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Quasi-symmetric 3-designs with a fixed block intersection number

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Abstract

Quasi-symmetric 3-designs with block intersection numbers x, y $(0 \leq x \leq y < k)$ are studied. It is proved that the parameter λ of a quasi-symmetric 3-(v, k, λ) design satisfies a quadratic whose coefficients are polynomial functions of k, x and y. We use this quadratic to prove that there exist finitely many quasi-symmetric 3-designs under either of the following two restrictions:

1. The block intersection number x is fixed.
2. The difference of block intersection numbers $y - x > 1$ is fixed.

1 Introduction

Let X be a finite set of v elements, called points, and β be a finite family of distinct k-subsets of X, called blocks. Then the pair $D = (X, \beta)$ is called a t-(v, k, λ) design if every t-subset of X occurs in exactly λ blocks. For $0 \leq x < k$, x is called an intersection number of D if there exists $B, B' \in \beta$ such that $|B \cap B'| = x$. A 2-design with two intersection numbers is said to be a quasi-symmetric design. We always denote these intersection numbers by x and y, and assume $0 \leq x < y < k$.

By Cameron's result a 3-design has at least two block intersection numbers. In [4] Cameron has given the classification of quasi-symmetric 3-designs with $x = 0$. In [12] Sane and Shrikhande began investigation of quasi-symmetric 3-designs with non-zero intersection numbers. A partial attempt to characterize the Witt design among the family of quasi-symmetric 3-designs with $x = 1$ was made. Several inequalities, divisibility conditions and characterizations were obtained, which led the author to formulate the following conjecture.

Conjecture ⋆ : Let D be a quasi-symmetric 3-design. Then one of the following cases occurs:

1. *$x = 0$ and D is a design in Cameron's family.*
2. *$x = 1$ and D is the Witt-Lüneburg design on 23 points or its residual.*
3. *D is the complement of some design in (i) or (ii).*

The obvious way to attack this conjecture is to fix x and try to find feasible parameter sets of quasi-symmetric 3-designs with smaller intersection number x.

In support of this conjecture the case $x = 1$ was settled by Calderbank and Morton [3], and the author and Sane [7]. In [8] the author obtained several bounds for the block intersection numbers of quasi-symmetric 3-designs in terms of functions of v and k. In those inequalities upper bounds are attained if and only if D is the Witt 4-(23,7,1) design or its complement, and in some inequalities lower bounds are attained if and only if D is the complement of a design in Cameron's classification. Observe that these are the possibilities given in the conjecture. Using those inequalities it was proved that quasi-symmetric 3-designs corresponding to $y = x + 1, x + 2$ are designs in Cameron's family or designs corresponding to the Witt design (or trivial design) or the complement of the above designs.

Our aim in this paper is to carry out further investigation of the quasi-symmetric 3-designs for fixed $x \geq 2$. We extended the techniques used in [7] to find the quadratic $f(\lambda) = 0$. We rely on the equations derived in [1], [8], [11] and [12], and prove that the parameter λ of quasi-symmetric 3-designs satisfies a quadratic $f(\lambda) = 0$, whose coefficients are polynomial functions of k, x and y. We use this quadratic to show that for fixed $x \geq 2$ there exist finitely many parametrically feasible quasi-symmetric 3-designs. We put $y = x + \theta$ and show that under condition $\theta \geq 2$ and $x + y \leq k$, x is bounded by a function of θ. Hence by the Theorem 2.6 and Theorem 3.4, for fixed $\theta \geq 2$, there exist finitely many quasi-symmetric 3-designs with intersection numbers x and $y = x + \theta$. The steps of the proofs can be followed to find feasible parameter sets of quasi-symmetric 3-designs for given values of x or θ. Using the equations of this paper the computer search was carried out to find the feasible parameter of quasi-symmetric 3-designs with $2 \leq x \leq 100$. The search revealed that the designs are complement of designs with $x = 0$ or $x = 1$.

2 Preliminaries

In this section we list some results required in Section 3. The reader can refer to [1], [5], [7], [8], [11] and [12] for details.

