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Hamilton decompositions of complete bipartite graphs with a 3-factor leave

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Abstract

We show that for any 2-factor U of $K_{n,n}$ with n odd, there exists a hamilton decomposition of $K_{n,n} - E(U)$ with a 1-factor leave. Settling this last open case now provides necessary and sufficient conditions for a hamilton decomposition of *any* complete multipartite graph with the edges of *any* 2-factor removed.

1 Introduction

In 1892, Walecki [7] published a result showing that K_n has a hamilton decomposition if and only if n is odd. Laskar and Auerbach [4] extended this result in 1974, proving that the complete p-partite graph $K_{m,\dots,m}$ (henceforth denoted as $K_m^{(p)}$) has a hamilton decomposition when $m(p-1)$ is even, and a hamilton decomposition with a 1-factor leave (that is, a hamilton decomposition of $K_m^{(p)} - F$ for some 1-factor F in $K_m^{(p)}$) when $m(p-1)$ is odd.

More recently, the problem of constructing a hamilton decompositon of various graphs with a 2-factor removed has been attacked. The first result of this type was produced in 1997 by Buchanan [2], who showed that for any 2-factor U of K_n, with n odd, there exists a hamilton decomposition of $K_n - E(U)$. Rodger and Leach in 2001 [6] solved the corresponding problem for $K_{n,n}$ with n even. Rodger and Leach [5] also produced a result showing that $K_m^{(p)} - E(U)$ has a hamilton decomposition, where $p \geq 3$ and U is a 2-factor of $K_m^{(p)}$ with each cycle having length at least $p+1$. Recently, Rodger [9] produced a result removing this restriction on the cycle lengths. Rodger's result is unique in this area in that is the only result that was obtained without using the technique of vertex amalgamation. The new observation Rodger made which allowed this advance was independently made by Bryant; Bryant used the observation to decompose K_n into any three given 2-factors and $(n-7)/2$ hamilton cycles [1]. Unlike the results in [5] and [6], Rodger's result in [9] also settles the cases in which the decomposition has a 1-factor leave. His result applies to all values of p except $p = 2$, so the only problem remaining open in this category is proving the existence of a hamilton decomposition of $K_{n,n} - E(U)$ with a 1-factor leave for n odd. We settle this problem here using two different methods. The first involves amalgamation and a recent result of Hilton and Johnson [3]; the second approach, which uses direct difference methods, is based on the new observation referred to above, and is a good deal simpler than the first.

In this paper, factorizations of graphs are represented as edge-colorings, $G(i)$ denotes the subgraph of G induced by the edges colored i, k_i denotes the regularity of $G(i)$, and κ_i denotes the edge-connectivity of $G(i)$. If the graph G is an amalgamation of H, then H is said to be a *disentanglement* of G.

2 Outline Factorizations

The main result of this paper relies on the following corollary of a result of Hilton and Johnson [3]:

Corollary 2.1 *Let $\ell \geq 1$, let G be an ℓ-edge-colored bipartite graph, with vertex bipartition $\{L, R\}$, let $\eta : V(G) \to \mathbb{N}$, let $\kappa_i \in \{0, 2\}$ and let $k_i \in \{1, 2\}$ for $1 \leq i \leq \ell$. If*

A1) For any pair $v, w \in V(G)$, the number of edges joining v and w is $\eta(v)\eta(w)$ if v and w are in different parts, and 0 if v and w are in the same part.

A2) $d_{G(i)}(w) = k_i\eta(w)$ for all $w \in V(G)$ and $1 \leq i \leq \ell$.

A3) $\sum_{w\in L} \eta(w) = \sum_{w \in R} \eta(w) = n$, and

A4) for $1 \leq i \leq \ell$, $G(i)$ has a κ_i-edge-connected, k_i-regular disentanglement.

then G is the amalgamation of an ℓ-edge-colored graph $H \cong K_{n,n}$ in which $H(i)$ is a k_i-regular, κ_i-edge-connected subgraph for $1 \leq i \leq \ell$.

