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On the spectrum of the quasi-Laplacian matrix of a graph*

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Abstract

Let G be a simple graph and let $A(G)$ and $D(G)$ be the adjacency matrix and degree matrix of G respectively. We call $Q(G) = A(G) + D(G)$ the quasi-Laplacian matrix of G. In this note, we study the spectrum of $Q(G)$ and give the upper and lower bounds of the k-th largest eigenvalue of $Q(G)$.

1 Introduction and notation

Let G be an undirected simple graph with vertices $\{v_1, v_2, \cdots, v_n\}$ and edges $\{e_1, e_2, \cdots, e_m\}$; we call G an (n, m)-graph. The adjacency matrix $A = A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge. Since G has no loops, the main diagonal of A contains only 0's. Suppose the valence or *degree* of vertex v_i equals d_i for $i = 1, 2, \cdots, n$, and let $D = D(G)$ be the diagonal matrix whose (i, i)-entry is d_i. Set

$$L(G) = D(G) - A(G) \quad \text{and} \quad Q(G) = D(G) + A(G).$$

The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. With its great important applications in graph theory, the Laplacian matrix has been studied extensively in the literature. For convenience, we call $Q(G)$ *the quasi-Laplacian matrix* of G. It is well-known that graph spectra have great importance in many fields. Several graph spectra have been defined in [1], such as the characteristic polynomial of $A(G)$,

$$P_G(\lambda) = |\lambda I - A|,$$

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and its spectrum

$$Sp_P(G) = [\lambda_1, \lambda_2, \cdots, \lambda_n]_P;$$

the characteristic polynomial of $L(G)$,

$$L_G = |\lambda I - L|,$$

and its spectrum

$$Sp_L(G) = [\lambda_1, \lambda_2, \cdots, \lambda_n]_L;$$

the characteristic polynomial of $Q(G)$,

$$Q(G) = |\lambda I - Q|,$$

with its corresponding spectrum

$$Sp_Q(G) = [\lambda_1, \lambda_2, \cdots, \lambda_n]_Q.$$

The spectra of $A(G)$ and $L(G)$ are well studied, but the spectrum of $Q(G)$ seems to be less well known. It is not until recently that some authors found that the spectrum of $Q(G)$ has strong connections with the structures of graphs (see [2], [3]). It is easy to see that $Q(G)$ is a symmetric, positive semi-definite matrix, so we can set that $0 \leq \lambda_n \leq \lambda_{n-1} \cdots \lambda_2 \leq \lambda_1$. Another two important properties of $Q(G)$ (see [3]) are stated in the following Lemmas 1.1 and 1.2.

We introduce some definitions first. If a component of G does not contain an odd cycle, we call it a *bipartite component* of G. Let $\omega = \omega(G)$, $\omega_0 = \omega_0(G)$ denote the number of components, bipartite components of G, respectively. For convenience, let us call a connected graph containing exactly one cycle, with that cycle having odd length, an *odd unicyclic* graph. (The graph may contain other edges and vertices as well, as long as they do not create another cycle or another component.) Thus, an odd unicyclic graph consists of an odd cycle together with (possibly trivial) trees growing out of each vertex in the cycle.

We call a spanning subgraph S of a connected graph G an *essential spanning subgraph* of G if either of the following conditions is true:

(i) G is a bipartite and S is a spanning tree of G;

(ii) G is not bipartite, $V(S) = V(G)$, and every component of S is an odd unicyclic graph.

Lemma 1.1 *[3] The rank of $Q(G)$ is $n - \omega_0$, where ω_0 is the number of bipartite components of G.*

Lemma 1.2 *[3] If G has no bipartite components, then*

$$\det\ Q = \sum_S 4^{\omega(S)},$$

where the sum is taken over all essential spanning subgraphs of G.

In this note, we are concerned mainly with the bound of the k-th largest eigenvalue of $Q(G)$. However, we want to say more about λ_1 and λ_n. Now $\lambda_1(Q)$ has a natural upper bound $\lambda_1 \leq 2\Delta$, where Δ is the largest vertex degree of G. However, $\lambda_1(Q)$ has a more precise upper bound which is similar to $\lambda_1(L)$ given in [5].

Proposition 1.1 *Let G be a graph of order n. Then*

(i) $\lambda_1(L) \leq \lambda_1(Q)$;

(ii) $\lambda_1(Q) \leq \max\{d_i + d_j : v_i v_j \in E(G)\}$.

If G is connected, then equality in (ii) holds if and only if G is a bipartite graph and the degree is constant on each class of vertices.

Proof. (i) Let $L(G) = [l_{ij}]$, and denote by $|L|$ the matrix obtained by replacing l_{ij} by its absolute value $|l_{ij}|$ $(1 \leq i \leq j \leq n)$. Then $Q(G) = |L(Q)|$. So, $\lambda_1(L) \leq \lambda_1(Q)$.

