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Long cycles through specified edges and vertices

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Abstract

Let k, m, s be integers with $k \geq 2$, $m \geq 0$, and $0 \leq s \leq k$. We show that if G is an $(m+k)$-connected graph, and F is a linear forest of G with m edges and s isolated vertices, then G has a cycle of length at least $\min\{|V(G)|, 2\delta(G) - m\}$ passing through F.

1 Introduction

All graphs considered in this paper are finite simple undirected graphs with no loops and no multiple edges. For a graph G, we let $V(G)$ and $E(G)$ denote the set of *vertices* and *edges* of G, respectively. For a vertex v of G, we let $\deg_G(v)$ denote the degree of v in G. The *minimum degree* $\delta(G)$ of G is defined by $\delta(G) = \min\{\deg_G(v)|v \in V(G)\}$. For $k \geq 1$, we define

$$\sigma_k(G) = \min\left\{\sum_{i=1}^{k} \deg_G(v_i) \mid v_1, \ldots, v_k \text{ are independent in } G\right\};$$

thus $\sigma_1(G) = \delta(G)$. By a *cycle*, we mean a connected graph C such that $\deg_C(v) = 2$ for all $v \in V(C)$. For a finite set X, the cardinality of X is denoted by $|X|$.

A graph F is called a *linear forest* if every component of F is a path (F may contain components consisting of a single vertex). For a linear forest F in a graph G, we say that a cycle C of G *passes through* F if $E(F) \subset E(C)$ and $V(F) \subset V(C)$. Define

$$S(F) = \{x \in V(F)|\deg_F(x) = 0\}.$$

There are many results about long cycles in graphs passing through specified edges and vertices. Among them is the following theorem, which is proved by Hu et al. in [5; Theorem 3]:

Theorem A *Let k, m, s be integers with $k \geq 2$, $m \geq 0$, and $0 \leq s \leq k-2$. Let G be an $(m+k)$-connected graph, and let F be a linear forest of G with $|E(F)| = m$ and $|S(F)| = s$. Then G has a cycle C of length at least* $\min\{|V(G)|, (2/(k+1))\sigma_{k+1}(G) - m\}$ *passing through F.*

As an immediately corollary of Theorem A, we obtain the following statement (note that $(1/k)\sigma_k(G) \geq \delta(G)$ for any graph G by the definition of $\delta(G)$ and $\sigma_k(G)$) :

Corollary B *Let k, m, s be integers with $k \geq 2$, $m \geq 0$, and $0 \leq s \leq k-2$. Let G be an $(m+k)$-connected graph, and let F be a linear forest of G with $|E(F)| = m$ and $|S(F)| = s$. Then G has a cycle C of length at least $\min\{|V(G)|, 2\delta(G) - m\}$ passing through F.*

In [5], it is shown that the lower bound $\min\{|V(G)|, 2\delta(G) - m\}$ on the length of C is best possible in Corollary B (and hence in Theorem A). In [5], it is also shown that for $k = 2$, the assumption that $0 \leq s \leq k-2$ cannot be replaced by the assumption that $0 \leq s \leq k-1$ in Theorem A (it does not seem to be known whether the same is true for $k \geq 3$). The purpose of this paper is to show that as for Corollary B, the conclusion holds under the assumption that $0 \leq s \leq k$:

Theorem 1 *Let k, m, s be integers with $k \geq 2$, $m \geq 0$, and $0 \leq s \leq k$. Let G be an $(m+k)$-connected graph, and let F be a linear forest of G with $|E(F)| = m$ and $|S(F)| = s$. Then G has a cycle of length at least $\min\{|V(G)|, 2\delta(G) - m\}$ passing through F.*

We here mention that the following theorem, which is the case where $m = 0$ in Theorem 1, was already proved by Locke in [6; Corollary 4.4] (for $k = 2$) and by Egawa and Glas and Locke in [3; Theorem 3] (for $k \geq 3$) :

Theorem C *Let k, d be integers with $d \geq k \geq 2$. Let G be a k-connected graph with $\delta(G) \geq d$, and let X be a subset of $V(G)$ with $|X| = k$. Then G has a cycle of length at least $\min\{|V(G)|, 2d\}$ passing through X.*

We also add that it was shown by Glas in [4] that for $k = 2$, the conclusion of Theorem C holds under the weaker assumption that $\sigma_2(G) \geq 2d$ (instead of the assumption that $\delta(G) \geq d$), and that it has recently been shown by Sakai in [7] and [8] that for $k \geq 3$, the same holds under the still weaker assumption that $\max\{\deg_G(x), \deg_G(y)\} \geq d$ for any two nonadjacent distinct vertices x, y of G.

Our notation is standard, and is mostly taken from [1] and [2]. Possible exceptions are as follows. Let G be a graph. For $x \in V(G)$, define $N_G(x) = \{y \in V(G) | xy \in E(G)\}$; thus $\deg_G(x) = |N_G(x)|$. For $X \subset V(G)$, we let $N_G(X) = \cup_{x \in X} N_G(x)$. For $X \subset V(G)$, we let $\langle X \rangle_G$ denote the graph induced by X in G, and define $G - X = \langle V(G) - X \rangle_G$. If X consists of a single vertex, say x, then we write $G - x$ for $G - X$. For $x, y \in V(G)$, a path having x as its initial vertex and y as its terminal vertex is called an (x, y)-*path*. For an (x, y)-path P, P^{-1} denotes the (y, x)-path obtained by tracing P in the inverse direction. For $x \in V(G)$ and $Y \subset V(G)$, an (x, y)-path P such that $V(P) \cap Y = \{y\}$ is called an (x, Y)-*path*; thus if $x \in Y$, then the path x of length 0 is the only (x, Y)-path. A subgraph is often identifed with its vertex set. For example, if H is a subgraph of G, then $N_G(H)$ means $N_G(V(H))$, and $G - H$ means $G - V(H)$.

If C is a cycle, we denote by $\overrightarrow{C}$ the cycle C with a given orientation. For $u, v \in V(C)$, we denote by $u\overrightarrow{C}v$ the segment of C obtained by tracing C from u to

v in the direction of $\overrightarrow{C}$ (if $u = v$, we let $u\overrightarrow{C}v = u$). Similarly, for a path P and $u, v \in V(P)$ such that u occurs before v on P, we let uPv denote the segment of P between u and v. If X is a cycle or a path, the length of X is denoted by $l(X)$.

A connected graph is called *separable* if it has a cut vertex; otherwise it is called *nonseparable*. For a separable graph G, a maximal nonseparable subgraph of G is called a *block* of G. A block of G which contains precisely one cut vertex of G is called an *endblock* of G. In the proof of Theorem 1, we make use of the following lemma proved in [3; Lemma 5]:

Lemma 1 *Let G be a nonseparable graph with at least two vertices, let u, v, x be vertices of G with $u \neq v$, and let d be an integer. Suppose that every vertex of G, except possibly u, v and one other vertex, has degree at least d. Suppose further that x has degree at least* $\min\{3, d\}$. *Then in G, there is a (u, v)-path which has length at least d and passes through x.*

2 Proof of Theorem 1

By Theorem C, Theorem 1 holds for $m = 0$. Thus let k, m, s be integers with $k \geq 2$, $m \geq 1$ and $0 \leq s \leq k$. We proceed by induction on s. By Corollary B, Theorem 1 holds for $s = 0$. Thus let $s > 0$, and assume that Theorem 1 is proved for $s - 1$. Let G, F be as in Theorem 1. Let C be a longest cycle such that $E(F) \subset E(C)$ and $|S(F) \cap V(C)| \geq s - 1$. By the induction hypothesis, $l(C) \geq \min\{|V(G)|, 2\delta(G) - m\}$. Thus if $S(F) \subset V(C)$, then the desired conclusion holds. Consequently we may assume $|S(F) \cap V(C)| = s - 1$. Write $S(F) - V(C) = \{y\}$. Let H be the connected component of $G - V(C)$ which contains y. We henceforth fix an orientation of C, and let $\overrightarrow{C}$ denote the cycle C with the orientation.