Proof: This follows from Theorem 4 of [3]

Any graph that satisfies $(A1 - A4)$ is called an *outline factorization.*

3 Constructing an Outline Factorization

In the proof of the following theorem, we will construct an outline factorization by first constructing a hamilton decomposition of $K_{n,n}$ with a 1-factor leave, amalgamating the vertices, and then swapping some edges between two of the color classes to form the 2-factor.

Theorem 3.1 *Let n be odd, let $H \cong K_{n,n}$, and let $V(H) = \{v_{i,j} | 1 \leq i \leq n, j \in \{L, R\}\}$. For any 2-factor U of H, there exists a hamilton decomposition of $H - E(U)$ with a 1-factor leave.*

Proof: Since H is bipartite, every edge is of the form $\{v_{i,L}, v_{j,R}\}$, and can be assigned a *difference* value of $(j - i) \mod n$. Let E_i be the set of edges having difference i. Let $c : E(H) \to \{1, \ldots, \frac{n+1}{2}\}$ be the $(\frac{n+1}{2})$-edge-coloring where

$$c_i = \begin{cases} \{E_{2i-3} \bigcup E_{2i-2}\} & \text{if } 1 \leq i \leq \frac{n-1}{2}, \\ E_{n-2} & \text{if } i = \frac{n+1}{2}, \end{cases}$$

is the set of edges colored i, with the index calculations being reduced modulo n. Colors 1 through $\frac{n-1}{2}$ induce hamilton cycles of H, and color $\frac{n+1}{2}$ induces a 1-factor.

Let U be any 2-factor of H. U consists of a set of disjoint cycles spanning H, and can be described completely by listing the lengths of the cycles. Therefore we can suppose that U consists of q cycles, the i^{th} cycle having length $s_i \geq 4$; so $\sum_{i=1}^{q} s_i = 2n$. If $q = 1$, then the problem reduces to finding a hamilton decomposition of $K_{n,n}$ with a 1-factor leave, which was settled by Laskar and Auerbach (see the Introduction); so we will assume that $q \geq 2$. We now amalgamate H to form a new graph G. Let $V(G) = \{w_{i,j} | 1 \leq i \leq q,\ j \in \{L, R\}\}$. In both parts, for each i, we amalgamate $s_i/2$ vertices of H to form one vertex of G. This is done using the amalgamating function $\psi : V(H) \to V(G)$ defined by $\psi(v_{i,j}) = w_{z,j}$ if and only if $\sum_{x=1}^{z-1} s_x/2 < i \leq \sum_{x=1}^{z} s_x/2$ and $j \in \{L, R\}$.

It is easy to show that G satisfies (A1-A4), and is thus an outline factorization of H. (A1) holds since each pair of vertices $v, w \in V(H)$ are joined by an edge if and only if they are in different parts. (A2) holds since $H(i)$ is k_i-regular. (A3) holds since the amalgamating function maps every vertex of H to exactly one vertex of G. Finally, (A4) holds, since $H(i)$ is a k_i-regular, κ_i-edge-connected disentanglement of $G(i)$ for $1 \leq i \leq \frac{n+1}{2}$, with

$$k_i = \begin{cases} 1 & \text{if } i = \frac{n+1}{2} \\ 2 & \text{otherwise} \end{cases}$$

$$\kappa_i = \begin{cases} 0 & \text{if } i = \frac{n+1}{2} \\ 2 & \text{otherwise.} \end{cases}$$

We will modify the edge-coloring of G to disconnect $G(1)$ into q components, each consisting of exactly 2 vertices. This is accomplished by swapping edges between color classes 1 and 2.