Lemma 2.1 *Let D be a t-design and λ_i be the number of blocks containing any i points. Then*

$$\lambda_i \begin{pmatrix} k-i \\ t-i \end{pmatrix} = \lambda \begin{pmatrix} v-i \\ t-i \end{pmatrix}.$$

By the above lemma it is clear that any t-design is also an i-design for $1 \leq i \leq t$. In a t-design D, $\lambda_0 = b$ denotes the number of blocks and $\lambda_1 = r$ the number of blocks through any point of D.

Lemma 2.2 *Let D be a proper quasi-symmetric 2-design, with the standard parameter set $(v, b, r, k, \lambda; x, y)$. Then the following relation holds:*

$$k(r-1)(x+y-1) - xy(b-1) = k(k-1)(\lambda-1). \tag{1}$$

Theorem 2.3 *For a fixed value of the block size k, there exist finitely many quasi-symmetric 3-designs with larger block intersection number $y \geq 2$.*

Theorem 2.4 *Let D be a quasi-symmetric 3-design. Then the parameters of D satisfy the following equation.*

$$xy(v-1)(v-2) - (x+y-1)k(k-1)(v-2) + k(k-1)^2(k-2) = 0. \tag{2}$$

Proposition 2.5 *Let D be a quasi-symmetric 3-design with $x > 1$.*
If $4xy - (x+y-1)^2 > 0$ then

$$k < 8xy/[4xy - (x+y-1)^2]. \tag{3}$$

Theorem 2.6 *Let D be a quasi-symmetric 3-design with the intersection numbers x and y, $1 \leq x < y$ and let D' denote the complement of D with block size k' and intersection numbers x' and y'. If $k - 1 \leq x + y$ then $x' + y' \leq k'$.*

Theorem 2.7 *Let D be a quasi-symmetric 3-design with the smaller block intersection number $x = 1$. Then D is either the Witt 4-$(23, 7, 1)$ design or its residual (treating 3-$(5, 3, 1)$ as trivial).*

Theorem 2.8 *Let D be a quasi-symmetric 3-design with the intersection numbers x and $y = x + 1$. Then D is a trivial design (i.e., $v = k + 2$ and $b = v(v-1)/2$).*

Theorem 2.9 *Let D be a quasi-symmetric 3-design with the intersection numbers x and $y = x + 2$. Then D is either the 3-$(22, 6, 1)$ or the 3-$(22, 7, 4)$ or the 4-$(23, 7, 1)$ design or the complement of one of these three designs.*

3 Main Results

Theorem 3.1 *Let D be a quasi-symmetric 3-design with the parameter set $(v, b, r, k, \lambda; x \geq 2, y)$. Then $f(\lambda) = A\lambda^2 + B\lambda + C = 0$, where A, B, C are polynomials in k, x and y given by*

$$\begin{aligned}
A &= (k-1)[(m^2-4n)k^2 + 8nk - 2n(m+2)];\\
B &= -[m(m^2-4n)k^3 - ((m^2-4n)(m+n) - 6mn)k^2\\
&\quad -2n(5n+3m)k + n^2(m+6)];\\
C &= -n^2(k-2)[k^2 - (m+1)k + n],
\end{aligned}$$

where $m = x + y - 1$ and $n = xy$.

Proof: Substituting $x+y-1=m, xy=n, r=(v-1)\lambda_2/(k-1)$ and $b=v(v-1)\lambda_2/k(k-1)$ in (1) we get

$$\lambda_2[k^2(v-1)m - v(v-1)n - k^2(k-1)^2] = -k(k-1)[k^2-k(m+1)+n]. \quad (4)$$

Multiplying (2) by λ_2 we get

$$\lambda_2[n(v-1)(v-2) - mk(k-1)(v-2) + k(k-1)^2(k-2)] = 0. \quad (5)$$

Adding (4) and (5) we get

$$\lambda_2\{(mk-2n)(v-1) - k(k-1)[2(k-1)-m]\} = -k(k-1)[k^2-k(m+1)+n]. \quad (6)$$

