AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **32** (2005), Pages 91–103

Supertough graphs need not be $K_{1,3}$-free

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Abstract

The computation of the maximum toughness among graphs with n vertices and m edges is completed for $n \leq 14$. In the process, some questions and conjectures are addressed. Most notably, an example on 14 vertices of a 5-regular $\frac{5}{2}$-tough graph with two $K_{1,3}$-centers is presented. This refutes a conjecture that r-regular $\frac{r}{2}$-tough graphs must be $K_{1,3}$-free.

1 Introduction

A graph $G = (V, E)$ is an (n, m)-graph if $|V| = n$ and $|E| = m$. The toughness [1] of a non-complete graph $G = (V, E)$ is

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)} \ : S \subseteq V \text{ and } \omega(G-S) > 1\},$$

where $\omega(G-S)$ is the number of components in the subgraph of G induced by $V - S$. A graph G is said to be t-tough if $\tau(G) \geq t$. A τ-set for G is a separating set S for which $\tau(G) = |S|/\omega(G-S)$. Among all (n, m)-graphs, the maximum toughness [1, 7, 3, 5] is denoted by $T_n(m)$. An (n, m)-graph G is said to be maximally tough if $\tau(G) = T_n(m)$ and supertough if

$$\tau(G) = \frac{1}{2}\left\lfloor \frac{2m}{n} \right\rfloor.$$

* Supported in part by a Bloomsburg University Research and Disciplinary Grant

All standard notation and terminology not presented here can be found in [11].

It is a simple consequence of the definitions that the toughness of a graph is at most half of its connectivity [1]. Consequently, $T_n(m)$ is at most half of the maximum connectivity among (n,m)-graphs [5]. Since Harary [8] showed, for $m \geq n$, that the maximum connectivity among (n,m)-graphs is $\lfloor 2m/n \rfloor$, supertough graphs are maximally tough. However, maximally tough graphs need not be supertough. For fixed n, the function T_n is obviously nondecreasing.

The search for graphs which are maximally tough or supertough has focused on the presence of $K_{1,3}$-centers. These are vertices with 3 non-adjacent neighbors. Graphs without $K_{1,3}$-centers are said to be $K_{1,3}$-free. Matthews and Sumner [10] show that the absence of $K_{1,3}$-centers in a graph eases the computation of its toughness.

Theorem 1.1 ([10]). *If a graph G is $K_{1,3}$-free, then $\tau(G) = \frac{\kappa(G)}{2}$.*

Using this theorem has been the standard technique for proving graphs to be supertough. Goddard and Swart conjectured that this is the only means for constructing regular supertough graphs.

Conjecture 1.2 ([7]). *For any $r \geq 3$, if G is r-regular and $\tau(G) = \frac{r}{2}$, then G is $K_{1,3}$-free.*

Jackson and Katerinis established this conjecture for $r = 3$.

Theorem 1.3 ([9]). *If G is cubic and $\tau(G) = \frac{3}{2}$, then G is $K_{1,3}$-free.*

We show in Section 3 that Conjecture 1.2 fails when $r = 5$. The other objective of this paper is to complete the computation of the values of $T_n(m)$ for all $n \leq 14$. Section 2 contains general results that settle many of those values. Section 3 addresses the remaining values and summarizes all of them. The most intricate computations are presented in Section 4.

2 Families of Maximally Tough Graphs

Most Harary graphs [8] turn out to be supertough.

Theorem 2.1 ([7],[5]). *Let $n \geq 3$, $n \leq m \leq \frac{n(n-1)}{2}$, and $r = \lfloor \frac{2m}{n} \rfloor$. If*

(i) r is even, or

(ii) $r \geq 2\lfloor \frac{n}{3} \rfloor$ (i.e. $m \geq n\lfloor \frac{n}{3} \rfloor$)

then $T_n(m) = \frac{r}{2}$.

For r odd, the first author [3] provides a further class of supertough graphs.

Theorem 2.2 ([1],[3]). *Let r be odd with $r \geq 3$, a even with $2 \leq a \leq r-1$, $k \geq 1$, and $n = ka(r+2-a)$. If $\frac{nr}{2} \leq m < \frac{n(r+1)}{2}$, then $T_n(m) = \frac{r}{2}$.*

Question 2.3 ([5]). *For r odd and n even, does the equality $T_n(\frac{nr}{2}) = \frac{r}{2}$ imply, for some even a with $2 \le a \le r-1$, that $a(r+2-a)$ divides n?*

The values of $T_n(m)$ for $\lfloor 2m/n \rfloor \le 2$ are quite easy to compute and are presented in [5]. The computation of $T_n(m)$ for $\lfloor 2m/n \rfloor = 3$ is almost completed in [4] with a family of nearly cubic graphs.