(ii) Let M be the *vertex-edge incidence* of G. It is well-known that $Q = MM^T$. Let $N = M^T M$. It is easy to show that the two matrices Q and N share common non-zero eigenvalues, therefore, $\lambda_1(Q) = \rho(N)$; here $\rho(N)$ is the spectral radius of N. Since the spectrum of G is the union of the spectra of its components, we only need to consider connected graphs. Then N is irreducible, by the classic theory of nonnegative matrices, $\rho(N) \leq$ maximum row sum of N. If $e = v_i v_j \in E(G)$, then the row sum in the row corresponding to e is $d_i + d_j$. Thus we complete the proof of the inequality.

If G is bipartite, then $\rho(N) =$ max row sum if and only if all row sums are equal; i.e. if and only if the condition of the proposition holds. Equivalently, equality holds if and only if the line graph of G is regular. □

With λ_n, it does not always equal to 0, which is different from λ_n of $L(G)$. As the next proposition shows, the value of $\lambda_n(Q)$ has a connection with the bipartite component of G.

Proposition 1.2 $\lambda_n(Q) = 0$ *if and only if* $\omega_0(G) > 0$.

Proof. $\lambda_n = 0 \Leftrightarrow \det\ (0I - Q) = \det\ Q = 0 \Leftrightarrow$ *rank* $Q < n \Leftrightarrow n - \omega_0 < n \Leftrightarrow \omega_0 > 0$. □

2 The k-th largest eigenvalue of $Q(G)$

In this section, we discuss the upper and lower bounds for the k-th largest eigenvalue λ_k of $Q(G)$.

Lemma 2.1 *[4] Let G be a (n, m)-graph with degree sequence $(d_1, d_2, \cdots, d_n)$. Then*

$$\sum_{i=1}^{n} {d_i}^2 \leq \frac{nm^2}{n-1},$$

with equality if and only if G is a star graph $K_{1,n-1}$.

Theorem 2.1 *Let G be a (n,m)-graph. Write*

$$M(G) = \min\{2m(n^2-2m), m[\frac{(n-2)^2}{n-1}m+2n)]\}.$$

Then for $k = 1, 2, \cdots, n$,

$$\lambda_k \leq \frac{1}{n}\left\{2m + \sqrt{\frac{n-k}{k}\cdot M(G)}\right\}, \qquad (1)$$

with the equality in (1) holding for some $1 \leq k_0 \leq n$ if and only if G is K_n or $K_{1,n-1}$.

Proof.

$$Tr(Q^2) = \sum_{i=1}^{k}{\lambda_i}^2 + \sum_{i=k+1}^{n}{\lambda_i}^2 \geq \frac{1}{k}(\sum_{i=1}^{k}\lambda_i)^2 + \frac{1}{n-k}(\sum_{i=k+1}^{n}\lambda_i)^2.$$

Let $N_k = \sum_{i=1}^{k}\lambda_i$; then $Tr(Q^2) \geq \frac{{N_k}^2}{k} + \frac{(2m-N_k)^2}{n-k}$. Therefore,

$$k\lambda_k \leq N_k \leq \frac{1}{n}\left\{2km + \sqrt{k(n-k)[nTr(Q^2)-4m^2]}\right\}.$$

Hence,

$$\lambda_k \leq \frac{1}{n}\left\{2m + \sqrt{\frac{n-k}{k}[nTr(Q^2)-4m^2]}\right\}.$$

Moreover, by Lemma 2.1, we have

$$\begin{aligned} nTr(Q^2) - 4m^2 &= n\sum_{i=1}^{n}{d_i}^2 + n\sum_{i=1}^{n}d_i - 4m^2 \\ &\leq n\cdot\frac{n}{n-1}\cdot m^2 + 2mn - 4m^2 \\ &= [\frac{(n-2)^2}{n-1}\cdot m + 2n]m. \end{aligned}$$

On the other hand, since $d_i \leq n-1$ for each $1 \leq i \leq n$,

$$\begin{aligned} nTr(Q^2) - 4m^2 &= n\sum_{i=1}^{n}{d_i}^2 + n\sum_{i=1}^{n}d_i - 4m^2 \\ &\leq n(n-1)\sum_{i=1}^{n}d_i + n\sum_{i=1}^{n}d_i - 4m^2 \\ &= 2mn^2 - 4m^2 \\ &= 2m(n^2-2m). \end{aligned}$$

Hence, $nTr(Q^2) - 4m^2 \leq M(G)$. So the result follows.

When $G = K_n$, the equality in (1) holds for $k = 1$. If the equality in (1) holds for some $1 \leq k_0 \leq n$, then $nTr(Q^2) - 4m^2 = M(G)$. Hence $\sum_{i=1}^{n} {d_i}^2 = \frac{nm^2}{n-1}$ or $\sum_{i=1}^{n} {d_i}^2 = (n-1)\sum_{i=1}^{n} d_i$. Therefore, in the former case, $G = K_{1,n-1}$ by Lemma 2.1; in the latter case, $G = K_n$ by $d_i = n-1$ for each i. □

For a regular graph, we have

Corollary 2.1 *Let G be a d-regular graph of order n. Then for $k = 1, 2, \cdots, n$,*

$$\lambda_k \leq \frac{1}{n}\left\{2m + \sqrt{\frac{n-k}{k} nd(n-d-1)}\right\}. \tag{2}$$

If G has no bipartite components, then we can obtain the lower bound of λ_k.

