AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **33** (2005), Pages 77–86

Constructions of 3-resolvable nested 3-designs and 3-wise balanced designs

Sarita Rudra   Shakti Banerjee

Devi Ahilya University
Indore-452001
India
shaktibn@yahoo.com

Sanpei Kageyama

Hiroshima University
Higashi-Hiroshima 739-8524
Japan
ksanpei@hiroshima-u.ac.jp

Abstract

Methods of constructing 3-resolvable nested 3-designs and nested 3-wise balanced designs from an affine resolvable 3-design are proposed with illustrations.

1 Introduction

Considering the nested structure, Preece [5] introduced a class of nested balanced incomplete block (BIB) designs in which there are two systems of blocks, the second system nested within the first. Each block from the first system is called a super-block and contains some blocks called sub-blocks from the second. We extend the work in Preece [5] by introducing some new nested t-designs.

A t-(v, k, λ_t) design (or simply a t-design) is a system with v points (or treatments) and b blocks containing k distinct points, each point appearing in r different blocks and every set of t distinct points appearing in exactly λ_t blocks. Then it follows (cf. Hedayat and Kageyama [3], Raghavarao [6]) that

$$\lambda_t \binom{v}{t} = b \binom{k}{t} \tag{1.1}$$

and for each $0 \leq s \leq t$ every t-(v, k, λ_t) design is an s-(v, k, λ_s) design with

$$\lambda_s = \lambda_t \binom{v-s}{t-s} / \binom{k-s}{t-s}.$$

Here $\lambda_1 = r$ and $\lambda_0 = b$. When $t = 2$, the design is called a BIB design with parameters v, b, r, k, λ $(= \lambda_2)$.

A nested t-design with parameters v, b, r, k, λ_t, $b_*, k_*, \lambda_{*t}, m$ is a block design for v points, each replicated r times with two systems of blocks such that:

(a) the second system is nested within the first with each block (super-block) from the first system containing exactly m blocks (sub-blocks) from the second system, i.e., $k = mk_*$ and $b_* = mb$;

(b) ignoring the second system leaves a t-design with b blocks each of k points and with λ_t occurrences for any t-tuple; and

(c) ignoring the first system leaves a t-design with b_* blocks each of k_* points and with λ_{*t} concurrences for any t-tuple.

(See also Dey [2].)

In a nested t-design it holds that $vr = bk = b_*k_*$ and $\lambda_{*t}\binom{v}{t} = b_*\binom{k_*}{t}$. Hence it follows that

$$\lambda_t = \lambda_{*t}\Big(m + \frac{m-1}{k_*-1}\Big)\Big(m + \frac{2(m-1)}{k_*-2}\Big)\cdots\Big(m + \frac{(t-1)(m-1)}{k_*-t+1}\Big).$$

Note that in a 3-design if $v = 2k$, then $r = 3\lambda_2 - 2\lambda_3$.

A nested block design is said to be α-resolvable if its super-blocks (and then sub-blocks) can be grouped into several resolution sets (also here called α-resolution sets) of some blocks such that every point appears in each resolution set precisely α times. A 1-resolvable design is simply called a resolvable design. An α-resolvable block design with ℓ resolution sets of β blocks each $(b = \ell\beta)$ is said to be affine α-resolvable if any pair of blocks belonging to the same resolution set contains q_1 points in common, whereas any pair of blocks belonging to different resolution sets contains q_2 points in common (Shrikhande and Ragharavao [8]). Note that $q_2 = \alpha k/\beta = k^2/v$ and in an affine α-resolvable BIB design $q_1 = k(\alpha-1)/(\beta-1) = k + \lambda - r$.

A block design is said to be t-wise balanced if v points are arranged in b blocks such that each block contains $k_1, k_2, \ldots,$ or k_p points that are all distinct $(k_i \le v, k_i \ne k_{i'})$ and every t-tuple of distinct points occurs in exactly λ_t blocks of the design. If b_i is the number of blocks of size k_i, $i = 1, 2, \ldots, p$, then for a t-wise balanced design

$$b = \sum_{i=1}^{p} b_i, \quad \lambda_t\binom{v}{t} = \sum_{i=1}^{p} b_i\binom{k_i}{t}.$$

This paper presents general methods of constructing 3-resolvable nested 3-designs and 3-resolvable nested 3-wise balanced designs from an affine resolvable 3-design. Throughout the paper, $v^*, b^*, r^*, k^*, \lambda_2^*, \lambda_3^*$ represent the parameters of the super-block structure, whereas $v_*, b_*, r_*, k_*, \lambda_{*2}, \lambda_{*3}$ represent the parameters of the sub-block structure.

The following lemmas are from Kageyama [4].

Lemma 1.1. *There does not exist an affine α-resolvable t-design for $\alpha \geq 2$ and $t \geq 3$.*

Lemma 1.2. *There does not exist an affine resolvable t-design for $t \geq 4$.*

Lemma 1.3. *Any affine resolvable 3-(v, k, λ_3) design is a 3-$(4\lambda_3 + 4, 2\lambda_3 + 2, \lambda_3)$ design.*

Because of Lemmas 1.1 to 1.3, we consider only affine resolvable 3-$(4\lambda_3 + 4, 2\lambda_3 + 2, \lambda_3)$ designs as a basic design in the paper.

2 Methods

Banerjee and Kageyama [1], using pairs of blocks belonging to different resolution sets of an affine resolvable BIB design, constructed an α-resolvable nested BIB design. A similar discussion is made to construct resolvable nested 3-designs by Rai, Banerjee and Kageyama [7]. However, the present approach cannot show the resolvability but a 3-resolvability can be obtained. This property is entirely different from the result of [7].