Now multiply (6) by $n(k-2)$ and write $\lambda_2(k-2) = \lambda(v-2)$. Then substitute the value of $n(v-1)(v-2)$ from (2) to get

$$v-2 = \frac{(k-2)\{(k-1)(km-2n)\lambda - [k^2-k(m+1)+n]n\}}{[k(m^2-2n)-(m-2)n]\lambda} \quad (7)$$

We write (2) as a quadratic in $v-2$ and then substitute the value of $v-2$ from (7) to obtain $A'\lambda^2 + B'\lambda + C' = 0$, where

$$\begin{aligned} A' &= (k-1)\{n(k-2)(k-1)(km-2n)^2 \\ &\quad -[k(k-1)m-n](km-2n)[k(m^2-2n)-(m-2)n] \\ &\quad +k(k-1)[k(m^2-2n)-(m-2)n]^2\}; \\ B' &= -n[k^2-k(m+1)+n]\{2n(k-2)(k-1)(km-2n) \\ &\quad -[k(k-1)m-n][k(m^2-2n)-(m-2)n]\}; \\ &\quad -2n(5n+3m)k+n^2(m+6)]; \\ C' &= n^3(k-2)[k^2-(m+1)k+n]^2. \end{aligned}$$

We rewrite A' as

$$\begin{aligned} A' &= (k-1)n\{k^4(4n-m^2) + k^3(m^3+m^2-4mn-12n) \\ &\quad -k^2(m^2n-12n-4n^2-10mn) - 2kn(m^2+3m+4n+2) + 2n^2(m+2) \\ &\quad +2n^2(m+2)\}. \end{aligned}$$

Now divide A' by $k^2-k(m+1)+n$ to get

$$A' = (k-1)n[k^2-k(m+1)+n]\{k^2(4n-m^2) - 8kn + 2n(m+2)\}.$$

Now it is easy to see that $n[k^2-k(m+1)+n]\{k^2(4n-m^2)-8kn+2n(m+2)\}$ is the common factor in A', B' and C'. Cancelling it and changing signs of A', B', C' we get A, B, C as given in the theorem.

Eq.7 of [7] is the particular case of the equation $f(\lambda) = 0$. The former turned out to be an important step in the classification of quasi-symmetric 3-designs with the smaller intersection number $x = 1$.

We record some small observations.

Theorem 3.2 *Let D be a quasi-symmetric 3-design with parameter set $(v, b, r, k, \lambda; x \geq 2, y)$ and $m = x + y - 1$, $n = xy$.*

1. *$k - 1$ divides $(m - n)n(m\lambda - n)$.*
2. *If $(m+1)(m^2 - 4n) - n^2 > 0$ then $k < (2m^2 + m^3 + 3n - 3mn)/n$.*

Proof: (1) Consider $f(\lambda) = 0$ modulo $k - 1$.

(2) In $f(\lambda) = 0$, the condition $(m+1)(m^2 - 4n) - n^2 > 0$ forces A to be positive while B and C are negative. Hence λ is the positive (larger) root of $f(\lambda) = 0$. Now we use Mathematica to obtain

$$\begin{aligned} f(m) &= -k^3n^2 + k^2n(2m^2 + m^3 + 3n - 3mn) \\ &\quad -kn(6m^2 + 2m^3 - 8mn + n(2+n)) \\ &\quad +n(2m^3 - m^2(-4+n) - 6mn + 2n^2), \end{aligned}$$

$$\begin{aligned} f(m+1) &= k^3(m^2 + m^3 - 4mn - n(4+n)) \\ &\quad +k^2(m^3(-1+n) + m(14-3n)n - (-12+n)n + m^2(-1+3n)) \\ &\quad -kn(12 + 10m^2 + 2m^3 - 8n + n^2 - 4m(-5+2n)) \\ &\quad +n(2m^3 + m(10-7n) - m^2(-8+n) + 2(2 - 3n + n^2)) \\ &= [m^2 + m^3 - 4mn - n(4+n)][k^3 + k^2(-1+n) + 2n - 2kn)] \\ &\quad +n(-m^2(-6+n) + m(10+n) + 2(2 + n + 2n^2))+ \\ &\quad k^2(8 + 2m^2 + 2n + n^2 + m(10+n)) - k(20m + 8m^2 + 3(4+n^2)). \end{aligned}$$

Under the conditions $(m+1)(m^2-4n)-n^2 > 0$ and $k \geq (2m^2+m^3+3n-3mn)/n$, it is easy to see that $f(m)$ and $f(m+1)$ have opposite signs. Hence λ lies strictly between m and $m+1$. This is a contradiction. This completes the proof.