Write $E(F) \cup (S(F) \cap V(C)) = \{f_1, f_2, \ldots, f_{m+s-1}\}$, where $f_1, \ldots, f_{m+s-1}$ occur in this order along $\overrightarrow{C}$. For j with $1 \leq j \leq m + s - 1$, if $f_j \in E(F)$, let $f_j = p_j q_j$ (p_j precedes q_j on $\overrightarrow{C}$), and if $f_j \in S(F)$, let $p_j = q_j = f_j$. Define $S_j = q_j \overrightarrow{C} p_{j+1}$ (we take $p_{m+s} = p_1$).

Claim 2.1 *Let u, v be distinct vertices in $V(C) \cap N_G(H)$. Then the following hold.*
(a) $l(u\overrightarrow{C}v) \geq 1$.
(b) *If $E(u\overrightarrow{C}v) \cap E(F) = \emptyset$, then $l(u\overrightarrow{C}v) \geq 2$.*

Proof. Statement (a) immediately follows from the assumption that $u \neq v$, and (b) follows from the maximality of C. □

Claim 2.2 *There exist two distinct vertices $x_1, x_2 \in V(C) \cap N_G(H)$ such that*
(a) $E(x_1\overrightarrow{C}x_2) \cap E(F) = \emptyset$ *and* $(V(x_1\overrightarrow{C}x_2) - \{x_1, x_2\}) \cap S(F) = \emptyset$,
(b) *there is an (x_1, x_2)-path Q_0 in $\langle V(H) \cup \{x_1, x_2\}\rangle_G$ which passes through y, and*
(c) $(V(x_1\overrightarrow{C}x_2) - \{x_1, x_2\}) \cap N_G(H) = \emptyset$.

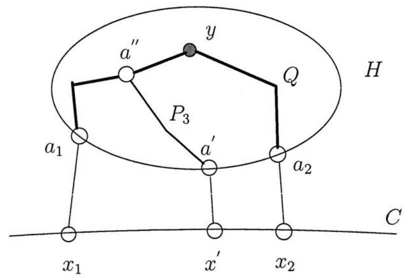

Fig. Claim 2.2

Proof. Since G is $(m+k)$-connected, $\delta(G) \geq m+k$, so $l(C) \geq \min\{|V(G)|, 2\delta(G) - m\} \geq m+k$. Thus by Menger's Theorem, there are $m+k$ $(y, V(C))$-paths which are pairwise disjoint except at y. Since $m+s-1 < m+k$, at least one of the segments S_j $(1 \leq j \leq m+s-1)$ contains the endvertices (different from y) of two of such paths, say P_1 and P_2. If we let x_1 and x_2 be the endvertices of P_1 and P_2, respectively, then the path $P_1^{-1}P_2$ is an (x_1, x_2)-path through y in $\langle V(H) \cup \{x_1, x_2\}\rangle_G$. Thus there exist two vertices x_1, $x_2 \in V(C) \cap N_G(H)$ which satisfy (a) and (b). Choose such vertices x_1, x_2 so that $x_1\overrightarrow{C}x_2$ is minimal, and let Q_0 be as in (b). Suppose $x' \in (V(x_1\overrightarrow{C}x_2) - \{x_1, x_2\}) \cap N_G(H)$, and let $a' \in V(H) \cap N_G(x')$. Set $Q = Q_0 - \{x_1, x_2\}$, and let a_1, a_2 be the endvertices of Q. We choose our notation so that $a_1x_1, a_2x_2 \in E(Q_0)$ (it is possible that $a_1 = a_2 = y$). Since H is connected, there exists an $(a', V(Q))$-path P_3 in H. Let a'' be the endvertex of P_3 on Q. If a'' is on a_1Qy, x' and x_2 satisfy (a) and (b); if a'' is on yQa_2, x_1 and x' satisfy (a) and (b). In either case, we get a contradiction to the minimality of $x_1\overrightarrow{C}x_2$. Thus x_1 and x_2 satisfy (c), as desired. $\square$

Throughout the rest of the proof of Theorem 1, we let x_1, x_2, Q_0 be as in Claim 2.2, and set $C_0 = Q_0x_2\overrightarrow{C}x_1$. Then C_0 passes through F by (a) and (b) of Claim 2.2.

Claim 2.3 *If $|V(C) \cap N_G(H)| \geq \delta(G)$, there exists a cycle of length at least $2\delta(G) - m$ passing through F.*

Proof. Write $V(C) \cap N_G(H) = \{x_1, \ldots, x_p\}$, where x_1, x_2 are as in Claim 2.2 and $x_1, \ldots, x_p$ occur in this order along $\overrightarrow{C}$. Set $I = \{i \,|1 \leq i \leq p, E(x_i\overrightarrow{C}x_{i+1}) \cap E(F) \neq \emptyset\}$ (we take $x_{p+1} = x_1$) and $J = \{1, 2, \ldots, p\} - I$. Note that $1 \in J$, and $l(Q_0)$ is at least two. Since $|I| \leq m$, it follows from Claim 2.1 that

$$\begin{aligned} l(C_0) =& \sum_{i=1}^{p} l(x_i\overrightarrow{C_0}x_{i+1}) = l(Q_0) + \sum_{i \in I} l(x_i\overrightarrow{C}x_{i+1}) + \sum_{i \in J - \{1\}} l(x_i\overrightarrow{C}x_{i+1}) \\ \geq& 2 + |I| + 2(|J| - 1) = 2(|I| + |J|) - |I| \geq 2\delta(G) - m. \quad \square \end{aligned}$$

We now divide the proof of Theorem 1 into two cases.

Case 1: H is separable. Let B_1, B_2 be two endblocks of H for which there exists a path in H which joins a vertex in $V(B_1)$ and a vertex in $V(B_2)$ and passes through y. Let b_1, b_2 be the cut vertices of H such that $b_1 \in V(B_1)$ and $b_2 \in V(B_2)$, respectively. Set

$$\begin{aligned} r &= |\ V(C) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))\ |, \\ q &= |(V(C) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))) \cup \{x_1, x_2\}|, \end{aligned}$$

and write

$$\begin{aligned} V(C) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2)) &= \{u_1, \ldots, u_r\}, \\ (V(C) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))) \cup \{x_1, x_2\} &= \{\bar{u}_1, \ldots, \bar{u}_q\} \end{aligned}$$

so that $u_1, \ldots, u_r$ and $\bar{u}_1, \ldots, \bar{u}_q$ occur in this order along $\overrightarrow{C}$, respectively (indices of u and $\bar{u}$ are to be read modulo r and q, repsecitvely. We start with a claim.