In this paper, we shall present two construction procedures:
(i) In an affine resolvable 3-design, consider any pair of blocks belonging to different resolution sets in the partition due to the affine resolvability. Consider the k^2/v $(= q_2)$ common points in these two blocks and consequently the remaining $k - q_2$ points in each of the two blocks and taking them as three sub-blocks nested in a super-block of size $2k - q_2$, we can obtain a nested 3-design (see Theorem 2.1 later).
(ii) On the same line, considering the k^2/v $(= q_2)$ common points and $2(k - q_2)$ distinct points from the two blocks of different resolution sets, as two sub-blocks nested in a super-block of size $2k - q_2$, we can obtain a nested 3-wise balanced design (see Theorem 2.2 later).

Theorem 2.1. *The existence of an affine resolvable 3-$(4\lambda_3 + 4, 2\lambda_3 + 2, \lambda_3)$ design with $\lambda_3 \geq 2$ implies the existence of a 3-resolvable nested 3-design with parameters $v^* = 4(\lambda_3 + 1), b^* = 4(2\lambda_3 + 1)(4\lambda_3 + 3)$, $r^* = 3(2\lambda_3 + 1)(4\lambda_3 + 3)$, $k^* = 3(\lambda_3 + 1)$, $\lambda_2^* = 3(2\lambda_3+1)(3\lambda_3+2)$, $\lambda_3^* = 3[\lambda_3(\lambda_3-1)/2+(\lambda_3+1)(4\lambda_3+1)]$, $b_* = 3b^*, k_* = \lambda_3+1$, $\lambda_{*2} = 3\lambda_3(2\lambda_3 + 1)$, $\lambda_{*3} = 3\lambda_3(\lambda_3 - 1)/2$.*

Proof. In the affine resolvable 3-$(4\lambda_3+4, 2\lambda_3+2, \lambda_3)$ design, any two blocks belonging to different resolution sets of two blocks each have $\lambda_3 + 1$ points in common, whereas any two blocks belonging to the same resolution set are disjoint.

Now take any two blocks belonging to different resolution sets and consider the $\lambda_3 + 1$ common points and the remaining $[(2\lambda_3 + 2) - (\lambda_3 + 1)]$, i.e., $(\lambda_3 + 1)$ points in each of the two blocks as three sub-blocks nested in a super-block of size $3(\lambda_3 + 1)$. Hence $k^* = 3(\lambda_3 + 1)$. Each pair of resolution sets contributes 4 blocks and there are

$4\lambda_3+3$ such resolution sets. Thus $b^* = 4\binom{4\lambda_3+3}{2} = 4(2\lambda_3+1)(4\lambda_3+3)$.

Take a pair of resolution sets among the $4\lambda_3+3$ resolution sets. Since there is a single block in each resolution set containing a particular point θ, the present procedure gives 3 blocks which will contain θ. Then $r^* = 3\binom{4\lambda_3+3}{2} = 3(2\lambda_3+1)(4\lambda_3+3)$.

Next, to obtain the value λ_t^* where $t = 2, 3$, we can partition the $4\lambda_3+3$ resolution sets into four types, (A), (B), (C), (D), containing three points, say, θ, ψ, δ, as follows.

$$\begin{array}{cccc} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{21} & \mathcal{A}_{22} \end{array}$$

$$\text{(A)} \quad \begin{pmatrix} \theta & \theta' \\ \psi & \psi' \\ \delta & \delta' \end{pmatrix} ; \begin{pmatrix} \theta & \theta' \\ \psi & \psi' \\ \delta & \delta' \end{pmatrix}$$

occurs λ_3 times, where $\{\theta', \psi', \delta'\}$ are different points from $\{\theta, \psi, \delta\}$, respectively;

$$\begin{array}{cccc} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{21} & \mathcal{B}_{22} \end{array}$$

$$\text{(B)} \quad \begin{pmatrix} \theta & \delta \\ \psi & \end{pmatrix} ; \begin{pmatrix} \theta & \delta \\ \psi & \end{pmatrix}$$

occurs $\lambda_2 - \lambda_3$ $(= \lambda_3 + 1)$ times;

$$\begin{array}{cccc} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{21} & \mathcal{C}_{22} \end{array}$$

$$\text{(C)} \quad \begin{pmatrix} \theta & \psi \\ \delta & \end{pmatrix} ; \begin{pmatrix} \theta & \psi \\ \delta & \end{pmatrix}$$

occurs $\lambda_3 + 1$ times;

$$\begin{array}{cccc} \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}$$

$$\text{(D)} \quad \begin{pmatrix} \psi & \theta \\ \delta & \end{pmatrix} ; \begin{pmatrix} \psi & \theta \\ \delta & \end{pmatrix}$$

occurs $\lambda_3 + 1$ times, where $\lambda_2 = 2\lambda_3 + 1$.

Under (A), any two resolution sets give three blocks which contain $\{\theta, \psi, \delta\}$ and $\binom{\lambda_3}{2}$ such pairs of combinations can be made, while in each of (B), (C) and (D) any two resolution sets give two blocks which contain $\{\theta, \psi, \delta\}$ and we can make $\binom{\lambda_3+1}{2}$ such pairs of combinations. Any combination of resolution sets between (A) and each of (B), (C) and (D) give two blocks containing $\{\theta, \psi, \delta\}$ and hence there can be $3 \times 2\lambda_3(\lambda_3+1)$ such combinations. Any two resolution sets from (B), (C) and (D), respectively, give one block containing $\{\theta, \psi, \delta\}$ and $(\lambda_3+1)^2$ such combinations are possible. Thus $\lambda_3^* = 3\binom{\lambda_3}{2} + 6\binom{\lambda_3+1}{2} + 6\lambda_3(\lambda_3+1) + 3(\lambda_3+1)^2 = 3[\lambda_3(\lambda_3-1)/2 + (\lambda_3+1)(4\lambda_3+1)]$.