Now we prove some finiteness results for quasi-symmetric 3-designs using the equation $f(\lambda) = 0$.

Proposition 3.3 *For a fixed positive integer y, there exist finitely many quasi-symmetric 3-designs with the larger block intersection number y.*

Proof: In view of Theorem 2.3 we show that for a fixed y, the block size k takes finitely many values.

If $4n - m^2 > 0$, then by (3) k is bounded. If $m^2 - 4n \geq 0$ then we observe that in $f(\lambda) = 0$, A is positive while B and C are negative. Hence λ is a positive (larger) root of $f(\lambda) = 0$. We have

$$\lambda \leq \left[(-B/A) + \sqrt{\Delta_1}\right] \Big/ 2,$$

where $\Delta_1 = (-B/A)^2 + 4(-C/A)$.

Now consider $m^2 - 4n = 0$. We get

$$\begin{aligned} A &= (k-1)[8k - 2(m+2)]; \\ B &= -[6mk^2 - 2(5n+3m)k + n(m+6)]; \\ C &= -n(k-2)[k^2 - k(m+1) + n]. \end{aligned}$$

If $-4B > 3mA$ then $(m-3)k < n-3$, and if $-8C > nkA$ then $(3m+2)k < 4n$; each in turn implies $k < y+1$, which is a contradiction. Hence $(-B/A) \leq (3m/4)$ and $(-C/A) \leq (nk/8)$. This gives

$$\lambda \leq \left\{(3m/4) + \sqrt{\left(9m^2/16\right) + (nk/2)}\right\} \Big/ 2. \tag{8}$$

Now $f(\lambda) = 0$ implies $(k-1)$ divides $n[(m-4)\lambda - (n-m)]$. If $(m-4)\lambda - (n-m) > 0$ then $n[(m-4)\lambda - (n-m)] \geq (k-1)$. Hence

$$\frac{(k-1) + n(n-m)}{n(m-4)} \leq \lambda. \tag{9}$$

By (8) and (9), k takes finitely many values.

If $(m-4)\lambda - (n-m) \leq 0$ then $\lambda \leq (n-m)/(m-4)$. Let $m^2 - 4n > 0$. Now $(-B/A) > m$ implies

$$k^2[2mn + n(m^2-4n)] - k[2n(m^2-4n) - 2n^2 + 6mn] + 2mn(m+2) - n^2(m+6) < 0,$$

which is a contradiction and $(-C/A) > n^2/(m^2-4n)$ implies $k[(m^2-4n)(m+1) + 8n] - n[(m^2-4n) + (m+2)] < 0$, which also gives a contradiction. Hence $(-B/A) \leq m$ and $(-C/A) \leq n^2/(m^2-4n)$, and

$$\lambda \leq \left\{m + \sqrt{m + \left[4n^2/(m^2-4n)\right]}\right\} \Big/ 2. \tag{10}$$

In both cases λ takes finitely many values. Again by $f(\lambda) = 0$, k takes finitely many values. This completes the proof.