Claim 2.4

(a) *Let $\lambda \in \{1, 2\}$, and let $z \in V(C) \cap N_G(B_\lambda - b_\lambda)$. Then there exists a (z, b_λ)-path P in $\langle V(B_\lambda) \cup \{z\}\rangle_G$ which has length at least $\delta(G) - r + 1$. Further if $y \in V(B_\lambda - b_\lambda)$, we can choose P so that P passes through y.*

(b) *Let $z_1 \in V(C) \cap N_G(B_1 - b_1)$ and $z_2 \in V(C) \cap N_G(B_2 - b_2)$, and suppose that $z_1 \neq z_2$. Then there exists a (z_1, z_2)-path in $\langle V(H) \cup \{z_1, z_2\}\rangle_G$ which passes through y and has length at least $2(\delta(G) - r + 1)$.*

(c) *Let $\lambda \in \{1, 2\}$, and let $z, z' \in V(C) \cap N_G(B_\lambda)$ with $z \neq z'$, and suppose that $|V(B_\lambda) \cap N_G(\{z, z'\})| \geq 2$. Then there exists a (z, z')-path in $\langle V(B_\lambda) \cup \{z, z'\}\rangle_G$ which has length at least $\delta(G) - r + 2$.*

(d) *Let $z \in V(C) \cap N_G(B_1 - b_1) \cap N_G(B_2 - b_2)$ and $x \in V(C) \cap N_G(H - (B_1 - b_1) - (B_2 - b_2))$, and suppose that $z \neq x$. Then there exists a (z, x)-path in $\langle V(H) \cup \{z, x\}\rangle_G$ which passes through y and has length at least $\delta(G) - r + 2$.*

Proof. (a) Take $a \in V(B_\lambda - b_\lambda) \cap N_G(z)$. Note that each vertex in $V(B_\lambda - b_\lambda)$ has degree at least $\delta(G) - r$ in B_λ. Hence by Lemma 1, B_λ contains an (a, b_λ)-path Q with length at least $\delta(G) - r$ and, in the case where $y \in V(B_\lambda - b_\lambda)$, we can choose Q so that Q passes through y. Now if we let $P = zaQ$, then P has the desired properties.

(b) By (a), for each $\lambda = 1, 2$, there exists a (z_λ, b_λ)-path P_λ in $\langle V(B_\lambda) \cup \{z_\lambda\}\rangle_G$ with length at least $\delta(G) - r + 1$ such that P_λ passes through y in the case where $y \in V(B_\lambda - b_\lambda)$. Let R be a (b_1, b_2)-path in $H - (B_1 - b_1) - (B_2 - b_2)$. By the choice of B_1 and B_2, we can choose R so that $y \in V(R)$ in the case where $y \notin V(B_1 - b_1) \cup V(B_2 - b_2)$. Then the path $P_1 R P_2^{-1}$ has the desired properties.

(c) By the assumption that $|V(B_\lambda) \cap N_G(\{z, z'\})| \geq 2$, we can take $a \in V(B_\lambda) \cap N_G(z)$ and $a' \in V(B_\lambda) \cap N_G(z')$ so that $a \neq a'$. By Lemma 1, B_λ has an (a, a')-path Q with length at least $\delta(G) - r$. Then the path $zaQa'z'$ has the desired property.

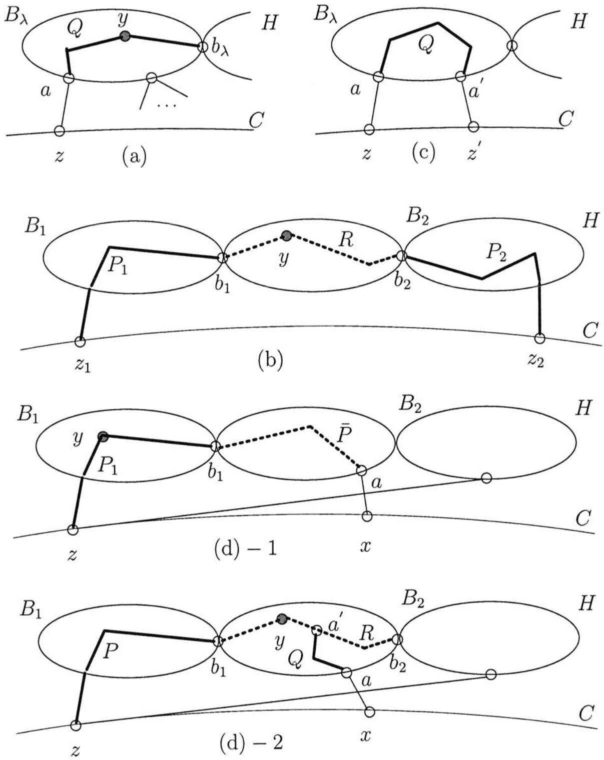

Fig. Claim 2.4

(d) Take $a \in V(H - (B_1 - b_1) - (B_2 - b_2)) \cap N_G(x)$. First assume $y \in V(B_1 - b_1) \cup V(B_2 - b_2)$. By symmetry, we may assume $y \in V(B_1 - b_1)$. Then by (a), there exists a (z, b_1)-path P in $\langle V(B_1) \cup \{z\}\rangle_G$ which passes through y and has length at least $\delta(G) - r + 1$. There also exists a (b_1, a)-path $\bar{P}$ in $H - (B_1 - b_1) - (B_2 - b_2)$. Then the path $P\bar{P}ax$ has the desired properties.

Next assume $y \in V(H - (B_1 - b_1) - (B_2 - b_2))$. By the choice of B_1 and B_2, we can take a (b_1, b_2)-path R in $H - (B_1 - b_1) - (B_2 - b_2)$ passing through y. Since $H - (B_1 - b_1) - (B_2 - b_2)$ is connected, there exists an $(a, V(R))$-path Q in $H - (B_1 - b_1) - (B_2 - b_2)$. Let a' be the endvertex of Q on R. Then at least one of the two paths b_1Ra' and $a'Rb_2$ passes through y. We may assume $y \in V(b_1Ra')$. By (a), there exists a (z, b_1)-path P in $\langle V(B_1) \cup \{z\}\rangle_G$ with length at least $\delta(G) - r + 1$. Then the path $Pb_1Ra'Q^{-1}ax$ has the desired properties. □

Let x_1, x_2, Q_0, C_0 be as in Claim 2.2 and the paragraph following the proof of

Claim 2.2. Also define

$$\begin{aligned}M =&\{(u_i, u_{i+1}) |\, V(B_\lambda - b_\lambda) \cap N_G(\{u_i, u_{i+1}\}) \neq \emptyset \text{ for each } \lambda = 1, 2,\\ &E(u_i \overrightarrow{C} u_{i+1}) \cap E(F) = \emptyset, (V(u_i \overrightarrow{C} u_{i+1}) - \{u_i, u_{i+1}\}) \cap S(F) = \emptyset\},\\ \widetilde{M} =&\{(u_i, u_{i+1}) |\, V(B_\lambda - b_\lambda) \cap N_G(\{u_i, u_{i+1}\}) \neq \emptyset \text{ for each } \lambda = 1, 2,\\ &E(u_i \overrightarrow{C} u_{i+1}) \cap E(F) = \emptyset, |(V(u_i \overrightarrow{C} u_{i+1}) - \{u_i, u_{i+1}\}) \cap S(F)| = 1\}.\end{aligned}$$

Further for $\lambda = 1, 2$, define

$$\begin{aligned}M_\lambda =&\{(u_i, u_{i+1}) |\, |V(B_\lambda - b_\lambda) \cap N_G(\{u_i, u_{i+1}\})| \geq 2,\\ &V(B_{3-\lambda} - b_{3-\lambda}) \cap N_G(\{u_i, u_{i+1}\}) = \emptyset,\\ &E(u_i \overrightarrow{C} u_{i+1}) \cap E(F) = \emptyset, (V(u_i \overrightarrow{C} u_{i+1}) - \{u_i, u_{i+1}\}) \cap S(F) = \emptyset\}.\end{aligned}$$

Claim 2.5 *Suppose that one of the following four conditions is satisfied;*

(a) $|M| \geq 1$;

(b) *there exists* $(u_i, u_{i+1}) \in \widetilde{M}$ *such that* $x_1 \overrightarrow{C} x_2 \not\subset u_i \overrightarrow{C} u_{i+1}$;

(c) *there exist* $(u_{i_1}, u_{i_1+1}), (u_{i_2}, u_{i_2+1}) \in M_1 \cup M_2$, $i_1 \neq i_2$, *such that* $x_1 \overrightarrow{C} x_2 \not\subset u_{i_1} \overrightarrow{C} u_{i_1+1}$ *and* $x_1 \overrightarrow{C} x_2 \not\subset u_{i_2} \overrightarrow{C} u_{i_2+1}$; *or*

(d) *there exists* $(u_i, u_{i+1}) \in \widetilde{M}$ *such that* $u_i, u_{i+1} \in N_G(B_1 - b_1) \cap N_G(B_2 - b_2)$.