To calculate λ_2^*, i.e., take a pair $\{\theta, \psi\}$ to consider the following:

(1) Any two resolution sets from (A). This family gives the values for $\mathcal{A}_{11}\mathcal{A}_{21}$, $\mathcal{A}_{11}\mathcal{A}_{22}$ and $\mathcal{A}_{12}\mathcal{A}_{21}$ combination, where $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{21}$, $\mathcal{A}_{22}$ denote blocks of resolution sets of A.

(2) Any two resolution sets from (B). This family gives the values for $\mathcal{B}_{11}\mathcal{B}_{21}$, $\mathcal{B}_{11}\mathcal{B}_{22}$ and $\mathcal{B}_{12}\mathcal{B}_{21}$ combination, where $\mathcal{B}_{11}$, $\mathcal{B}_{12}$, $\mathcal{B}_{21}$, $\mathcal{B}_{22}$ denote blocks of resolution sets of B.

(3) Any two resolution sets from (C). This family gives the values for $\mathcal{C}_{11}\mathcal{C}_{22}$ and $\mathcal{C}_{12}\mathcal{C}_{21}$ combination, where $\mathcal{C}_{11}$, $\mathcal{C}_{12}$, $\mathcal{C}_{21}$, $\mathcal{C}_{22}$ denote blocks of resolution sets of C.

(4) Any two resolution sets from (D). This family gives the values for $\mathcal{D}_{11}\mathcal{D}_{22}$ and $\mathcal{D}_{12}\mathcal{D}_{21}$ combination, where $\mathcal{D}_{11}$, $\mathcal{D}_{12}$, $\mathcal{D}_{21}$, $\mathcal{D}_{22}$ denote blocks of resolution sets of D.

(5) Any two resolution sets between (A) and (B). This family gives the values for $\mathcal{A}_{11}\mathcal{B}_{11}$, $\mathcal{A}_{11}\mathcal{B}_{12}$ and $\mathcal{A}_{21}\mathcal{B}_{11}$ combination, where $\mathcal{A}_{11}$, $\mathcal{A}_{21}$, $\mathcal{B}_{11}$, $\mathcal{B}_{12}$ denote blocks of A and B, respectively.

(6) Any two resolution sets between (A) and (C). This family gives the values for $\mathcal{A}_{11}\mathcal{C}_{11}$ and $\mathcal{A}_{11}\mathcal{C}_{12}$ combination, where $\mathcal{A}_{11}$, $\mathcal{C}_{11}$, $\mathcal{C}_{12}$ denote blocks of A and C, respectively.

(7) Any two resolution sets between (A) and (D). This family gives the values for $\mathcal{A}_{11}\mathcal{D}_{11}$ and $\mathcal{A}_{11}\mathcal{D}_{12}$ combination, where $\mathcal{A}_{11}$, $\mathcal{D}_{11}$, $\mathcal{D}_{12}$ denote blocks of A and D, respectively.

(8) Any two resolution sets between (B) and (C). This family gives the values for $\mathcal{B}_{11}\mathcal{C}_{11}$ and $\mathcal{B}_{11}\mathcal{C}_{12}$ combination, where $\mathcal{B}_{11}$, $\mathcal{C}_{11}$, $\mathcal{C}_{12}$ denote blocks of B and C, respectively.

(9) Any two resolution sets between (B) and (D). This family gives the values for $\mathcal{B}_{11}\mathcal{D}_{11}$ and $\mathcal{B}_{11}\mathcal{D}_{12}$ combination, where $\mathcal{B}_{11}$, $\mathcal{D}_{11}$, $\mathcal{D}_{12}$ denote blocks of B and D, respectively.

(10) Any two resolution sets between (C) and (D). This family gives the values for $\mathcal{C}_{11}\mathcal{D}_{11}$ and $\mathcal{C}_{12}\mathcal{D}_{12}$ combination, where $\mathcal{C}_{11}$, $\mathcal{D}_{11}$, $\mathcal{D}_{12}$ denote blocks of C and D, respectively.

This yields that $\lambda_2^* = 3\binom{\lambda_3}{2} + 7\binom{\lambda_3+1}{2} + 7\lambda_3(\lambda_3+1)^2 + 6(\lambda_3+1)^2 = 3(2\lambda_3+1)(3\lambda_3+2)$.