Any vertex $w_{i,L}$ is incident with $2\eta(w_{i,L}) = s_i$ edges of color 1. By the definition of the edge-coloring c, exactly one of these edges joins $w_{i,L}$ to $w_{i-1,R}$, and the rest join $w_{i,L}$ to $w_{i,R}$. Since $\eta(w_{i,L}) \geq 2$, $w_{i,L}$ is incident with at least 4 edges of color 1, and thus there are at least 3 edges of color class 1 joining $w_{i,L}$ to $w_{i,R}$. Let S_1 be the set of all q edges of color 1 joining $w_{i,L}$ to $w_{i-1,R}$ for $1 \leq i \leq q$. Similarly, any vertex $w_{i,L}$ is incident with $2\eta(w_{i,L}) = s_i \geq 4$ edges of color 2. Exactly three of these edges join $w_{i,L}$ to $w_{i+1,R}$. (If $s_i = 4$, then there is *exactly* one edge colored 2 joining $w_{i,L}$ to $w_{i,R}$.) For each $1 \leq i \leq q$, pick an edge colored 2 joining $w_{i,L}$ to $w_{i,R}$ and put it in the set S_2. Recolor the edges of G with $c' : E(G) \to \{1, \ldots, \frac{n+1}{2}\}$, defined by

$$c'(e) = \begin{cases} 1 & \text{if } e \in S_2 \\ 2 & \text{if } e \in S_1 \\ c(e) & \text{otherwise,} \end{cases}$$

and denote this edge-colored version of G by G'.

Now $G'(1)$ has q components, the i^{th} component being induced by the two vertices $w_{i,L}$ and $w_{i,R}$. Furthermore, $\eta(w_{i,L}) + \eta(w_{i,R}) = s_i$. This recoloring has not changed the degree of any vertex in any color class, so G' still satisfies (A1-A3). However, since $G(1)$ is connected and $G'(1)$ is disconnected, G' does not satisfy (A4) for the original values of κ_i, $1 \leq i \leq \frac{n+1}{2}$. However, we can show that G' does satisfy (A4) for

$$\kappa_i' = \begin{cases} 0 & \text{if } i \in \{1, \frac{n+1}{2}\} \\ 2 & \text{otherwise.} \end{cases}$$

For $3 \leq i \leq \frac{n+1}{2}$, $H(i)$ is a k_i-regular, κ_i'-edge-connected disentanglement of $G'(i)$. This is clear, since colors 3 through $\frac{n+1}{2}$ were unaffected by the recoloring. To show that $G'(2)$ satisfies (A4) we need the following corollary of Nash-Williams [8]:

Corollary 3.1 *Let k and l be nonnegative integers, with l even. Let C be a graph in which the degree of each vertex is a multiple of k. Then C has an l-edge-connected k-regular disentanglement if and only if C is l-edge-connected.*

Each vertex of $G'(2)$ has even degree, since the degrees were unchanged by the recoloring. To show that $G'(2)$ is 2-edge-connected, note that $w_{i,L}$ is adjacent to $w_{i-1,R}$ for $1 \leq i \leq q$. So the vertex sets $\{w_{i,L}, w_{j,R} | i \text{ is even}, j \text{ is odd}\}$ and $\{w_{i,L}, w_{j,R} | i \text{ is odd}, j \text{ is even}\}$ induce two cycles that, together, span $V(G')$. Because n is odd, the 2-factor U must contain at least one cycle with length at least 6. Thus, for at least one value of i, $w_{i,L}$ was joined to $w_{i,R}$ by at least 3 edges of $G(2)$ before the recoloring. Only one of those edges was recolored, thus $w_{i,L}$ is joined to $w_{i,R}$ by at least two edges of $G'(2)$. These edges join the two cycles, and $G'(2)$ is 2-edge-connected. Thus by Corollary 3.1, $G'(2)$ has a disentanglement that is 2-regular and 2-edge-connected.