Then there exists a cycle of length at least $2\delta(G) - m$ *passing through* F.

Proof. (a) Let $(u_i, u_{i+1}) \in M$. By symmetry, we may assume $u_i \in V(C) \cap N_G(B_1 - b_1)$ and $u_{i+1} \in V(C) \cap N_G(B_2 - b_2)$. By Claim 2.4 (b), there exists a (u_i, u_{i+1})-path Q_1 in $\langle V(H) \cup \{u_i, u_{i+1}\}\rangle_G$ which passes through y and has length at least $2(\delta(G) - r + 1)$. Now set $C_1 = Q_1 u_{i+1} \overrightarrow{C} u_i$. Then C_1 passes through F and, arguing as in the proof of Claim 2.3, we obtain

$$\begin{aligned}l(C_1) &= l(Q_1) + \sum\nolimits_{1 \leq h \leq r, h \neq i} l(u_h \overrightarrow{C} u_{h+1})\\ &\geq 2(\delta(G) - r + 1) + 2(r - 1) - m = 2\delta(G) - m\end{aligned}$$

by Claim 2.1.

(b) As in (a), we may assume $u_i \in V(C) \cap N_G(B_1 - b_1)$ and $u_{i+1} \in V(C) \cap N_G(B_2 - b_2)$. Let Q_1 be as in (a). Note that both $u_i \overrightarrow{C} u_{i+1} - \{u_i, u_{i+1}\}$ and $Q_1 - \{u_i, u_{i+1}\}$ contain precisely one vertex of $S(F)$. Hence it follows from the maximality of C that $l(u_i \overrightarrow{C} u_{i+1}) \geq l(Q_1) \geq 2(\delta(G) - r + 1)$. Note that $q \geq r$. Let $u_i = \bar{u}_{h_1}, x_1 = \bar{u}_{h_2}$. Then $u_{i+1} = \bar{u}_{h_1+1}$ by the assumption that $x_1 \overrightarrow{C} x_2 \not\subset u_i \overrightarrow{C} u_{i+1}$ and Claim 2.2 (c), and $x_2 = \bar{u}_{h_2+1}$ by Claim 2.2 (c). Recall that C_0 passes through F. Further since the length of Q_0 is at least two, we now obtain

$$\begin{aligned}l(C_0) &= l(Q_0) + l(u_i \overrightarrow{C} u_{i+1}) + \sum\nolimits_{1 \leq h \leq q, h \neq h_1, h_2} l(\bar{u}_h \overrightarrow{C} \bar{u}_{h+1})\\ &\geq 2 + 2(\delta(G) - r + 1) + 2(q - 2) - m \geq 2\delta(G) - m.\end{aligned}$$

(c) By Claim 2.4 (c) and the maximality of C, $l(u_{i_1}\overrightarrow{C}u_{i_1+1}), l(u_{i_2}\overrightarrow{C}u_{i_2+1}) \geq \delta(G) - r + 2$. Let $u_{i_1} = \bar{u}_{h_1}, u_{i_2} = \bar{u}_{h_2}, x_1 = \bar{u}_{h_3}$. Then as in the proof of (b), $u_{i_1+1} = \bar{u}_{h_1+1}, u_{i_2+1} = \bar{u}_{h_2+1}, x_2 = \bar{u}_{h_3+1}$ by the assumption that $x_1\overrightarrow{C}x_2 \not\subset u_{i_1}\overrightarrow{C}u_{i_1+1}$ and $x_1\overrightarrow{C}x_2 \not\subset u_{i_2}\overrightarrow{C}u_{i_2+1}$ and by Claim 2.2 (c). Therefore

$$
\begin{aligned}
l(C_0) &= l(Q_0) + l(u_{i_1}\overrightarrow{C}u_{i_1+1}) + l(u_{i_2}\overrightarrow{C}u_{i_2+1}) \\
&\qquad + \sum\nolimits_{1 \leq h \leq q, h \neq h_1, h_2, h_3} l(\bar{u}_h\overrightarrow{C}\bar{u}_{h+1}) \\
&\geq 2 + 2(\delta(G) - r + 2) + 2(q - 3) - m \geq 2\delta(G) - m.
\end{aligned}
$$

(d) In view of (b), we may assume $x_1\overrightarrow{C}x_2 \subset u_i\overrightarrow{C}u_{i+1}$. Since $u_i\overrightarrow{C}u_{i+1} \neq x_1\overrightarrow{C}x_2$ by the definition of $\widetilde{M}$ and Claim 2.2 (a), this implies that at least one of x_1 and x_2 belongs to $V(u_i\overrightarrow{C}u_{i+1}) - \{u_i, u_{i+1}\}$. By symmetry, we may assume $x_2 \in V(u_i\overrightarrow{C}u_{i+1}) - \{u_i, u_{i+1}\}$. By the definition of $u_1, \ldots, u_r$, this means $x_2 \in N_G(H - (B_1 - b_1) - (B_2 - b_2))$. Hence by Claim 2.4 (d), there exist a (u_i, x_2)-path P' in $\langle V(H) \cup \{u_i, x_2\}\rangle_G$ and an (x_2, u_{i+1})-path P'' in $\langle V(H) \cup \{x_2, u_{i+1}\}\rangle_G$ which pass through y and have length at least $\delta(G) - r + 2$. Since $|(V(u_i\overrightarrow{C}u_{i+1}) - \{u_i, u_{i+1}\}) \cap S(F)| = 1$ by the definition of $\widetilde{M}$, we have $(V(u_i\overrightarrow{C}x_2) - \{u_i, x_2\}) \cap S(F) = \emptyset$ or $(V(x_2\overrightarrow{C}u_{i+1}) - \{x_2, u_{i+1}\}) \cap S(F) = \emptyset$. We may assume $(V(u_i\overrightarrow{C}x_2) - \{u_i, x_2\}) \cap S(F) = \emptyset$ (we do not make use of x_1 in the rest of the proof of the claim; so the roles of u_i and u_{i+1} are symmetric). By the maximality of C, $l(x_2\overrightarrow{C}u_{i+1}) \geq l(P'') \geq \delta(G) - r + 2$. Now set $C_2 = P'x_2\overrightarrow{C}u_i$. Then C_2 passes through F, and

$$
\begin{aligned}
l(C_2) &= l(P') + l(x_2\overrightarrow{C}u_{i+1}) + \sum\nolimits_{1 \leq h \leq r, h \neq i} l(u_h\overrightarrow{C}u_{h+1}) \\
&\geq 2(\delta(G) - r + 2) + 2(r - 1) - m \geq 2\delta(G) - m. \quad \square
\end{aligned}
$$

We return to the proof of the theorem for Case 1. In view of Claim 2.5 (a), we may assume that $M = \emptyset$.