Now, if the sub-blocks of size λ_3+1, nested in a super-block of size $3(\lambda_3+1)$, are considered, a sub-structure with $3b^*$ blocks is obtained. Thus $b_* = 3b^*$. In the design of sub-blocks any two resolution sets among (A) only contribute to λ_{*3}. Thus $\lambda_{*3} = 3\binom{\lambda_3}{2}$. Other than λ_3^* blocks, where the triplet $\{\theta, \psi, \delta\}$ occurs, any two resolution sets from (B), (C) and (D) contribute three blocks each for the pair $\{\theta, \psi\}$. Thus $\lambda_{*2} = 3\binom{\lambda_3}{2} + 9\binom{\lambda_3+1}{2} = 3\lambda_3(2\lambda_3+1)$. We next show that the resulting design is 3-resolvable. The present procedure gives $\binom{4\lambda_3+3}{2}$ resolution sets containing four blocks each such that in the partition any of the v^* points are replicated three times. The nested structure also preserves a 3-resolvability. This completes the proof. □

Example 2.1. Consider an affine resolvable 3-design with parameters $v = 12, b = 22, r = 11, k = 6$, $\lambda_2 = 5$, $\lambda_3 = 2$, whose blocks are given by {(1, 3, 4, 5, 9, 11), (2, 6, 7, 8, 10, 12)}, {(2, 4, 5, 6, 10, 1), (3, 7, 8, 9, 11, 12)}, {(3, 5, 6, 7, 11, 2), (4, 8, 9, 10, 1, 12)}, {(4, 6, 7, 8, 1, 3), (5, 9, 10, 11, 2, 12)}, {(5, 7, 8, 9, 2, 4), (6, 10, 11, 1, 3, 12)}, {(6, 8, 9, 10, 3, 5), (7, 11, 1, 2, 4, 12)}, {(7, 9, 10, 11, 4, 6), (8, 1, 2, 3, 5, 12)}, {(8, 10, 11, 1, 5, 7), (9, 2, 3, 4, 6, 12)}, {(9, 11, 1, 2, 6, 8), (10, 3, 4, 5, 7, 12)}, {(10, 1, 2, 3, 7, 9), (11, 4, 5, 6, 8, 12)}, {(11, 2, 3, 4, 8, 10), (1, 5, 6, 7, 9, 12)}. Then Theorem 2.1 yields a 3-resolvable nested 3-design with parameters $v^* = 12, b^* = 220$, $r^* = 165$, $k^* = 9$, $\lambda_2^* = 120$, $\lambda_3^* = 84$, $b_* = 660, k_* = 3$, $\lambda_{*2} = 30$, $\lambda_{*3} = 3$. The nested structure is given as follows, where ( ) and [ ] show sub-blocks and super-blocks, respectively, and then 12 sub-blocks in each brace form a 3-resolution set in which each point appears three times, i.e.,

{[(3, 9, 11), (1, 4, 5), (2, 6, 10)], [(1, 4, 5), (3, 9, 11), (7, 8, 12)], [(7, 8, 12), (2, 6, 10), (1, 4, 5)], [(2, 6, 10), (3, 9, 11), (7, 8, 12)]},

{[(1, 4, 9), (3, 5, 11), (2, 6, 7)], [(3, 5, 11), (1, 4, 9), (8, 10, 12)], [(8, 10, 12), (3, 5, 11), (2, 6, 7)], [(2, 6, 7), (1, 4, 9), (8, 10, 12)]},

{[(5, 9, 11), (1, 3, 4), (6, 7, 8)], [(1, 3, 4), (5, 9, 11), (2, 10, 12)], [(2, 10, 12), (1, 3, 4), (6, 7, 8)], [(6, 7, 8), (5, 9, 11), (2, 10, 12)]},

{[(1, 3, 11), (4, 5, 9), (2, 7, 8)], [(4, 5, 9), (1, 3, 11), (6, 10, 12)], [(6, 10, 12), (4, 5, 9), (2, 7, 8)], [(2, 7, 8), (1, 3, 11), (6, 10, 12)]},

{[(1, 4, 11), (3, 5, 9), (6, 8, 10)], [(3, 5, 9), (1, 4, 11), (2, 7, 12)], [(2, 7, 12), (3, 5, 9), (6, 8, 10)], [(6, 8, 10), (1, 4, 11), (2, 7, 12)]},

{[(1, 3, 5), (4, 9, 11), (6, 7, 10)], [(4, 9, 11), (1, 3, 5), (2, 8, 12)], [(2, 8, 12), (6, 7, 10), (4, 9, 11)], [(6, 7, 10), (1, 3, 5), (2, 8, 12)]},

{[(3, 4, 9), (1, 5, 11), (7, 8, 10)], [(1, 5, 11), (3, 4, 9), (2, 6, 12)], [(2, 6, 12), (1, 5, 11), (7, 8, 10)], [(7, 8, 10), (3, 4, 9), (2, 6, 12)]},

{[(3, 4, 5), (1, 9, 11), (2, 6, 8)], [(1, 9, 11), (3, 4, 5), (7, 10, 12)], [(7, 10, 12), (1, 9, 11), (2, 6, 8)], [(2, 6, 8), (3, 4, 5), (7, 10, 12)]},

{[(4, 5, 11), (1, 3, 9), (2, 7, 10)], [(1, 3, 9), (4, 5, 11), (6, 8, 12)], [(6, 8, 12), (1, 3, 9), (2, 7, 10)], [(2, 7, 10), (4, 5, 11), (6, 8, 12)]},

{[(1, 5, 9), (3, 4, 11), (2, 8, 10)], [(3, 4, 11), (1, 5, 9), (6, 7, 12)], [(6, 7, 12), (3, 4, 11), (2, 8, 10)], [(2, 8, 10), (1, 5, 9), (6, 7, 12)]},

{[(1, 4, 10), (2, 5, 6), (3, 7, 11)], [(2, 5, 6), (1, 4, 10), (8, 9, 12)], [(8, 9, 12), (2, 5, 6), (3, 7, 11)], [(3, 7, 11), (1, 4, 10), (8, 9, 12)]},

{[(2, 5, 10), (1, 4, 6), (3, 7, 8)], [(1, 4, 6), (2, 5, 10), (9, 11, 12)], [(9, 11, 12), (1, 4, 6), (3, 7, 8)], [(3, 7, 8), (2, 5, 10), (9, 11, 12)]},