We now need to show that $G'(1)$ has a disentanglement that is a disconnected 2-factor of K, with the i^{th} cycle having length s_i. This argument also uses Corollary 3.1. Let C be the i^{th} component of $G'(1)$. C contains exactly 2 vertices, $w_{i,L}$ and $w_{i,R}$, each of which has degree s_i. Since s_i is the length of a cycle in a bipartite graph, we know that s_i is even and at least 4. Therefore C is at least 4-edge-connected, and by Corollary 3.1 has a 2-edge-connected, 2-regular disentanglement. Thus $K(1)$ consists of q cycles, the i^{th} cycle having length s_i.

Finally, we need to show that $G'(\frac{n+1}{2})$ has a 1-regular disentanglement. Clearly each vertex of $G'(\frac{n+1}{2})$ has a degree that is a multiple of 1, and $G'(\frac{n+1}{2})$ is 0-edge connected. By Corollary 3.1, $G'(\frac{n+1}{2})$ has a 1-regular disentanglement.

So

$$\bigcup_{2\leq i\leq \frac{n-1}{2}} K(i)$$

is a hamilton decomposition of K, $K(1) \cong U$, and $K(\frac{n+1}{2})$ is the 1-factor leave.

It is possible to prove Theorem 3.1 in a more direct manner, as the following proof shows.

Proof: We define the sets E_i for $0 \leq i \leq n-1$ as was done in the previous proof, and let $c : E(H) \to \{1, \ldots, \frac{n+1}{2}\}$ be the $(\frac{n+1}{2})$-edge-coloring defined by

$$c_i = \begin{cases} E_{2i-2} \cup E_{2i-1} & \text{if } 2 \leq i \leq \frac{n-1}{2}, \\ E_1 \cup E_{n-1} & \text{if } i = 1, \\ E_0 & \text{if } i = 0. \end{cases}$$

$H(i)$ is a hamilton cycle for $1 \leq i \leq \frac{n-1}{2}$, and $H(0)$ is a 1-factor of H. Somewhat surprisingly, as we will show below, $H(0) \cup H(1)$ contains every 2-factor of $K_{n,n}$, and so the result follows.

To find the 2-factor with cycles of lengths $s_1, s_2, \ldots, s_q$, we will swap some edges between c_0 and c_1. Let $t_i = \frac{1}{2}(s_1+s_2+\cdots+s_i)$ for $1 \leq i \leq q$ and let $V(K_{n,n}) = \{Z_n \times \{L, R\}\}$ with the obvious vertex partition. For $1 \leq i \leq q$, let S_0 contain $\{(t_i, L), (t_i + 1, R)\} \cup \{(t_i, R)(t_i+1, L)\}$ and let S_1 contain $\{(t_i, L), (t_i, R)\} \cup \{(t_i+1, L), (t_i+1, R)\}$. We now recolor H with the $(\frac{n+1}{2})$-edge-coloring $c' : E(H) \to \{1, \ldots, \frac{n+1}{2}\}$ defined by

$$c'(e) = \begin{cases} 0 & \text{if } e \in S_0, \\ 1 & \text{if } e \in S_1, \\ c(e) & \text{otherwise.} \end{cases}$$

After this recoloring, $H(1)$ is the prescribed 2-factor, $H(0)$ is the 1-factor leave, and the the remaining colors all induce hamilton cycles.

4 Conclusion

Theorem 3.1 settles the last remaining case in the problem of constructing hamilton decompositions of $K_m^{(p)}$ with a given 2-factor removed. In particular, Theorem 3.1 settles the case when $p = 2$ with a 1-factor leave. Rodger and Leach settled the case with $p = 2$ and no leave in [6], and Rodger settled the cases with $p > 2$ both with and without 1-factor leaves in [9]. Together, these results allow us to state the following general theorem:

Theorem 4.1 *Let $p \geq 2$ and $m \geq 1$, and let U be a 2-factor of $K_m^{(p)}$. Then there exists a hamilton decomposition of $K_m^{(p)} - E(U)$ if and only if $m(p-1)$ is even, and there exists a hamilton decomposition of $K_m^{(p)} - E(U)$ with a 1-factor leave if and only if $m(p-1)$ is odd.*

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