Claim 2.6 *Let $1 \leq j \leq m + s - 1$, and suppose that $V(S_j) \cap N_G(B_1 - b_1) \neq \emptyset$ and $V(S_j) \cap N_G(B_2 - b_2) \neq \emptyset$. Then*

$$
\begin{aligned}
V(S_j) \cap N_G(B_1 - b_1) &= V(S_j) \cap N_G(B_2 - b_2) \text{ and} \\
|V(S_j) \cap N_G(B_1 - b_1)| &= |V(S_j) \cap N_G(B_2 - b_2)| = 1.
\end{aligned}
$$

Proof. By way of contradiction, suppose that $|V(S_j) \cap N_G(B_1 - b_1)| \geq 2$ or $|V(S_j) \cap N_G(B_2 - b_2)| \geq 2$ or $V(S_j) \cap N_G(B_1 - b_1) \neq V(S_j) \cap N_G(B_2 - b_2)$. Then there exist $w_1, w_2 \in V(S_j)$ with $w_1 \neq w_2$ and $w_1\overrightarrow{C}w_2 \subset S_j$ such that either $w_1 \in N_G(B_1 - b_1)$ and $w_2 \in N_G(B_2 - b_2)$ or $w_1 \in N_G(B_2 - b_2)$ and $w_2 \in N_G(B_1 - b_1)$. We may assume that we have chosen w_1 and w_2 so that $w_1\overrightarrow{C}w_2$ is minimal. Then $(V(w_1\overrightarrow{C}w_2) - \{w_1, w_2\}) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2)) = \emptyset$, which implies $(w_1, w_2) \in M$ because $w_1\overrightarrow{C}w_2 \subset S_j$. This contradicts the assumption that $M = \emptyset$. $\square$

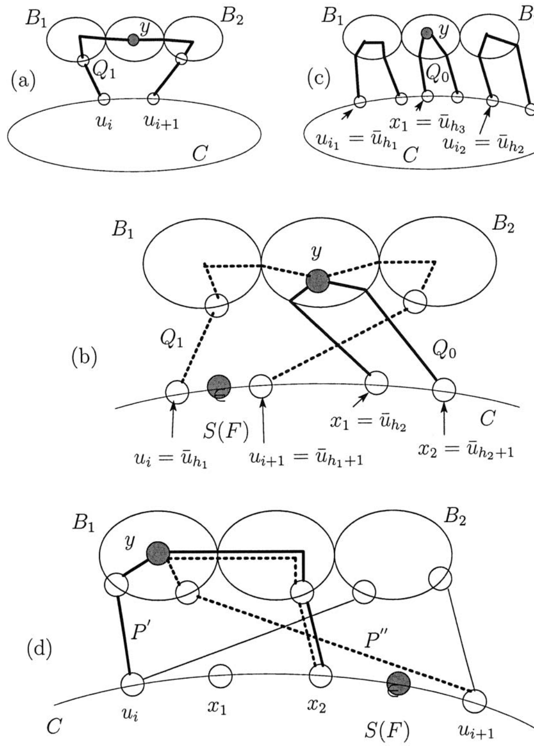

Fig. Claim 2.5

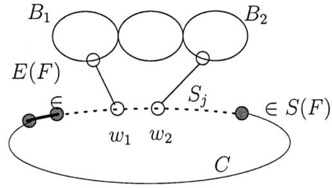

Fig. Claim 2.6

Now define

$$D_\lambda \;=\; \{u_i \,|\, |V(B_\lambda - b_\lambda) \cap N_G(u_i)| \geq 2\}$$

for $\lambda = 1, 2$. Then for each $\lambda = 1, 2$,

$$\begin{aligned}&\Big(((V(C) \cap N_G(B_\lambda - b_\lambda)) - D_\lambda) \cap N_G(a)\Big)\\ &\cap\Big(((V(C) \cap N_G(B_\lambda - b_\lambda)) - D_\lambda) \cap N_G(b)\Big) = \emptyset \qquad (1)\end{aligned}$$

for every $a, b \in V(B_\lambda - b_\lambda)$ with $a \neq b$. We divide the proof into two subcases according to which of the two quantities $|(V(C) \cap N_G(B_\lambda - b_\lambda)) - D_\lambda|$ and $|V(B_\lambda - b_\lambda)|$ is the larger.

Subcase 1.1 $|(V(C) \cap N_G(B_\lambda - b_\lambda)) - D_\lambda| \geq |V(B_\lambda - b_\lambda)|$ for some $\lambda \in \{1, 2\}$.

By symmetry, we may assume $|(V(C) \cap N_G(B_1 - b_1)) - D_1| \geq |V(B_1 - b_1)|$. If $|D_1| \geq \delta(G) - |V(B_1 - b_1)|$, then $|V(C) \cap N_G(H)| \geq |V(C) \cap N_G(B_1 - b_1)| = |(V(C) \cap N_G(B_1 - b_1)) - D_1| + |D_1| \geq |V(B_1 - b_1)| + (\delta(G) - |V(B_1 - b_1)|) = \delta(G)$, and hence we obtain a cycle with the desired properties by Claim 2.3. Thus we may assume $\delta(G) - |V(B_1 - b_1)| > |D_1|$. Then for every $a \in V(B_1 - b_1)$, $|((V(C) \cap N_G(B_1 - b_1)) - D_1) \cap N_G(a)| \geq |V(C) \cap N_G(a)| - |D_1| \geq \delta(G) - (|V(B_1)| - 1) - |D_1| > 0$ because $N_G(a) \subset V(B_1 - a) \cup (V(C) \cap N_G(a))$. Hence by (1),

$$\begin{aligned}&|V(C) \cap N_G(B_1 - b_1)|\\ &= |(V(C) \cap N_G(B_1 - b_1)) - D_1| + |D_1|\\ &\geq \{\delta(G) - (|V(B_1)| - 1) - |D_1|\}|V(B_1 - b_1)| + |D_1|\\ &= \delta(G) - 1 - |D_1|\\ &+(\delta(G) - |V(B_1)| - |D_1|)(|V(B_1)| - 2) + |D_1|\\ &\geq \delta(G) - 1. \qquad (2)\end{aligned}$$

Now if $V(C) \cap N_G(B_2 - b_2) \not\subset V(C) \cap N_G(B_1 - b_1)$, then $|V(C) \cap N_G(H)| \geq \delta(G)$ by (2), and hence the desired conclusion follows from Claim 2.3. Thus we may assume $V(C) \cap N_G(B_2 - b_2) \subset V(C) \cap N_G(B_1 - b_1)$. By Claim 2.6, this implies $|V(S_j) \cap N_G(B_2 - b_2)| \leq 1$ for each $1 \leq j \leq m + s - 1$. Since $|V(C) \cap N_G(B_2 - b_2)| \geq m + k - 1$ by the assumption that G is $(m + k)$-connected and since $s \leq k$, this forces $s = k$ and $|V(S_j) \cap N_G(B_2 - b_2)| = 1$ for each j. By Claim 2.6, this in turn implies $|V(S_j) \cap N_G(B_1 - b_1)| = 1$ for each j, and hence $|V(C) \cap N_G(B_1 - b_1)| = m + k - 1$. Since $\delta(G) \geq m + k$, this together with (2) implies $\delta(G) = m + k$, and hence

$$|V(C) \cap N_G(H)| \geq m + k = \delta(G),$$

by the assumption that G is $(m + k)$-connected. Therefore we obtain a desired cycle by Claim 2.3.

Subcase 1.2 $|(V(C) \cap N_G(B_\lambda - b_\lambda)) - D_\lambda| < |V(B_\lambda - b_\lambda)|$ for each $\lambda \in \{1, 2\}$.