{[(1, 6, 10), (2, 4, 5), (7, 8, 9)], [(2, 4, 5), (1, 6, 10), (3, 11, 12)], [(3, 11, 12), (2, 4, 5), (7, 8, 9)], [(7, 8, 9), (1, 6, 10), (3, 11, 12)]},

{[(1, 2, 4), (5, 6, 10), (3, 8, 9)], [(5, 6, 10), (1, 2, 4), (7, 11, 12)], [(7, 11, 12), (5, 6, 10), (3, 8, 9)], [(3, 8, 9), (1, 2, 4), (7, 11, 12)]},

{[(1, 2, 5), (4, 6, 10), (7, 9, 11)], [(4, 6, 10), (1, 2, 5), (3, 8, 12)], [(3, 8, 12), (4, 6, 10), (7, 9, 11)], [(7, 9, 11), (1, 2, 5), (3, 8, 12)]},

{[(2, 4, 6), (1, 5, 10), (7, 8, 11)], [(1, 5, 10), (2, 4, 6), (3, 9, 12)], [(3, 9, 12), (1, 5, 10), (7, 8, 11)], [(7, 8, 11), (2, 4, 6), (3, 9, 12)]},

{[(4, 5, 10), (1, 2, 6), (8, 9, 11)], [(1, 2, 6), (4, 5, 10), (3, 7, 12)], [(3, 7, 12), (1, 2, 6), (8, 9, 11)], [(8, 9, 11), (4, 5, 10), (3, 7, 12)]},

{[(4, 5, 6), (1, 2, 10), (3, 7, 9)], [(1, 2, 10), (4, 5, 6), (8, 11, 12)], [(8, 11, 12), (1, 2, 10), (3, 7, 9)], [(3, 7, 9), (4, 5, 6), (8, 11, 12)]},

{[(1, 5, 6), (2, 4, 10), (3, 8, 11)], [(2, 4, 10), (1, 5, 6), (7, 9, 12)], [(7, 9, 12), (2, 4, 10), (3, 8, 11)], [(3, 8, 11), (1, 5, 6), (7, 9, 12)]},

{[(2, 5, 11), (3, 6, 7), (1, 4, 8)], [(3, 6, 7), (2, 5, 11), (9, 10, 12)], [(9, 10, 12), (3, 6, 7), (1, 4, 8)], [(1, 4, 8), (2, 5, 11), (9, 10, 12)]},

{[(3, 6, 11), (2, 5, 7), (4, 8, 9)], [(2, 5, 7), (3, 6, 11), (1, 10, 12)], [(1, 10, 12), (2, 5, 7), (4, 8, 9)], [(4, 8, 9), (3, 6, 11), (1, 10, 12)]},

{[(2, 7, 11), (3, 5, 6), (8, 9, 10)], [(3, 5, 6), (2, 7, 11), (1, 4, 12)], [(1, 4, 12), (3, 5, 6), (8, 9, 10)], [(8, 9, 10), (2, 7, 11), (1, 4, 12)]},

{[(2, 3, 5), (6, 7, 11), (4, 9, 10)], [(6, 7, 11), (2, 3, 5), (1, 8, 12)], [(1, 8, 12), (6, 7, 11), (4, 9, 10)], [(4, 9, 10), (2, 3, 5), (1, 8, 12)]},

{[(2, 3, 6), (5, 7, 11), (1, 8, 10)], [(5, 7, 11), (2, 3, 6), (4, 9, 12)], [(4, 9, 12), (5, 7, 11), (1, 8, 10)], [(1, 8, 10), (2, 3, 6), (4, 9, 12)]},

{[(3, 5, 7), (2, 6, 11), (1, 8, 9)], [(2, 6, 11), (3, 5, 7), (4, 10, 12)], [(4, 10, 12), (2, 6, 11), (1, 8, 9)], [(1, 8, 9), (3, 5, 7), (4, 10, 12)]},

{[(5, 6, 11), (2, 3, 7), (1, 9, 10)], [(2, 3, 7), (5, 6, 11), (4, 8, 12)], [(4, 8, 12), (2, 3, 7), (1, 9, 10)], [(1, 9, 10), (5, 6, 11), (4, 8, 12)]},

{[(5, 6, 7), (2, 3, 11), (4, 8, 10)], [(2, 3, 11), (5, 6, 7), (1, 9, 12)], [(1, 9, 12), (2, 3, 11), (4, 8, 10)], [(4, 8, 10), (5, 6, 7), (1, 9, 12)]},

{[(1, 3, 6), (4, 7, 8), (2, 5, 9)], [(4, 7, 8), (1, 3, 6), (10, 11, 12)], [(10, 11, 12), (4, 7, 8), (2, 5, 9)], [(2, 5, 9), (1, 3, 6), (10, 11, 12)]},

{[(1, 4, 7), (3, 6, 8), (5, 9, 10)], [(3, 6, 8), (1, 4, 7), (2, 11, 12)], [(2, 11, 12), (3, 6, 8), (5, 9, 10)], [(5, 9, 10), (1, 4, 7), (2, 11, 12)]},

{[(1, 3, 8), (4, 6, 7), (9, 10, 11)], [(4, 6, 7), (1, 3, 8), (2, 5, 12)], [(2, 5, 12), (4, 6, 7), (9, 10, 11)], [(9, 10, 11), (1, 3, 8), (2, 5, 12)]},

{[(3, 4, 6), (1, 7, 8), (5, 10, 11)], [(1, 7, 8), (3, 4, 6), (2, 9, 12)], [(2, 9, 12), (1, 7, 8), (5, 10, 11)], [(5, 10, 11), (3, 4, 6), (2, 9, 12)]},