Theorem 2.4 ([6],[4]). *Let $n \ge 5$.*

(a) If $n \equiv 0$ or $5 \mod 6$, then

$$T_n(m) = \tfrac{3}{2} \qquad \text{for } \lceil \tfrac{3n}{2} \rceil \le m < 2n.$$

(b) If $n \equiv 1, 3$ or $4 \mod 6$, then

$$T_n(m) = \begin{cases} \frac{3\lfloor \frac{n}{6} \rfloor + 1}{2\lfloor \frac{n}{6} \rfloor + 1} & \text{for } m = \lceil \frac{3n}{2} \rceil, \\ \frac{3}{2} & \text{for } \lceil \frac{3n}{2} \rceil + 1 \le m < 2n. \end{cases}$$

(c) If $n \equiv 2 \mod 6$, then

$$T_n(m) = \begin{cases} \frac{3\lfloor \frac{n}{6} \rfloor + 1}{2\lfloor \frac{n}{6} \rfloor + 1} & \text{for } m = \lceil \frac{3n}{2} \rceil, \\ \frac{3}{2} & \text{for } \lceil \frac{3n}{2} \rceil + 2 \le m < 2n. \end{cases}$$

Moreover, $T_8(13) = \frac{4}{3}$.

For $n \equiv 1, 2, 3$, or $4 \mod 6$ and $m = \lceil 3n/2 \rceil$ or $\lceil 3n/2 \rceil + 1$, those (n,m)-graphs provided in [4] that are maximally tough but not supertough all have $K_{1,3}$-centers. In Theorem 2.4, only a computation in part (c) remains open.

Conjecture 2.5 ([4]). *If $n \ge 8$ and $n \equiv 2 \mod 6$, then $T_n(\lceil \frac{3n}{2} \rceil + 1) = \frac{3\lfloor \frac{n}{6} \rfloor + 1}{2\lfloor \frac{n}{6} \rfloor + 1}$.*

Conjecture 2.5 is settled above in the case $n = 8$ and is settled below (Theorem 3.6) in the case $n = 14$. The potential for extending the arguments we use is discussed at the end in Remark 4.2.

3 Computing $T_n(m)$ for $n \le 14$

The computation of $T_n(m)$ for $n \le 12$ is completed in [5] and [6] with the exception that one open question is left for $n = 11$. In fact, the value $T_{11}(29)$ should not have been listed in [5], and what should have appeared in its place is that the $(11,30)$-graph provided in [3] gives $T_{11}(30) = \frac{5}{2}$. The open question remaining for $n \le 12$ is thus corrected here.

Question 3.1 ([5]). *What are $T_{11}(28)$ and $T_{11}(29)$?*

It turns out that both values are the same.

Theorem 3.2. $T_{11}(28) = T_{11}(29) = \frac{7}{3}$.

Theorem 3.2 is a consequence of the following lemma whose proof is given in Section 4.

Lemma 3.3. *If G is a 5-connected $(11, 29)$-graph, then G contains $2K_1 + K_2$ as an induced subgraph.*

Proof of Theorem 3.2. The $\frac{7}{3}$-tough $(11, 28)$-graph given in [6] establishes that $\frac{7}{3} \leq T_{11}(28) \leq T_{11}(29)$. To establish equalities throughout, suppose to the contrary that there is an $(11, 29)$-graph G with $\tau(G) > \frac{7}{3}$. It follows that $\kappa(G) = 5$. The complement of the induced subgraph $2K_1 + K_2$ guaranteed by Lemma 3.3 now serves as a disconnecting set S with

$$\frac{|S|}{\omega(G - S)} = \frac{7}{3}.$$

This contradiction establishes our result. □

The graphs pictured in Figures 1 through 3 help us to establish several values of $T_n(m)$ for $n = 13$ or 14. The example in Figure 3 refutes Conjecture 1.2, since it is a 5-regular $\frac{5}{2}$-tough graph and the two topmost pictured vertices are $K_{1,3}$-centers. That example also answers Question 2.3 in the negative, since $n = 14$ is neither divisible by 10 nor 12. The graph in Figure 1 is also supertough and not $K_{1,3}$-free, since the central pictured vertex is a $K_{1,3}$-center. Although that graph is not regular, it is nearly so.