Fix $\lambda \in \{1,2\}$ for the moment. By (1), there is a vertex $g_\lambda \in V(B_\lambda - b_\lambda)$ satisfying

$$\{(V(C) \cap N_G(B_\lambda - b_\lambda)) - D_\lambda\} \cap N_G(g_\lambda) = \emptyset,$$

that is to say,

$$V(C) \cap N_G(g_\lambda) \subset D_\lambda. \tag{3}$$

Since G is $(m+k)$-connected, there exist $m+k$ $(g_\lambda, V(C))$-paths $P_{\lambda,1}, \ldots, P_{\lambda,m+k}$ which are pairwise disjoint except at g_λ. For each h, let $t_{\lambda,h}$ denote the endvertex of $P_{\lambda,h}$ differrent from g_λ. Thus $t_{\lambda,h} \in V(P_{\lambda,h}) \cap V(C)$. At most one of the paths $P_{\lambda,h}$ $(1 \le h \le m+k)$ passes through b_λ. We choose our labeling so that $b_\lambda \notin V(P_{\lambda,h})$ for each $1 \le h \le m+k-1$. Then $t_{\lambda,h} \in V(C) \cap N_G(B_\lambda - b_\lambda)$ for each $1 \le h \le m+k-1$.

Claim 2.7 *Let $1 \le j \le m+s-1$ and $1 \le h_1, h_2 \le m+k-1$ with $h_1 \ne h_2$, and suppose that $t_{\lambda,h_1}\overrightarrow{C}t_{\lambda,h_2} \subset S_j$. Then there exists $(u_i, u_{i+1}) \in M_\lambda$ such that $u_i\overrightarrow{C}u_{i+1} \subset t_{\lambda,h_1}\overrightarrow{C}t_{\lambda,h_2}$.*

Proof. We first show that $|V(B_\lambda - b_\lambda) \cap N_G(\{t_{\lambda,h_1}, t_{\lambda,h_2}\})| \ge 2$. If t_{λ,h_1} or t_{λ,h_2}, say t_{λ,h_1}, belongs to $N_G(g_\lambda)$, then by (3) and the definition of D_λ, $|V(B_\lambda - b_\lambda) \cap N_G(\{t_{\lambda,h_1}, t_{\lambda,h_2}\})| \ge |V(B_\lambda - b_\lambda) \cap N_G(t_{\lambda,h_1})| \ge 2$; if $t_{\lambda,h_1}, t_{\lambda,h_2} \notin N_G(g_\lambda)$, then letting a_1 be the vertex preceding t_{λ,h_1} on P_{λ,h_1} and a_2 be the vertex preceding t_{λ,h_2} on P_{λ,h_2}, we obtain $|V(B_\lambda - b_\lambda) \cap N_G(\{t_{\lambda,h_1}, t_{\lambda,h_2}\})| \ge |\{a_1, a_2\}| \ge 2$ by the choice of $P_{\lambda,1}, \ldots, P_{\lambda,m+k-1}$. Choose $w_1, w_2 \in V(t_{\lambda,h_1}\overrightarrow{C}t_{\lambda,h_2}) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))$ with $w_1 \ne w_2$, $w_1\overrightarrow{C}w_2 \subset t_{\lambda,h_1}\overrightarrow{C}t_{\lambda,h_2}$, $|(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap N_G(\{w_1, w_2\})| \ge 2$ and $V(B_\lambda - b_\lambda) \cap N_G(\{w_1, w_2\}) \ne \emptyset$ so that $w_1\overrightarrow{C}w_2$ is minimal. By the symmetry of the roles of w_1 and w_2, we may assume $V(B_\lambda - b_\lambda) \cap N_G(w_1) \ne \emptyset$. Suppose that $(V(w_1\overrightarrow{C}w_2) - \{w_1, w_2\}) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2)) \ne \emptyset$, and take $w \in (V(w_1\overrightarrow{C}w_2) - \{w_1, w_2\}) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))$. If $V(B_{3-\lambda} - b_{3-\lambda}) \cap N_G(w) \ne \emptyset$, then we have $|(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap N_G(\{w_1, w\})| \ge |V(B_\lambda - b_\lambda) \cap N_G(w_1)| + |V(B_{3-\lambda} - b_{3-\lambda}) \cap N_G(w)| \ge 2$ and $V(B_\lambda - b_\lambda) \cap N_G(\{w_1, w\}) \supset V(B_\lambda - b_\lambda) \cap N_G(w_1) \ne \emptyset$, which contradicts the minimality of $w_1\overrightarrow{C}w_2$. If $V(B_{3-\lambda} - b_{3-\lambda}) \cap N_G(w) = \emptyset$, then $V(B_\lambda - b_\lambda) \cap N_G(w) \ne \emptyset$ and hence we have $V(B_\lambda - b_\lambda) \cap N_G(\{w_1, w\}) \ne \emptyset$ and $V(B_\lambda - b_\lambda) \cap N_G(\{w, w_2\}) \ne \emptyset$ and, from $|(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap N_G(\{w_1, w_2\})| \ge 2$, we get $|(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap N_G(\{w_1, w\})| \ge 2$ or $|(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap N_G(\{w, w_2\})| \ge 2$, which again contradicts the minimality of $w_1\overrightarrow{C}w_2$. Thus $(V(w_1\overrightarrow{C}w_2) - \{w_1, w_2\}) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2)) = \emptyset$, and hence there exists i with $1 \le i \le r$ such that $w_1 = u_i$ and $w_2 = u_{i+1}$. Since we are assuming $M = \emptyset$ (see the paragraph preceding Claim 2.6), this forces $V(B_{3-\lambda} - b_{3-\lambda}) \cap N_G(\{w_1, w_2\}) = \emptyset$ because $w_1\overrightarrow{C}w_2 \subset S_j$. Consequently $(w_1, w_2) \in M_\lambda$, as desired. □

We are now in a position to complete the discussion for Case 1. If $|M_1| + |M_2| \ge 3$, then some two members of $M_1 \cup M_2$ satisfy the condition in Claim 2.5 (c), and hence we obtain a desired cycle by Claim 2.5 (c).

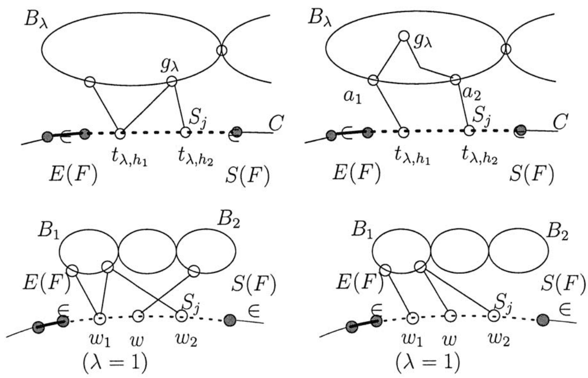

Fig. Claim 2.7

Thus we may assume that $|M_1| + |M_2| \leq 2$. For convenience, for each $1 \leq j \leq m + s - 1$, we define

$$T_j = \begin{cases} S_j & (\text{ if } f_{j+1} \in E(F)) \\ S_j - f_{j+1} & (\text{ if } f_{j+1} \in S(F)) \end{cases}$$