{[(3, 4, 7), (1, 6, 8), (2, 9, 11)], [(1, 6, 8), (3, 4, 7), (5, 10, 12)], [(5, 10, 12), (1, 6, 8), (2, 9, 11)], [(2, 9, 11), (3, 4, 7), (5, 10, 12)]},

{[(4, 6, 8), (1, 3, 7), (2, 9, 10)], [(1, 3, 7), (4, 6, 8), (5, 11, 12)], [(5, 11, 12), (1, 3, 7), (2, 9, 10)], [(2, 9, 10), (4, 6, 8), (5, 11, 12)]},

{[(1, 6, 7), (3, 4, 8), (2, 10, 11)], [(3, 4, 8), (1, 6, 7), (5, 9, 12)], [(5, 9, 12), (3, 4, 8), (2, 10, 11)], [(2, 10, 11), (1, 6, 7), (5, 9, 12)]},

{[(2, 4, 7), (5, 8, 9), (3, 6, 10)], [(5, 8, 9), (2, 4, 7), (1, 11, 12)], [(1, 11, 12), (5, 8, 9), (3, 6, 10)], [(3, 6, 10), (2, 4, 7), (1, 11, 12)]},

{[(2, 5, 8), (4, 7, 9), (6, 10, 11)], [(4, 7, 9), (2, 5, 8), (1, 3, 12)], [(1, 3, 12), (4, 7, 9), (6, 10, 11)], [(6, 10, 11), (2, 5, 8), (1, 3, 12)]},

{[(2, 4, 9), (5, 7, 8), (1, 10, 11)], [(5, 7, 8), (2, 4, 9), (3, 6, 12)], [(3, 6, 12), (5, 7, 8), (1, 10, 11)], [(1, 10, 11), (2, 4, 9), (3, 6, 12)]},

{[(4, 5, 7), (2, 8, 9), (1, 6, 11)], [(2, 8, 9), (4, 5, 7), (3, 10, 12)], [(3, 10, 12), (2, 8, 9), (1, 6, 11)], [(1, 6, 11), (4, 5, 7), (3, 10, 12)]}, {[(4, 5, 8), (2, 7, 9), (1, 3, 10)], [(2, 7, 9), (4, 5, 8), (6, 11, 12)], [(6, 11, 12), (2, 7, 9), (1, 3, 10)], [(1, 3, 10), (4, 5, 8), (6, 11, 12)]},

{[(5, 7, 9), (2, 4, 8), (3, 10, 11)], [(2, 4, 8), (5, 7, 9), (1, 6, 12)], [(1, 6, 12), (2, 4, 8), (3, 10, 11)], [(3, 10, 11), (5, 7, 9), (1, 6, 12)]},

{[(3, 5, 8), (6, 9, 10), (4, 7, 11)], [(6, 9, 10), (3, 5, 8), (1, 2, 12)], [(1, 2, 12), (6, 9, 10), (4, 7, 11)], [(4, 7, 11), (3, 5, 8), (1, 2, 12)]},

{[(3, 6, 9), (5, 8, 10), (1, 7, 11)], [(5, 8, 10), (3, 6, 9), (2, 4, 12)], [(2, 4, 12), (5, 8, 10), (1, 7, 11)], [(1, 7, 11), (3, 6, 9), (2, 4, 12)]},

{[(3, 5, 10), (6, 8, 9), (1, 2, 11)], [(6, 8, 9), (3, 5, 10), (4, 7, 12)], [(4, 7, 12), (6, 8, 9), (1, 2, 11)], [(1, 2, 11), (3, 5, 10), (4, 7, 12)]},

{[(5, 6, 8), (3, 9, 10), (1, 2, 7)], [(3, 9, 10), (5, 6, 8), (4, 11, 12)], [(4, 11, 12), (3, 9, 10), (1, 2, 7)], [(1, 2, 7), (5, 6, 8), (4, 11, 12)]},

{[(5, 6, 9), (3, 8, 10), (2, 4, 11)], [(3, 8, 10), (5, 6, 9), (1, 7, 12)], [(1, 7, 12), (3, 8, 10), (2, 4, 11)], [(2, 4, 11), (5, 6, 9), (1, 7, 12)]},

{[(4, 6, 9), (7, 10, 11), (1, 5, 8)], [(7, 10, 11), (4, 6, 9), (2, 3, 12)], [(2, 3, 12), (1, 5, 8), (7, 10, 11)], [(1, 5, 8), (4, 6, 9), (2, 3, 12)]},

{[(4, 7, 10), (6, 9, 11), (1, 2, 8)], [(6, 9, 11), (4, 7, 10), (3, 5, 12)], [(3, 5, 12), (6, 9, 11), (1, 2, 8)], [(1, 2, 8), (4, 7, 10), (3, 5, 12)]},

{[(4, 6, 11), (7, 9, 10), (1, 2, 3)], [(7, 9, 10), (4, 6, 11), (5, 8, 12)], [(5, 8, 12), (7, 9, 10), (1, 2, 3)], [(1, 2, 3), (4, 6, 11), (5, 8, 12)]},

{[(6, 7, 9), (4, 10, 11), (2, 3, 8)], [(4, 10, 11), (6, 7, 9), (1, 5, 12)], [(1, 5, 12), (4, 10, 11), (2, 3, 8)], [(2, 3, 8), (6, 7, 9), (1, 5, 12)]},