Theorem 2.2 *Let G be a graph of order n, with no bipartite components in it. Then for $k = 2, \cdots, n-1$,*

$$\lambda_k \geq \frac{1}{n-k}\left\{(n-1)(2^{n-k}\sum_{S} 4^{\omega(S)})^{\frac{1}{n-1}} - 2m\right\}. \tag{3}$$

Proof.

Det $Q = (\lambda_1 \cdots \lambda_{k-1})(\lambda_k \cdots \lambda_n) \leq (\frac{\lambda_1 + \cdots + \lambda_{k-1}}{k-1})^{k-1}(\frac{\lambda_k + \cdots + \lambda_{n-1}}{n-k})^{n-k}$.

Letting $M_k = \frac{\lambda_k + \cdots + \lambda_{n-1}}{n-k}$, then $M_k \leq \lambda_k$ and

$$\det\ Q \leq (\frac{2m-(n-k)M_k}{k-1})^{k-1} \cdot {M_k}^{n-k}.$$

Note that, if $a > 0, b > 0$ *and* $0 < p < 1$, then $a^p b^{1-p} \leq pa + (1-p)b$ with equality holding if and only if $a = b$. Hence,

$$\begin{aligned}
(2^{n-k}\ \det\ Q)^{\frac{1}{n-1}} &\leq (\frac{2m-(n-k)M_k}{k-1})^{\frac{k-1}{n-1}}(2M_k)^{\frac{n-k}{n-1}} \\
&\leq \frac{k-1}{n-1} \cdot \frac{2m-(n-k)M_k}{k-1} + \frac{n-k}{n-1} \cdot 2M_k \\
&= \frac{2m+(n-k)M_k}{n-1}.
\end{aligned}$$

By Lemma 1.2, we obtain,

$$\begin{aligned}
\lambda_k \geq M_k &\geq \frac{1}{n-k}\left\{(n-1)(2^{n-k} \cdot \det\ Q)^{\frac{1}{n-1}} - 2m\right\} \\
&= \frac{1}{n-k}\left\{(n-1)(2^{n-k} \cdot \sum_{S} 4^{\omega(S)})^{\frac{1}{n-1}} - 2m\right\}.
\end{aligned}$$

□

Theorem 2.3 *Let G be a graph of order n,with no bipartite components in it. Then for* $k = 2, \cdots, n$,

$$\lambda_k \geq \left\{ \sum_S 4^{\omega(S)} (\frac{k-1}{2m})^{k-1} \right\}^{\frac{1}{n-k+1}}. \quad (4)$$

Proof. Since,

$$\begin{aligned} \det\ Q = (\lambda_1 \cdots \lambda_{k-1})(\lambda_k \cdots \lambda_n) &\leq (\frac{\lambda_1 + \cdots + \lambda_{k-1}}{k-1})^{k-1} \lambda_k^{n-k+1} \\ &\leq (\frac{\lambda_1 + \cdots + \lambda_n}{k-1})^{k-1} \cdot \lambda_k^{n-k+1} \\ &= (\frac{2m}{k-1})^{k-1} \cdot \lambda_k^{n-k+1}. \end{aligned}$$

recalling Lemma 1.2, the result follows immediately. □

Example 2.1 *Let G be a* $(4,5)$*-graph as in Fig. 1.*

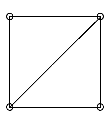

Fig. 1

$$Q(G) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

It is easy to see that $\sum_S 4^{\omega(S)} = 16$, by a simple computation, $Sp_Q(G) = [\ 3 + \sqrt{5},\ 2,\ 2,\ 3 - \sqrt{5}\ ]$. The following table shows the bounds of eigenvalues of $Q(G)$ by the above theorems.

	$\lambda_1 \approx 5.236$	$\lambda_2 = 2$	$\lambda_3 = 2$	$\lambda_4 \approx 0.764$
$Th.2.1$	$\lambda_1 \leq 5.854$	$\lambda_2 \leq 4.436$	$\lambda_3 \leq 3.618$	$\lambda_4 \leq 2.5$
$Th.2.2$		$\lambda_2 \geq 1$	$\lambda_3 \geq -0.475$	
$Th.2.3$		$\lambda_2 \geq 1.17$	$\lambda_3 \geq 0.8$	$\lambda_4 \geq 0.432$

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References

[1] D.M. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[2] J. van den Heuvel, Hamilton Cycles and Eigenvalues of Graphs, *Linear Algebra Appl.* 226/228 (1995), 723–730.

[3] Jerrold W. Grossman, D.M. Kulkarni and I.E. Schochetman, Algebraic Graph Theory Without Orientation, *Linear Algebra Appl.* 212/213 (1994), 289–307.

[4] X.D. Zhang and J.S. Li, On the k-th Largest Eigenvalue of the Laplacian Matrix of a Graph, *Acta Mathematicae Applicatae Sinica* 17 (2001), 183–190.

[5] W.N. Anderson and T.D. Morley, Eigenvalues of the Laplacian of a graph, *Linear and Multilinear Algebra* 18 (1985), 141–145.

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