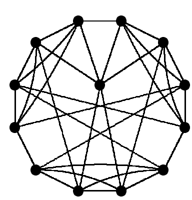

Figure 1: A $(13, 33)$-graph with toughness $\frac{5}{2}$

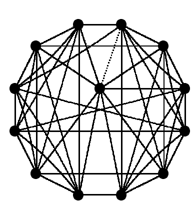

Figure 2: A $(13, 46)$-graph with toughness $\frac{10}{3}$ is pictured with solid edges. The addition of the dotted edge gives a $(13, 47)$-graph with toughness $\frac{7}{2}$

Theorem 3.4. $T_{13}(46) = \frac{10}{3}$.

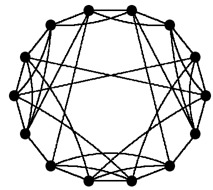

Figure 3: A $(14, 35)$-graph with toughness $\frac{5}{2}$

The proof of Theorem 3.4 is given in Section 4. However, a simple consequence of Theorem 3.4 is proven here.

Corollary 3.5. *If G is a 7-connected $(13, 46)$-graph, then G contains three independent vertices.*

Proof. By Theorem 3.4, any $(13, 46)$-graph must have a disconnecting set S with

$$\frac{|S|}{\omega(G-S)} \leq \frac{10}{3}.$$

If G is also 7-connected, then $|S| \geq 7$, and this forces $\omega(G-S) \geq 3$. □

Theorem 3.6. $T_{14}(22) = \frac{7}{5}$.

The proof of Theorem 3.6 is given in Section 4. The values of $T_n(m)$ for $n = 13, 14$ and $\lceil 3n/2 \rceil \leq m < n\lfloor n/3 \rfloor$ are listed in Table 1.

m	$n = 13$	Justification		m	$n = 14$	Justification
20	$\frac{7}{5}$	Theorem 2.4		21	$\frac{7}{5}$	Theorem 2.4
$21-25$	$\frac{3}{2}$	Theorem 2.4		22	$\frac{7}{5}$	Theorem 3.6
$26-32$	2	Theorem 2.1		$23-27$	$\frac{3}{2}$	Theorem 2.4
$33-38$	$\frac{5}{2}$	Figure 1		$28-34$	2	Theorem 2.1
$39-45$	3	Theorem 2.1		$35-41$	$\frac{5}{2}$	Figure 3
46	$\frac{10}{3}$	Theorem 3.4		$42-48$	3	Theorem 2.1
$47-51$	$\frac{7}{2}$	Figure 2		$49-55$	$\frac{7}{2}$	Theorem 2.2

Table 1: Maximum Toughness Values for $n = 13$ or 14

4 Proofs for the Trickier Values

This section contains the proofs of Lemma 3.3 and Theorems 3.4 and 3.6. In these proofs we employ some further standard notation. For a vertex v in a given graph, $N(v)$ denotes the set of neighbors of v, and $N[v] = \{v\} \cup N(v)$. For a set U of vertices, $\langle U \rangle$ denotes the subgraph induced by U.

Proof of Lemma 3.3. Suppose G is a 5-connected $(11, 29)$-graph. It follows that G has degree sequence $8, 5, \ldots, 5$ or $7, 6, 5, \ldots, 5$ or $6, 6, 6, 5, \ldots, 5$.

Claim 1: No set of 3 vertices of degree 5 in G is independent.

> Suppose to the contrary that a set A of 3 degree 5 vertices is independent. Since G does not contain an induced subgraph $2K_1 + K_2$, each of the 8 vertices of $G - A$ is adjacent to at least 2 vertices of A. This gives $16 \leq \sum_{u \in A} \deg(u) = 15$, a contradiction. Q.E.D.

Claim 2: There are vertices y_1, y_2, x such that y_1 has maximum degree, y_2 has maximum degree among the remaining vertices, and x is a degree 5 common neighbor of y_1 and y_2.

> If $\deg(y_1) \geq 7$, then there are not enough vertices in G for $N(y_1)$ and $N(y_2)$ to be disjoint. The only potential problem is when $\deg(y_1) = \deg(y_2) = 6$ and y_1 and y_2 are adjacent. However, that problem cannot exist for every pair of the three degree 6 vertices. Q.E.D.