{[(5, 7, 10), (1, 8, 11), (2, 6, 9)], [(1, 8, 11), (5, 7, 10), (3, 4, 12)], [(3, 4, 12), (1, 8, 11), (2, 6, 9)], [(2, 6, 9), (5, 7, 10), (3, 4, 12)]},

{[(5, 8, 11), (1, 7, 10), (2, 3, 9)], [(1, 7, 10), (5, 8, 11), (4, 6, 12)], [(4, 6, 12), (1, 7, 10), (2, 3, 9)], [(2, 3, 9), (5, 8, 11), (4, 6, 12)]},

{[(1, 5, 7), (8, 10, 11), (2, 3, 4)], [(8, 10, 11), (1, 5, 7), (6, 9, 12)], [(6, 9, 12), (8, 10, 11), (2, 3, 4)], [(2, 3, 4), (1, 5, 7), (6, 9, 12)]},

{[(6, 8, 11), (1, 2, 9), (3, 7, 10)], [(1, 2, 9), (6, 8, 11), (4, 5, 12)], [(4, 5, 12), (1, 2, 9), (3, 7, 10)], [(3, 7, 10), (6, 8, 11), (4, 5, 12)]},

{[(1, 6, 9), (2, 8, 11), (3, 4, 10)], [(2, 8, 11), (1, 6, 9), (5, 7, 12)], [(5, 7, 12), (2, 8, 11), (3, 4, 10)], [(3, 4, 10), (1, 6, 9), (5, 7, 12)]},

{[(1, 7, 9), (2, 3, 10), (4, 8, 11)], [(2, 3, 10), (1, 7, 9), (5, 6, 12)], [(5, 6, 12), (2, 3, 10), (4, 8, 11)], [(4, 8, 11), (1, 7, 9), (5, 6, 12)]}.

Here, the first ten 3-resolution sets in the nested 3-design come from pairing the *first* resolution set of the starting affine resolvable 3-design with, successively, the remaining resolution sets of the design, while the next nine 3-resolution sets in the nested 3-design come from pairing the *second* resolution set of the starting affine resolvable 3-design with, successively, the remaining resolution sets of the design, and so on. Thus we have fifty-five 3-resolution sets in all.

Remark.

1) In Theorem 2.1, we assume $\lambda_3 \geq 2$. This is because we want to have the resulting

3-design. For example, the smallest example of an affine resolvable 3-design is a 3-(8,4,1) design with $b = 14, r = 7, \lambda_2 = 3$, which yields the resulting design with blocks of size 2, i.e., λ_{*3} being not positive.
2) Each block in the sub-block structure is repeated three times. Then we can take only one of the three blocks to produce a new 3-design with the parameters of one third of the original 3-design consisting of sub-blocks. However this design is neither nested nor resolvable.

Theorem 2.2. *The existence of an affine resolvable* 3-$(4\lambda_3 + 4, 2\lambda_3 + 2, \lambda_3)$ *design implies the existence of a* 3*-resolvable nested* 3*-wise balanced design with parameters* $v^* = 4(\lambda_3 + 1), b^* = 4(2\lambda_3 + 1)(4\lambda_3 + 3)$, $r^* = 3(4\lambda_3 + 3)(2\lambda_3 + 1)$, $k^* = 3(\lambda_3 + 1)$, $\lambda_2^* = 3(2\lambda_3 + 1)(3\lambda_3 + 2)$, $\lambda_3^* = 3[\lambda_3(\lambda_3 - 1)/2 + (\lambda_3 + 1)(4\lambda_3 + 1)]$, $b_* = 2b^*, b_{*1} = b_{*2} = b^*, k_{*1} = 2(\lambda_3 + 1)$, $k_{*2} = \lambda_3 + 1$,

$$\lambda_{*t} = \sum_{i=1}^{2} b_{*i} \frac{\binom{k_{*i}}{t}}{\binom{v^*}{t}}, \quad t = 2, 3.$$

Proof. The parameters for the super-block structure are obvious. Since, in the affine resolvable 3-$(4\lambda_3 + 4, 2\lambda_3 + 2, \lambda_3)$ design, there are $\lambda_3 + 1$ common points between any two blocks belonging to different resolution sets, consider these $\lambda_3 + 1$ common points and the remaining $2[(2\lambda_3 + 2) - (\lambda_3 + 1)]$ points from the two blocks as two sub-blocks nested in a super-block of size $3(\lambda_3 + 1)$. Thus there are b^* sub-blocks of $k_{*1} = 2(\lambda_3 + 1)$ and $k_{*2} = \lambda_3 + 1$ each, i.e., $b_{*1} = b_{*2} = b^*$. Also by the relation (1.1) it is clear that

$$\lambda_{*t} = \sum_{i=1}^{2} b_{*i} \frac{\binom{k_{*i}}{t}}{\binom{v^*}{t}}, \quad t = 2, 3.$$

It can be noted from the nested structure that it preserves a 3-resolvability. This completes the proof. □

