mathcal{T}) = \bigcup_{T\in\mathcal{T}} E(T)$, R' be the graph induced by $E(R)\setminus E(\mathcal{T})$ and B' be the graph induced by $E(B)\setminus E(\mathcal{T})$. Since both of R' and B' have at most $n/2$ components, using Facts 2 and 3, we can find $(|E(R')| + |E(B')| - l)/2$ pairwise edge-disjoint monochromatic paths of length two in $R' \cup B'$, where $l \leq 2 \cdot n/2 = n$. Let $\mathcal{P}$ be the set of such paths, $E(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} E(P)$, and $I = E(G)\setminus(E(\mathcal{T}) \cup E(\mathcal{P}))$. Then $|I| = l \leq n$ and

$$\begin{aligned}
|\mathcal{P}| &= \frac{|E(G)| - |E(\mathcal{T})| - |I|}{2} \\
&\geq \frac{\frac{n(n-1)}{2} - 3m - l}{2} \\
&\geq \frac{n^2 - 6m - 3n}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
w(G) &= \sum_{T\in\mathcal{T}} w(T) + \sum_{P\in\mathcal{P}} w(P) + \sum_{e\in I} w(e) \\
&\leq 6|\mathcal{T}| + 6|\mathcal{P}| + 6|I| \\
&\leq 6m + 6 \cdot \frac{n^2 - 6m - 3n}{4} + 6n \\
&= \frac{3}{2}n^2 - 3m + \frac{3}{2}n.
\end{aligned}$$

Then,

$$WR(3;n) \geq \frac{6}{w(G)} = \frac{4}{n^2 - 2N(n;3) + n}.$$

□

In [3], concerning the Turán graph $T_2(n)$ and its complement, the following conjecture is given.

Conjecture 1 (Erdős).

$$N(n,3) = \left(\frac{1}{12} + o(1)\right) n^2.$$

If this conjecture is true, then Proposition 5 implies

$$\begin{aligned} WR(3;n) &\geq \frac{4}{n^2 - 2\left(\frac{1}{12} + o(1)\right) n^2} \\ &= \left(\frac{24}{5} + o(1)\right) \frac{1}{n^2}. \end{aligned}$$

The coefficient of n^{-2} in this lower bound is the same as the coefficient of n^{-2} in the upper bound of Proposition 3 for $s = 3$. Considering this, we state the following conjecture.

Conjecture 2.

$$WR(3;n) = \left(\frac{24}{5} + o(1)\right) \frac{1}{n^2}.$$

In fact, the lower bound of $N(n,3)$ is known as follows.

Theorem 3 (Erdős, Faudree, Gould, Jacobson and Lehel [3]).

$$N(n,3) \geq \left(\frac{3}{55} + o(1)\right) n^2.$$

By using Proposition 5, we have

$$\begin{aligned} WR(3;n) &\geq \frac{4}{n^2 - 2\left(\frac{3}{55} + o(1)\right) n^2} \\ &= \left(\frac{220}{49} + o(1)\right) \frac{1}{n^2}. \end{aligned}$$

which improves the lower bound in Proposition 4.

Acknowledgments

This work was partially supported by the JSPS Research Fellowships for Young Scientists (to J.F.) and the 21st Century COE Program; Integrative Mathematical Sciences: Progress in Mathematics Motivated by Social and Natural Sciences.

References

[1] B. Bollobás, *Modern Graph Theory.* Graduate Texts in Mathematics, 184. *Springer-Verlag*, 1998.

[2] J.A. Bondy and Z. Tuza, A weighted generalization of Turán's theorem, *J. Graph Theory*, **25** (1997), 267–275.

[3] P. Erdős, R.J. Faudree, R.J. Gould, M.S. Jacobson and J. Lehel, Edge disjoint monochromatic triangles in 2-colored graphs, *Discrete Math.* **231** (2001), 135–141.

[4] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sós and E. Szemerédi, Turán-Ramsey theorems and simple asymptotically extremal structures, *Combinatorica* **13** (1993), no. 1, 31–56.

[5] Z. Füredi and A. Kündgen, Turán problems for integer-weighted graphs, *J. Graph Theory*, **40** (2002), 195–225.

[6] M. Las Vergnas, A note on matchings in graphs, *Cahiers Centre Etudes Recherche Oper.*, **17** (1975), 257–260.

[7] D.P. Sumner, Graphs with 1-factors, *Proc. Amer. Math. Soc.*, **42** (1974), 8–12.

(Received 17 Sep 2004; revised 6 Oct 2005)