Case 1: x is adjacent to all vertices of degree greater than 5.
Since $\deg(y_1) \geq 6$, there is a neighbor z of y_1 outside of $N[x]$. Since the vertices of $G - N[x]$ have degree 5, and they are not adjacent to x, it follows from Claim 1 that $\langle G - N[x] \rangle$ is complete. Let x_1, x_2, y_3 be the vertices of $N(x) - \{y_1, y_2\}$, where $\deg(x_1) = \deg(x_2) = 5$. Since x_1 and x_2 are not adjacent to z and $\deg(z) = 5$, it follows from Claim 1 that x_1 is adjacent to x_2. The known structure of G is pictured in Figure 4.

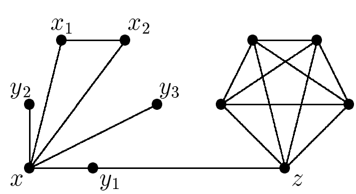

Figure 4: Building an $(11, 29)$-graph in Case 1.

Since $\kappa(G) = 5$, the 5 vertices of $G - N[x]$ must each have distinct neighbors in $N(x)$. (Otherwise, those neighbors would form a disconnecting set with fewer than 5 vertices.) So, z is the only vertex of $G - N[x]$ that is adjacent to y_1. If $\deg(y_1) \geq 7$, then there are not enough available neighbors for y_1. So assume that $\deg(y_1) = \deg(y_2) = \deg(y_3) = 6$. Since y_1, y_2, y_3 each have only one neighbor in $G - N[x]$, they each must be adjacent to the four other vertices in $N(x)$. However, this gives $\deg(x_1), \deg(x_2) \geq 6$, a contradiction.

Case 2: x is not adjacent to all vertices of degree greater than 5, and hence the degree sequence is $6, 6, 6, 5, \ldots, 5$.
Let x_1, x_2, x_3 be the degree 5 neighbors of x. For the vertices outside of $N(x)$, let u

be the degree 6 vertex, and let u_1, u_2, u_3, u_4 be the degree 5 vertices. It follows from Claim 1 that $\langle\{u_1, u_2, u_3, u_4\}\rangle$ is complete. Let

$$k = |N(u) \cap \{u_1, u_2, u_3, u_4\}|.$$

If $k \leq 2$, then, x, u, and two vertices from $\{u_1, u_2, u_3, u_4\} - N(u)$ induce the subgraph $2K_1 + K_2$. Hence, $k \geq 3$. Say $u_1, u_2, u_3 \in N(u)$. Since the set $(N(\{u_1, u_2, u_3\}) \cap N(x)) \cup \{u, u_4\}$ disconnects G, $|N(\{u_1, u_2, u_3\}) \cap N(x)| \geq 3$. Moreover, equality must hold since $\deg(u_1) = \deg(u_2) = \deg(u_3) = 5$. That is,

$$\text{each vertex in } \{u_1, u_2, u_3\} \text{ has a unique neighbor in } N(x). \tag{4.1}$$

Since every pair of vertices from $\{x_1, x_2, x_3\}$ must therefore have a vertex of $\{u_1, u_2, u_3\}$ that is not adjacent to the pair, it follows from Claim 1 that $\langle\{x_1, x_2, x_3\}\rangle$ is complete. The known structure of G is pictured in Figure 5.

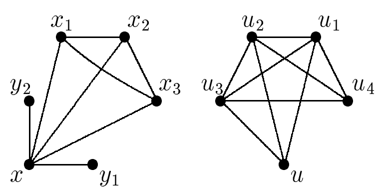

Figure 5: Building an $(11, 29)$-graph in Case 2.

Let F be the set of edges between $G - N[x]$ and $N(x)$. Since the set of endpoints of F in $N(x)$ disconnects G, and $\kappa(G) = 5$, it must be that

$$\text{the set of endpoints of } F \text{ in } N(x) \text{ is all of } N(x). \tag{4.2}$$

Since the number of edges in $\langle G - N[x]\rangle$ is $6 + k$, and

$$26 = \sum_{v \in G - N[x]} \deg(v) = 2(6 + k) + |F|,$$

it follows that $|F| = 14 - 2k$. Hence, the number of edges in $\langle N[x]\rangle$ is

$$29 - (6 + k) - (14 - 2k) = 9 + k.$$

Let D be the set of edges in $\langle N[x]\rangle$ incident with $\{y_1, y_2\}$. From Figure 5, we see that $|D| = (9 + k) - 6 = 3 + k$.