Example 2.2. Consider an affine resolvable 3-design with parameters $v = 8, b = 14, r = 7, k = 4$, $\lambda_2 = 3$, $\lambda_3 = 1$, whose blocks are given by [(1, 2, 3, 5), (4, 6, 7, 8)], [(2, 3, 4, 6), (5, 7, 1, 8)], [(3, 4, 5, 7), (6, 1, 2, 8)], [(4, 5, 6, 1), (7, 2, 3, 8)], [(5, 6, 7, 2), (1, 3, 4, 8)], [(6, 7, 1, 3), (2, 4, 5, 8)], [(7, 1, 2, 4), (3, 5, 6, 8)]. Then Theorem 2.2 yields a 3-resolvable nested 3-wise balanced design with parameters $v^* = 8, b^* = 84$, $r^* = 63$, $k^* = 6$, $\lambda_2^* = 45$, $\lambda_3^* = 30$, $b_* = 168, b_{*1} = b_{*2} = 84, k_{*1} = 4$, $k_{*2} = 2$, $\lambda_{*2} = 21$, $\lambda_{*3} = 6$. The nested structure is given as follows, where ( ) and [ ] show sub-blocks and super-blocks, respectively, and then 8 sub-blocks in each brace form a 3-resolution set, namely, there are twenty-one 3-resolution sets in all below

{ [(1,4,5,6), (2,3)], [(2,3,7,8), (1,5)], [(2,3,7,8), (4,6)], [(1,4,5,6), (7,8)] },
{ [(1,2,4,7), (3,5)], [(3,5,6,8), (1,2)], [(3,5,6,8), (4,7)], [(1,2,4,7), (6,8)] },
{ [(2,3,4,6), (1,5)], [(1,5,7,8), (2,3)], [(1,5,7,8), (4,6)], [(2,3,4,6), (7,8)] },
{ [(1,3,6,7), (2,5)], [(2,4,5,8), (1,3)], [(2,4,5,8), (6,7)], [(1,3,6,7), (4,8)] },
{ [(2,5,6,7), (1,3)], [(1,3,4,8), (2,5)], [(1,3,4,8), (6,7)], [(2,5,6,7), (4,8)] },

{ [(3,4,5,7), (1,2)], [(1,2,6,8), (3,5)], [(1,2,6,8), (4,7)], [(3,4,5,7), (6,8)] },
{ [(2,5,6,7), (3,4)], [(1,3,4,8), (2,6)], [(1,3,4,8), (5,7)], [(2,5,6,7), (1,8)] },
{ [(1,2,3,5), (4,6)], [(4,6,7,8), (2,3)], [(4,6,7,8), (1,5)], [(1,2,3,5), (7,8)] },
{ [(3,4,5,7), (2,6)], [(1,2,6,8), (3,4)], [(1,2,6,8), (5,7)], [(3,4,5,7), (1,8)] },
{ [(1,2,4,7), (3,6)], [(3,5,6,8), (2,4)], [(3,5,6,8), (1,7)], [(1,2,4,7), (5,8)] },
{ [(1,3,6,7), (2,4)], [(2,4,5,8), (3,6)], [(2,4,5,8), (1,7)], [(1,3,6,7), (5,8)] },
{ [(1,3,6,7), (4,5)], [(2,4,5,8), (3,7)], [(2,4,5,8), (1,6)], [(1,3,6,7), (2,8)] },
{ [(2,3,4,6), (5,7)], [(1,5,7,8), (3,4)], [(1,5,7,8), (2,6)], [(2,3,4,6), (1,8)] },
{ [(1,4,5,6), (3,7)], [(2,3,7,8), (4,5)], [(2,3,7,8), (1,6)], [(1,4,5,6), (2,8)] },
{ [(1,2,3,5), (4,7)], [(4,6,7,8), (3,5)], [(4,6,7,8), (1,2)], [(1,2,3,5), (6,8)] },
{ [(1,2,4,7), (5,6)], [(3,5,6,8), (1,4)], [(3,5,6,8), (2,7)], [(1,2,4,7), (3,8)] },
{ [(3,4,5,7), (1,6)], [(1,2,6,8), (4,5)], [(1,2,6,8), (3,7)], [(3,4,5,7), (2,8)] },
{ [(2,5,6,7), (1,4)], [(1,3,4,8), (5,6)], [(1,3,4,8), (2,7)], [(2,5,6,7), (3,8)] },
{ [(1,2,3,5), (6,7)], [(4,6,7,8), (2,5)], [(4,6,7,8), (1,3)], [(1,2,3,5), (4,8)] },
{ [(1,4,5,6), (2,7)], [(2,3,7,8), (5,6)], [(2,3,7,8), (1,4)], [(1,4,5,6), (3,8)] },
{ [(2,3,4,6), (1,7)], [(1,5,7,8), (2,4)], [(1,5,7,8), (3,6)], [(2,3,4,6), (5,8)] }.

References

[1] S. Banerjee and S. Kageyama, Existence of α-resolvable nested incomplete block designs, *Utilitas Math.* **38** (1990), 237–243.

[2] A. Dey, *Theory of Block Designs*, Wiley Eastern, New Delhi, 1986.

[3] A. Hedayat and S. Kageyama, The family of t-designs − Part I, *J. Statist. Plann. Inference* **4** (1980), 173–212 .

[4] S. Kageyama, On μ-resolvable and affine μ-resolvable t-designs, *Essays in Probability and Statistics*, S. Ikeda et al. (eds.), Shinko Tsusho Co. Ltd, Japan (1976), 17–30.

[5] D. A. Preece, Nested balanced incomplete block designs. *Biometrika* **54** (1967), 479–486.

[6] D. Raghavarao, *Constructions and Combinational Problems in Design of Experiments*, Dover, New York, 1988.

[7] S. Rai, S. Banerjee and S. Kageyama, A construction of resolvable nested 3-designs, *J. Combin. Designs* **12** (2004), 466–470.

[8] S. S. Shrikhande and D. Raghavarao, Affine resolvable incomplete block designs, *Contributions to Statistics*, Presented to Professor P. C. Mahalanobis on his 70th Birthday, Pergamon Press (1963), 471–480.

(Received 27 Nov 2003)