(when $j = m + s - 1$, we take $f_{m+s} = f_1$). The reason why we consider T_j besides S_j is that $V(C)$ is the disjoint union of the $V(T_j)$ $(1 \leq j \leq m + s - 1)$, while $V(S_j) \cap V(S_{j+1}) \neq \emptyset$ for j with $f_{j+1} \in S(F)$. We first consider the case where $M_\lambda = \emptyset$ for some $\lambda \in \{1, 2\}$. We may assume $M_1 = \emptyset$. Let $t_{1,1}, \ldots, t_{1,m+k-1}$ be as in the first paragraph of Subcase 1.2. Since $M_1 = \emptyset$, it follows from Claim 2.7 that $t_{1,1}, \ldots, t_{1,m+k-1}$ belong to distinct $V(T_j)$ $(j = 1, \ldots, m + s - 1)$. Since $k \geq s$, this implies that

$$s = k \tag{4}$$

and $V(T_j) \cap N_G(B_1 - b_1) \neq \emptyset$ for each j. By Claim 2.6, this implies $|V(T_j) \cap N_G(B_2 - b_2)| \leq 1$ for each j. Since G is $(m+k)$-connected, this forces $|V(T_j) \cap N_G(B_2 - b_2)| = 1$ for each j. Consequently again by Claim 2.6, $V(T_j) \cap N_G(B_1 - b_1) = V(T_j) \cap N_G(B_2 - b_2)$ and $|V(T_j) \cap N_G(B_1 - b_1)| = |V(T_j) \cap N_G(B_2 - b_2)| = 1$ for each j. Now take j such that $f_j \in S(F) \cap V(C)$ (note that $|S(F) \cap V(C)| = s - 1 = k - 1 \geq 1$ by (4)). Write $V(T_{j-1}) \cap N_G(B_1 - b_1) = V(T_{j-1}) \cap N_G(B_2 - b_2) = \{z\}$ and $V(T_j) \cap N_G(B_1 - b_1) = V(T_j) \cap N_G(B_2 - b_2) = \{z'\}$ (when $j = 1$, we take $T_0 = T_{m+s-1}$). Then $z' \neq f_j$ by Claim 2.6, and hence $(z, z') \in \widetilde{M}$. Note that (z, z') satisfies the condition in Claim 2.5 (d). Therefore we obtain a desired cycle by Claim 2.5 (d).

We are left with the case where $|M_1| = |M_2| = 1$. Write $M_1 = \{(u_i, u_{i+1})\}$ and $M_2 = \{(u_{i'}, u_{i'+1})\}$, and let j_1, j_2 be the indices such that $u_i \overrightarrow{C} u_{i+1} \subset S_{j_1}$ and $u_{i'} \overrightarrow{C} u_{i'+1} \subset S_{j_2}$. By Claim 2.6, $V(S_{j_1}) \cap N_G(B_2 - b_2) = \emptyset$ and $V(S_{j_2}) \cap N_G(B_1 - b_1) = \emptyset$; in particular, $j_1 \neq j_2$. Since $M_1 = \{(u_i, u_{i+1})\}$, we see from Claim 2.7 that

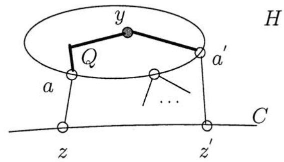

Fig. Claim 2.8

$|V(T_{j_1}) \cap \{t_{1,1}, \dots, t_{1,m+k-1}\}| \le 2$ and $|V(T_j) \cap \{t_{1,1}, \dots, t_{1,m+k-1}\}| \le 1$ for each j with $j \ne j_1, j_2$. Since $k \ge s$ and $V(T_{j_2}) \cap N_G(B_1 - b_1) = \emptyset$, this implies (4) holds, $|V(T_{j_1}) \cap \{t_{1,1}, \dots, t_{1,m+k-1}\}| = 2$, and $|V(T_j) \cap \{t_{1,1}, \dots, t_{1,m+k-1}\}| = 1$ for each j with $j \ne j_1, j_2$; in particular, $V(T_j) \cap N_G(B_1 - b_1) \ne \emptyset$ for each j with $j \ne j_2$. Similarly $V(T_j) \cap N_G(B_2 - b_2) \ne \emptyset$ for each j with $j \ne j_1$. By Claim 2.6, this implies $V(T_j) \cap N_G(B_1 - b_1) = V(T_j) \cap N_G(B_2 - b_2)$ and $|V(T_j) \cap N_G(B_1 - b_1)| = |V(T_j) \cap N_G(B_2 - b_2)| = 1$ for each $j \ne j_1, j_2$. Now take j such that $f_j \in S(F) \cap V(C)$. Let z be the vertex in $V(T_{j-1}) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))$ closest to f_j on S_{j-1} (when $j = 1$, we take $T_0 = T_{m+s-1}$ and $S_0 = S_{m+s-1}$), and let z' be the vertex in $V(T_j) \cap (N_G(B_1 - b_1) \cup N_G(B_2 - b_2))$ closest to f_j on T_j. We have $z \in N_G(B_1 - b_1)$ and $z' \in N_G(B_2 - b_2)$, or $z \in N_G(B_2 - b_2)$ and $z' \in N_G(B_1 - b_1)$. By Claim 2.6, this implies $z' \ne f_j$, and hence $(z, z') \in \widetilde{M}$. Note that either the two members of $M_1 \cup M_2$ satisfy the condition in Claim 2.5 (c), or else (z, z') satisfies the condition in Claim 2.5 (b). Therefore we obtain a desired cycle by (c) or (b) of Claim 2.5.

Case 2 H is nonseparable.

Let $r' = |V(C) \cap N_G(H)|$ and $V(C) \cap N_G(H) = \{v_1, \dots, v_{r'}\}$ so that $v_1, \dots, v_{r'}$ occur in this order along C (indices are to be read modulo r'). Define

$$\begin{aligned} M' =& \{(v_i, v_{i+1}) \mid |V(H) \cap N_G(\{v_i, v_{i+1}\})| \ge 2, E(v_i \overrightarrow{C} v_{i+1}) \cap E(F) = \emptyset, \\ & (V(v_i \overrightarrow{C} v_{i+1}) - \{v_i, v_{i+1}\}) \cap S(F) = \emptyset\}, \\ \widetilde{M'} =& \{(v_i, v_{i+1}) \mid |V(H) \cap N_G(\{v_i, v_{i+1}\})| \ge 2, E(v_i \overrightarrow{C} v_{i+1}) \cap E(F) = \emptyset, \\ & |(V(v_i \overrightarrow{C} v_{i+1}) - \{v_i, v_{i+1}\}) \cap S(F)| \le 1\} \end{aligned}$$

(so $M' \subset \widetilde{M'}$).

Claim 2.8 *Let $z, z' \in V(C) \cap N_G(H)$ with $z \ne z'$, and suppose that $|V(H) \cap N_G(\{z, z'\})| \ge 2$. Then there exists a (z, z')-path in $\langle V(H) \cap \{z, z'\} \rangle_G$ which passes through y and has length at least $\delta(G) - r' + 2$.*

Proof. By the assumption that $|V(H) \cap N_G(\{z, z'\})| \ge 2$, we can take $a \in V(H) \cap N_G(z)$ and $a' \in V(H) \cap N_G(z')$ so that $a \ne a'$. By Lemma 1, H has an (a, a')-path Q which passes through y and has length at least $\delta(G) - r'$. Then the path $zaQa'z'$ has the desired properties. □

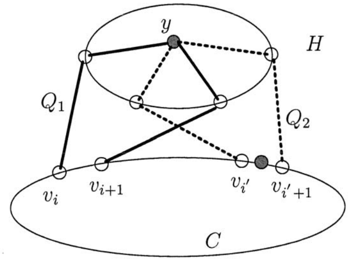

Fig. Claim 2.9

Claim 2.9 *If* $|M'| \geq 1$ *and* $|\widetilde{M}'| \geq 2$*, then there exists a cycle of length at least* $2\delta(G) - m$ *passing through* F*.*

Proof. Let $(v_i, v_{i+1}) \in M'$ and $(v_{i'}, v_{i'+1}) \in \widetilde{M}'$ with $i \neq i'$. By Claim 2.8, there exists a (v_i, v_{i+1})-path Q_1 in $\langle V(H) \cup \{v_i, v_{i+1}\}\rangle_G$ which passes through y and has length at least $\delta(G) - r' + 2$, and there exists a $(v_{i'}, v_{i'+1})$-path Q_2 in $\langle V(H) \cup \{v_{i'}, v_{i'+1}\}\rangle_G$ which passes through y and has length at least $\delta(G) - r' + 2$. By the maximality of C, $l(v_{i'}\overrightarrow{C}v_{i'+1}) \geq l(Q_2) \geq \delta(G) - r' + 2$. Set $C_1 = Q_1 v_{i+1}\overrightarrow{C}v_i$. Then C_1 passes through F and, arguing as in the proof of Claim 2.5, we obtain

$$\begin{aligned} l(C_1) &= l(Q_1) + l(v_{i'}\overrightarrow{C}v_{i'+1}) + \sum\nolimits_{1 \leq h \leq r', h \neq i, i'} l(v_h\overrightarrow{C}v_{h+1}) \\ &\geq (\delta(G) - r' + 2) + (\delta(G) - r' + 2) + 2(r' - 2) - m \\ &= 2\delta(G) - m. \quad \square \end{aligned}$$