Theorem 3.4 *For a fixed positive integer $x > 0$, there exist finitely many quasi-symmetric 3-designs with the smaller block intersection number x.*

Proof: By Proposition 3.3 it is sufficient to show that for fixed x, the parameter y takes finitely many values. By Theorem 2.7 we assume that $x > 1$. If $y < x^2 + x + 5$ then by Proposition 3.3 there exist finitely many quasi-symmetric 3-designs with the smaller block intersection number x. Now we take $y \geq x^2 + x + 5$. Since $(k-x) \geq 2(y-x)$ we get $k \geq y + x^2 + 5$ and $n^2 - m(m^2-4n) < 0$. By (10) we get

$$\lambda \leq \left\{m + \sqrt{m^2 + 4m}\right\} \Big/ 2 < m + 1. \tag{11}$$

Consider $f(\lambda)$. If we assume $(-B/A) \leq (y-x+1)$ then

$$\begin{aligned} &k^3(2x-2)(m^2-4n) - k^2[n(m^2-4n) + 2n(y-x+1) + (2x-2)(m^2-10n)] \\ &-2kn[5n + 3m - (m+6)(y-x+1)] + n^2(m+6) - 2n(m+2)(y-x+1) \leq 0, \end{aligned}$$

which is a contradiction. Hence

$$(-B/A) > (y-x+1). \tag{12}$$

By (11) and (12) we get

$$(y-x)+1<(-B/A)\leq\lambda\leq(y-x)+2x-1.$$

Substituting $\lambda=(y-x)+\alpha$, where $2\leq\alpha\leq 2x-1$, in $f(\lambda)=0$ and using $y-x$ divides $k-x$, we get $y-x$ divides $x^2(x-1)\alpha(\alpha-1)$. Thus for a fixed x, α takes finitely many values, hence y takes finitely many values. The proof is then complete using Proposition 3.3.

Now we fix the difference $\theta=y-x$. Observe that the difference between the block intersection numbers of a quasi-symmetric design D is equal to the corresponding difference for its complement D'. Moreover, by Theorem 2.6, a design obtained by considering $x+y\geq k-1$ is the complement of a design where $x+y\leq k$. Thus, it is enough to consider a quasi-symmetric 3-design with $\theta=y-x$ and $x+y\leq k$.

Theorem 3.5 *For a fixed $\theta\geq 2$, there exist finitely many quasi-symmetric 3-designs with intersection numbers x and $y=x+\theta$.*

Proof: If $x=0$ then the theorem is proved by Cameron. By Theorem 2.7 we assume that $x>1$. Now take $y=x+\theta$ and $x+y=2x+\theta\leq k$. In view of Theorem 3.4 it is sufficient to show that for a fixed θ, x takes finitely many values. For a sufficiently large x, by (3) we get $k<2x+2(\theta^2-\theta+1)$. We write equation (2) as quadratic in $v-2$ and take its discriminant.

$$\begin{aligned}\Delta &= (xy)^2-2xy(x+y-1)k(k-1)\\ &\quad +k(k-1)^2\left\{-k[4xy-(x+y-1)^2]+8xy\right\}\geq 0.\end{aligned}$$

Now

$$-k[4xy-(x+y-1)^2]+8xy\leq 2x(\theta^2+1)+\theta(\theta-1)^2.$$

This gives

$$\begin{aligned}&-2xy(x+y-1)+(k-1)\left\{-k[4xy-(x+y-1)^2]+8xy\right\}\\ &\leq -4x^3+2x^2(2\theta^2-3\theta+3)+2x(2\theta^4-\theta^3+1)+\theta(\theta-1)^2(2\theta^2-2\theta+1)].\end{aligned}$$

Finally we get

$$\begin{aligned}\Delta &< x^2(x+\theta)^2+(2x+\theta)(2x+\theta-1)[-4x^3+2x^2(2\theta^2-3\theta+3)\\ &\quad +2x(2\theta^4-\theta^3+1)+\theta(\theta-1)^2(2\theta^2-2\theta+1)].\end{aligned}\tag{13}$$

It is clear from (13) that for a sufficiently large x, $\Delta<0$. Hence x takes finitely many values. Application of Theorem 3.4 completes the proof.

The level of difficulty in proving the *Conjecture* $\star$ is somewhat rather hard to tell at present. This will also involve simplifying the proofs of Ito and Bremner, which give classification of quasi-symmetric 4-designs. It is hoped that the result of [8] and this paper will prove to be a substantial contribution in the final proof of the *Conjecture* $\star$.

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