Subcase 2a: $k = 4$.

Since $|F| = 6$, it follows from (4.2) that at most 3 edges of F can be incident with $\{y_1, y_2\}$. So, $12 = \deg(y_1) + \deg(y_2) \leq (|D| + 1) + 3 = 11$, a contradiction.

Subcase 2b: $k = 3$.

Let $R = (N(u_4) \cup N(u)) \cap N(x)$. If $|R| \leq 4$, then there is some vertex z in $N(x) - (N(u_4) \cup N(u))$, and the vertices u_4, u, x, z induce $2K_1 + K_2$. So $|R| \geq 5$. Therefore, the 5 edges in F incident with $\{u_4, u\}$ each have a distinct neighbor in $N(x)$. This together with (4.1) tells us that at most 4 edges of F can be incident with $\{y_1, y_2\}$. So, $\deg(y_1) + \deg(y_2) \leq (|D| + 1) + 4 = 11$, a contradiction. □

Proof of Theorem 3.4. The graph in Figure 2 establishes that $T_{13}(46) \geq \frac{10}{3}$. To establish equality, suppose to the contrary that there is a $(13, 46)$-graph G with $\tau(G) > \frac{10}{3}$. It follows that $\kappa(G) = 7$ and G has degree sequence $8, 7, \ldots, 7$.

Claim 1: No set of 3 vertices in G is independent.

If 3 vertices of G are independent, then the other 10 vertices form a disconnecting set S such that

$$\frac{|S|}{\omega(G-S)} = \frac{10}{3},$$

a contraction. Q.E.D.

Claim 2: If v is any vertex of degree 7, then $\langle G - N[v] \rangle$ is complete.

Let a and b be any two distinct vertices of $\langle G - N[v] \rangle$. Since v is not adjacent to these vertices, Claim 1 tells us that they must be adjacent to each other. Q.E.D.

Claim 3: If u and v are non-adjacent of degree 7, then $|N(u) \cap N(v)| = 3$.

It follows from Claim 1 that $|N(u) \cup N(v)| = 11$. Hence,
$11 = |N(u)| + |N(v)| - |N(u) \cap N(v)| = 14 - |N(u) \cap N(v)|$. Q.E.D.

Let x be a vertex of degree 7 that is adjacent to the vertex of degree 8. The vertex of degree 8 must then have a degree 7 neighbor z outside of $N[x]$. By Claim 3, $|N(x) \cap N(z)| = 3$. Let $W = \{w_1, w_2, w_3\}$ represent $N(x) \cap N(z)$. The vertex of degree 8 is in W. By Claim 2, $\langle G - N[x] \rangle$ and $\langle G - N[z] \rangle$ are complete. Let $X = \{x_1, x_2, x_3, x_4\}$ represent $N[x] - W$, and let $Z = \{z_1, z_2, z_3, z_4\}$ represent $N[z] - W$. This accounts for all of the vertices of G and the 26 straight solid edges pictured in Figure 6.

Claim 4: For all $v \in X \cup Z$, we have $1 \leq |N(v) \cap W| \leq 2$.

If there is an $x' \in X$ that is adjacent to each vertex of W, then the 6 vertices in $N(x') \cap N(x)$ disconnect G, a contradiction. A symmetric argument shows that no vertex in Z can be adjacent to all of W. Now suppose that there is an $x' \in X$ that is adjacent to no vertex of W. Since $\deg(x') = 7$, there is some $z' \in Z$ that is not adjacent to x'. By Claim 2, the vertices $\{w_1, w_2, w_3, z, z'\}$ of $G - N[x']$ must induce a complete subgraph. However, this makes z' adjacent to all of W and contradicts our earlier observation. Symmetrically, every vertex of Z must be adjacent to some vertex of W. Q.E.D.

Claim 5: There is a matching between X and Z.

The result follows from Hall's Matching Theorem once we show that each subset A of X has neighbors in a subset of Z of size at least $|A|$. Since each $x' \in A$ is adjacent to some $z' \in Z$ and each $z' \in Z$ adjacent to some $x' \in A$, the desired property holds for $|A| = 1, 4$.