Now define

$$D' = \{v_i \mid |V(H) \cap N_G(v_i)| \geq 2\}.$$

Then

$$\Big(((V(C) \cap N_G(H)) - D') \cap N_G(a)\Big) \cap \Big(((V(C) \cap N_G(H)) - D') \cap N_G(b)\Big) = \emptyset \qquad (5)$$

for every $a, b \in V(H)$ with $a \neq b$. We divide the proof into two subcases according to which of the two quantities $|(V(C) \cap N_G(H)) - D'|$ and $|V(H)|$ is the larger.

Subcase 2.1 $|(V(C) \cap N_G(H)) - D'| \geq |V(H)|$ (this includes the case where $|V(H)| = 1$).

If $|D'| \geq \delta(G) - |V(H)|$, then $|V(C) \cap N_G(H)| = |(V(C) \cap N_G(H)) - D'| + |D'| \geq |V(H)| + (\delta(G) - |V(H)|) = \delta(G)$, and hence we obtain a desired cycle by Claim 2.3. Thus we may assume $\delta(G) - |V(H)| > |D'|$. Then for every $a \in V(H)$, $|((V(C) \cap$

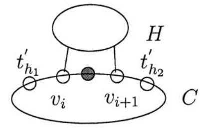

Fig. Claim 2.10

$N_G(H)) - D^{'}) \cap N_G(a)| \geq |V(C) \cap N_G(a)| - |D^{'}| \geq \delta(G) - (|V(H)| - 1) - |D^{'}| > 0$ because $N_G(a) \subset V(H-a) \cup (V(C) \cap N_G(a))$. Hence by (5),

$$\begin{aligned} &|V(C) \cap N_G(H)| \\ &= |(V(C) \cap N_G(H)) - D^{'}| + |D^{'}| \\ &\geq \{\delta(G) - (|V(H)| - 1) - |D^{'}|\}|V(H)| + |D^{'}| \\ &= \delta(G) - |D^{'}| \\ &+(\delta(G) - |V(H)| - |D^{'}|)(|V(H)| - 1) + |D^{'}| \\ &\geq \delta(G). \end{aligned}$$

Therefore we again obtain a desired cycle by Claim 2.3.

Subcase 2.2 $|(V(C) \cap N_G(H)) - D^{'}| < |V(H)|$.

By (5), there is a vertex $g^{'} \in V(H)$ satisfying

$$\left\{(V(C) \cap N_G(H)) - D^{'}\right\} \cap N_G(g^{'}) = \emptyset,$$

that is to say,

$$V(C) \cap N_G(g^{'}) \subset D^{'}.$$

Since G is $(m+k)$-connected, there exist $m+k$ $(g^{'}, V(C))$-paths $P^{'}_1, \ldots, P^{'}_{m+k}$ which are pairwise disjoint except at $g^{'}$. For each h, let $t^{'}{}_h$ denote the endvertex of $P^{'}_h$ differrent from $g^{'}$. Thus $t^{'}{}_h \in V(P^{'}_h) \cap V(C)$. We omit the proof of the following claim because it is similar to and easier than that of Claim 2.7.

Claim 2.10 *Let* $1 \leq h_1, h_2 \leq m+k$ *with* $h_1 \neq h_2$, *and suppose that* $E(t^{'}{}_{h_1}\overrightarrow{C}t^{'}{}_{h_2}) \cap E(F) = \emptyset$ *and* $|(V(t^{'}{}_{h_1}\overrightarrow{C}t^{'}{}_{h_2}) - \{t^{'}{}_{h_1}, t^{'}{}_{h_2}\}) \cap S(F)| \leq 1$. *Then there exists* $(v_i, v_{i+1}) \in \widetilde{M}^{'}$ *such that* $v_i\overrightarrow{C}v_{i+1} \subset t^{'}{}_{h_1}\overrightarrow{C}t^{'}{}_{h_2}$; *in particular, if* $(V(t^{'}{}_{h_1}\overrightarrow{C}t^{'}{}_{h_2}) - \{t^{'}{}_{h_1}, t^{'}{}_{h_2}\}) \cap S(F) = \emptyset$, *then* $(v_i, v_{i+1}) \in M^{'}$. □

We are now in a position to complete the discussion for Case 2. In view of Claim 2.9, we may assume $|M^{'}| \leq 1$. Let T_j be as in Subcase 1.2. Since $s \leq k$, there exists j_0 such that $|V(T_{j_0}) \cap \{t^{'}{}_1, \ldots, t^{'}{}_{m+k}\}| \geq 2$. Take $t^{'}{}_{h_1}, t^{'}{}_{h_2} \in V(T_{j_0}) \cap \{t^{'}{}_1, \ldots, t^{'}{}_{m+k}\}$ with $t^{'}{}_{h_1} \neq t^{'}{}_{h_2}$ so that $t^{'}{}_{h_1}\overrightarrow{C}t^{'}{}_{h_2} \subset T_{j_0}$. By Claim 2.10, there exists $(v_i, v_{i+1}) \in M^{'}$ such that $v_i\overrightarrow{C}v_{i+1} \subset t^{'}{}_{h_1}\overrightarrow{C}t^{'}{}_{h_2}$; thus $M^{'} = \{(v_i, v_{i+1})\}$. Then by Claim 2.10,

$V(T_{j_0}) \cap \{t'_1, \ldots, t'_{m+k}\} = \{t'_{h_1}, t'_{h_2}\}$ and $|V(T_j) \cap \{t'_1, \ldots, t'_{m+k}\}| \leq 1$ for each $j \neq j_0$. Since $s \leq k$, this implies $s = k$ and $|V(T_j) \cap \{t'_1, \ldots, t'_{m+k}\}| = 1$ for each $j \neq j_0$. In particular, $V(T_j) \cap \{t'_1, \ldots, t'_{m+k}\} \neq \emptyset$ for each j. Now take j such that $f_j \in S(F) \cap V(C)$ (note that we get $S(F) \cap V(C) \neq \emptyset$ from $s = k$). Let t'_{h_3} be the vertex in $V(T_{j-1}) \cap \{t'_1, \ldots, t'_{m+k}\}$ closest to f_j on S_{j-1} (when $j = 1$, we take $T_0 = T_{m+s-1}$ and $S_0 = S_{m+s-1}$), and let t'_{h_4} be the vertex in $V(T_j) \cap \{t'_1, \ldots, t'_{m+k}\}$ closest to f_j on T_j (it is possible that $t'_{h_3} = t'_{h_2}$ or $t'_{h_4} = t'_{h_1}$). By Claim 2.10, there exists $(v_{i'}, v_{i'+1}) \in \widetilde{M}'$ such that $v_{i'} \overrightarrow{C} v_{i'+1} \subset t'_{h_3} \overrightarrow{C} t'_{h_4}$. Then $(v_i, v_{i+1}) \neq (v_{i'}, v_{i'+1})$. Thus we have $|M'| = 1$ and $|\widetilde{M}'| \geq 2$, and we therefore obtain a desired cycle by Claim 2.9. This concludes the discussion for Case 2, and completes the proof of Theorem 1.

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