Case 1: $|A| = 2$.
Without loss of generality, suppose to the contrary that $A = \{x_1, x_2\}$ is only adjacent to z' from Z. Since Claim 4 tells us that z' has a neighbor in W, it follows that x_1, x_2 are the only vertices of X adjacent to z'. Thus, the 6 vertices $\{z', w_1, w_2, w_3, x_3, x_4\}$ disconnect G, a contradiction.
Case 2: $|A| = 3$. Let x' be the element of $X - A$.
If A is only adjacent to a two element subset of Z, then x' must be adjacent to the other two, say z_1, z_2. Since $\{z_1, z_2\}$ is not adjacent to A, reversing the roles of X and Z reduces this to Case 1. Q.E.D.

By reindexing Z if necessary, we may assume, for each i, that x_i is adjacent to z_i. We now know that G contains the 30 solid edges pictured in Figure 6. Let j be

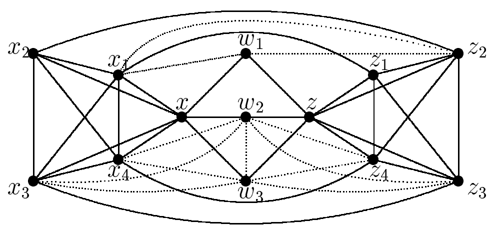

Figure 6: Building a $(13, 46)$-graph with $\tau > \frac{10}{3}$.

the number of edges joining X and Z that are not pictured as solid edges in Figure 6, and let k be the number of edges in $\langle W \rangle$.

Claim 6: $j \leq 1$.

Suppose to the contrary that $j \geq 2$. Without loss of generality, say x_1 is adjacent to z_2. It follows from Claim 4 that no further edges incident with x_1 or z_2 can join X with Z. So x_2 is adjacent to z_j for some $j \neq 2$. If $j \neq 1$, then $N(z_j) \cap N(x_1)$ contains $\{z_1, z_2, x_2, x_j\}$, contradicting Claim 3. So it must be that x_2 is adjacent to z_1. If x_1 and x_2 are adjacent to the same vertex w of W then the 6 vertices $\{x_3, x_4, x, z_1, z_2, w\}$ disconnect G, a contradiction. Without loss of generality, say that x_i is adjacent to w_i, for $i = 1, 2$. Since $\langle G - N[x_1] \rangle$ is complete, z_3 is adjacent to w_2 and w_3. Since $\langle G - N[x_2] \rangle$ is complete, z_3 is adjacent to w_1. That z_3 is adjacent to all of W contradicts Claim 4. Q.E.D.

Claim 1 tells us that $k \geq 1$, and Claim 6 tell that $j \leq 1$. The number of edges between W and $X \cup Z$ is $46 - (30 + k + j) = 16 - k - j$. From the equation

$$8 + 7 + 7 = \sum_{i=1}^{3} \deg(w_i) = 6 + 2k + (16 - k - j)$$

it follows that $j = k = 1$. Without loss of generality, say that x_1 is adjacent to z_2 and that w_2 is adjacent to w_3. Now x_1 has one yet unspecified neighbor, and it must be in W. Since $\langle G - N[x_1]\rangle$ is complete, it must be that x_1 is adjacent to w_1. A similar argument shows that z_2 is adjacent to w_1. Moreover, the vertices z_3, z_4, w_2, w_3 in $G - N[x_1]$ must induce a complete subgraph, as must the vertices x_3, x_4, w_2, w_3 in $G - N[z_2]$. This accounts for all of the edges (solid and dotted) shown in Figure 6. Since x_2 and z_1 each need two additional neighbors in W, it follows that $\deg(w_2) + \deg(w_3) > 15$, a contradiction. □

Our proof of Theorem 3.6 uses a coloring lemma.

Lemma 4.1. *Let G be a $(14, 22)$-graph with $\tau(G) > \frac{4}{3}$. Then, G is 3-colorable.*

Proof. Note that G must be 3-connected and have degree sequence $5, 3, \ldots, 3$ or $4, 4, 3, \ldots, 3$. If G has degree sequence $5, 3, \ldots, 3$, then Theorem 5 of [2] tells us that G is 3-colorable. So it suffices to assume now that G has degree sequence $4, 4, 3, \ldots, 3$. Brooks' Theorem tells us that G is 4 colorable. Let x and y be the two degree 4 vertices.

Claim 1: If $\{v_1, v_2\}$ is a pair of degree 3 vertices of G, then $|N(v_1) \cap N(v_2)| \leq 1$.

Suppose to the contrary that u_1 and u_2 are distinct vertices in $N(v_1) \cap N(v_2)$. It must be that v_1 is not adjacent to v_2, since otherwise $\{u_1, u_2\}$ would form a 2-element disconnecting set for G. It now follows that $S = N(v_1) \cup N(v_2)$ forms a disconnecting set and

$$\tau(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{4}{3},$$

a contradiction. Q.E.D.

Case 1: x and y are not adjacent.
Suppose to the contrary that G is not 3-colorable. Form a $(13, 22)$-graph G' by identifying x and y to a single vertex. Note that G' must be 2-connected, have degree sequence $8, 3, \ldots, 3$, and be 4-colorable. Since a 3-coloring of G' can easily be used to give a 3-coloring of G, it must be that G' has chromatic number 4. Theorem 1 of [2] then tells us that G' is critical. However, by Theorem 4 of [2], there is no critical graph on 13 vertices containing a vertex of degree 8. Thus G' cannot exist.

Case 2: x and y are adjacent.
Let $\{x_1, x_2, x_3, y\} = N(x)$ and $\{y_1, y_2, y_3, x\} = N(y)$. It follows from Claim 1 that $|\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\}| \leq 1$, since any pair of vertices in the intersection would have x and y as common neighbors, contradicting that claim. We may assume $x_1 \notin \{y_1, y_2, y_3\}$. Let $\{z_1, z_2, x\} = N(x_1)$. It also follows from Claim 1 that

$$|\{z_1, z_2\} \cap \{y_1, y_2, y_3\}| \leq 1, \tag{4.3}$$

since otherwise the pair $\{z_1, z_2\}$ would contradict that claim. Note that x_1 and y are not adjacent. Form a graph G' on 13 vertices from G by identifying x_1 and y

to a common vertex, say y'. It follows from (4.3) that $|N(x_1) \cap N(y)| \leq 2$ and so $\deg_{G'}(y') = 5$ or 6.

Subcase 2a: $\deg_{G'}(y') = 6$.
Here G' is a $(13, 21)$-graph with degree sequence $6, 3, \ldots, 3$. Since $G' - \{y'\}$ can be 3-colored, G' can be 4-colored. Suppose toward a contradiction that G' has chromatic number 4. Since G is 3-connected, G' must be 2-connected. Theorem 1 of [2] then tells us that G' is critical. It therefore follows from Theorem 4 of [2] that G' must be one of the four graphs pictured in Figure 7. In any case, using the labels from

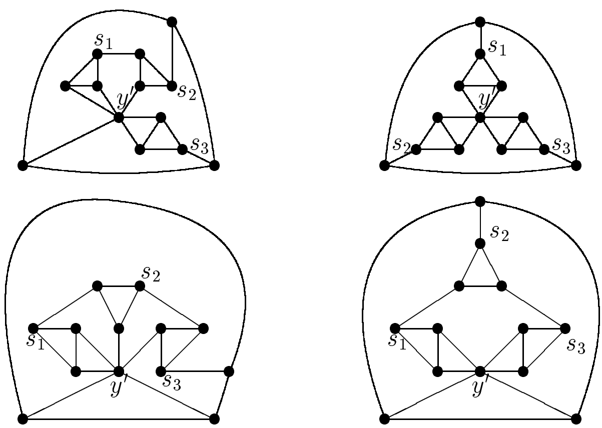

Figure 7: All critical 4-chromatic graphs with degree sequence $6, 3, \ldots, 3$.

Figure 7, the disconnecting set $S = \{s_1, s_2, s_3, x, y\}$ gives

$$\tau(G) \leq \frac{|S|}{\omega(G-S)} = \frac{5}{4} < \frac{4}{3},$$

a contradiction. Hence, G' must be 3-colorable. Since x_1 and y are not adjacent, a 3-coloring for G is easily obtainable from one for G'.

Subcase 2b: $\deg_{G'}(y') = 5$.
Say $z_1 = y_1$ and $\{x_1, y, w\} = N(z_1)$. Here, z_1 has degree 2 in G'. Form G'' from G' by replacing z_1 in G' by K_4 minus an edge as shown in Figure 8. Observe that

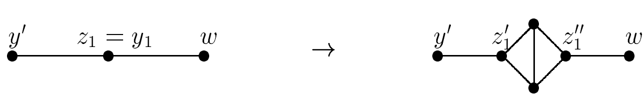

Figure 8: Forming G'' from G' by eliminating degree 2.

G'' has 16 vertices and degree sequence $5, 3, \ldots, 3$, where $\deg_{G'}(y') = 5$. Since G'' is certainly connected, Theorem 5 of [2] tells us that G'' can be 3-colored. Note that the vertices z_1' and z_1'' labeled in Figure 8 must receive the same color. By coloring z_1 in G the common color of z_1' and z_1'', and coloring x_1 and y in G the color of y', it is clear that the 3-coloring of G'' can be used to specify a 3-coloring of G. □

Proof of Theorem 3.6. Let G be a maximally tough $(14, 22)$-graph. Theorem 2.4(c) tells us that $\tau(G) \geq \frac{7}{5}$. Consequently, $\kappa(G) = 3$ and G has degree sequence $5, 3, \ldots, 3$ or $4, 4, 3, \ldots, 3$. It remains to show that $\tau(G) \leq \frac{7}{5}$.

Claim 1: If G has 6 independent vertices, then $\tau(G) < \frac{7}{5}$.

Taking S to be the other 8 vertices gives

$$\tau(G) \leq \frac{|S|}{\omega(G - S)} = \frac{8}{6} < \frac{7}{5}. \qquad \text{Q.E.D.}$$

We know from Lemma 4.1 that G has a 3-coloring. If one of the color classes has 6 or more vertices, then by Claim 1, $\tau(G) < \frac{7}{5}$. So we need only consider the case in which each color class has 5 or fewer vertices. That is, we may denote the color classes by A, B, and C so that $|A| = 4$ and $|B| = |C| = 5$. We may further assume that each vertex of A is adjacent to at least one vertex of B and to at least one vertex of C. Otherwise, an offending vertex from A could be recolored to enlarge B or C to size 6. Define A_B to be the set of vertices in A that are adjacent to exactly one vertex of B, and define A_C similarly. Note that $A_B \cup A_C \neq A$ only if at least one vertex of degree greater than 3 is in A.

Case 1: $A_B \cup A_C = A$.
Either $|A_B| \leq 2$ or $|A_C| \leq 2$. So assume that $|A_B| \leq 2$. For $S = A_B \cup B$, we have $|S| \leq 7$ and $\omega(G - S) = \omega(A_C \cup C) = |C| = 5$. Thus, $\tau(G) \leq \frac{7}{5}$.

Case 2: $A_B \cup A_C = A - \{w\}$, where w is a vertex of degree greater than 3.
Here $|A_B \cup A_C| = 3$, and either $|A_B| \leq 1$ or $|A_C| \leq 1$. So assume $|A_B| \leq 1$. For $S = A_B \cup \{w\} \cup B$, we have $|S| \leq 7$ and $\omega(G - S) = \omega(A_C \cup C) = |C| = 5$. Thus, $\tau(G) \leq \frac{7}{5}$.

Case 3: $A_B \cup A_C = A - \{w, v\}$, where w and v are vertices of degree 4.
Each of w and v must be adjacent to exactly 2 vertices of B and 2 vertices of C. Now count the edges between A and $B \cup C$. There are 3 each from the 2 vertices in $A \setminus \{w, v\}$ and 4 each from w and v. This accounts for 14 and leaves 8 edges between B and C. We may assume that each vertex of B is adjacent to at least one vertex in C or by recoloring we could enlarge C to 6 vertices. Thus, with 8 edges between B and C, there are at least 2 vertices x and y in B that are adjacent to only 1 vertex in C. For $S = A \cup (B \setminus \{x, y\})$, we have $|S| \leq 4 + 3 = 7$ and $\omega(G - S) = \omega(C \cup \{x, y\}) = |C| = 5$. Thus, $\tau(G) \leq \frac{7}{5}$. □

Remark 4.2. *By using a bit more machinery than that used to prove Lemma 4.1, we can further prove that, for any $n \geq 20$, if $n \equiv 2 \mod 6$, then any $(n, \frac{3}{2}n + 1)$-graph with $\tau(G) > \frac{4}{3}$ is 3-colorable. Then, using a much more intricate argument than that used in the proof of Theorem 3.6, we can establish Conjecture 2.5 in the case $n = 20$. However, our argument does not extend to $n \geq 26$. Since our techniques are overly complicated for $n = 20$ and fail for $n = 26$, we only answer Conjecture 2.5 here for $n \leq 14$.*

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(Received 6 Aug 2003)