

# Decomposing Graphs into Forests of Paths

## with Size Less Than Three

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### Abstract

A forest in which every component is path is called a path forest. A family of path forests whose edge sets form a partition of the edge set of a graph  $G$  is called a path decomposition of a graph  $G$ . The minimum number of path forests in a path decomposition of a graph  $G$  is the path number of  $G$  and denoted by  $p(G)$ . If we restrict the number of edges in each path to be at most  $x$  then we obtain a special decomposition. The minimum number of path forests in this type of decomposition is denoted by  $p_x(G)$ . In this paper we study  $p_2(G)$ . We note here that if we restrict the size to be one, the number  $p_1(G)$  is just the chromatic index of  $G$ .

In this paper, we study the special type of path decomposition and we obtain the answers for  $p_2(G)$  when  $G$  is a complete graph, a tree and some other graphs.

### 1. Introduction.

A path decomposition is a special case of an edge decomposition and is the type of decomposition we will study in this paper. There are many interesting and important results and problems in this area. A good survey of them is provided by

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Chung and Graham. [2] Among other things, the chromatic index (the minimum number of matchings required to decompose a graph), the arboricity (the minimum number of forests needed to decompose a graph), the linear arboricity (the minimum number of path forests required to decompose a graph) or the tree number (the minimum number of trees needed to decompose a graph) have all been studied. [1,3,4] In some cases, exact formulas for these numbers have been found. An  $x$ -path coloring of  $G$  is an edge-coloring of  $G$  so that each component of each color class is a path of length at most  $x$ . Let  $p_x(G) = \min\{c \mid G \text{ has an } x\text{-path coloring with } c \text{ colors}\}$ . So  $p_1(G) = \chi'(G)$ . In this paper, we will estimate  $p_2(G)$ .

Here, in Section 2, we will obtain some general upper and lower bounds for  $p_2(G)$  when  $G$  is a graph, a complete graph and a complete bipartite graph. In Section 3 we find  $p_2(T)$  when  $T$  is a tree, and some relationships between  $p_2(K_n)$  and  $p_2(K_{n,n})$  are also discussed. Finally in Section 4 we find  $p_2(K_n)$  and some results concerning  $p_2(K_{n,n})$  are mentioned.

Throughout this paper we consider only simple undirected graphs without loops and multiple edges. Let  $V(G)$ ,  $v(G)$ ,  $E(G)$  and  $e(G)$  denote the vertex set, the number of vertices, the edge set and the number of edges (or size) of  $G$ , respectively. Let  $d_G(v)$  denote the degree of a vertex  $v$  in the graph  $G$  and  $\Delta(G)$  denote the maximum degree of  $G$ .

## 2. Lower bounds on $p_2(G)$ .

Given a graph  $G$ , it is not difficult to see that  $G$  can be decomposed into at least  $\lceil e(G)/\frac{2}{3}v(G) \rceil$  path forests with the size of each path less than three. Thus we have the following

Proposition 2.1.  $p_2(G) \geq \lceil e(G)/\frac{2}{3}v(G) \rceil$ .

For some graphs, this estimate is quite good, but there are also some graphs

for which this estimate is pretty far from the exact answer. We can use Proposition 2.1 to obtain bounds on  $p_2(G)$  for two well-known classes of graphs: complete graphs and complete bipartite graphs.

Corollary 2.2.  $p_2(K_{3v}) \geq \lceil \frac{3}{4}(3v-1) \rceil$ ,  $p_2(K_{3v+1}) \geq \lceil \frac{3}{4}(3v+1) \rceil$  and  $p_2(K_{3v+2}) \geq \lceil \frac{(3v+2)(3v+1)}{2(2v+1)} \rceil$ .

Corollary 2.3.  $p_2(K_{3v,3v}) \geq \lceil \frac{9}{4}v \rceil$ ,  $p_2(K_{3v+1,3v+1}) \geq \lceil \frac{(3v+1)^2}{4v+1} \rceil$  and  $p_2(K_{3v+2,3v+2}) \geq \lceil \frac{(3v+2)^2}{4v+2} \rceil$ .

The above estimates of  $K_n$  and  $K_{n,n}$  are actually very good. In Section 4, we will see that, for almost all  $n$ , the lower bounds in Proposition 2.2 for  $p_2(K_n)$  are also upper bounds, so this gives the answers for  $p_2(K_n)$ .

Now let us look at  $\Delta(G)$  of a graph  $G$ . It is clear that  $p_2(G) \geq \lceil \Delta(G)/2 \rceil$ . This will produce a better estimate for the graphs, such as stars, friendship graphs, trees, ... etc. Thus combining this with the ideas of the above propositions we have

Proposition 2.4.  $p_2(G) \geq \max \{ \Delta(G)/2, e(G)/m_2(G) \}$  where  $m_2(G)$  is the size of maximum path forests in  $G$  in which the size of each path is less than three.

If  $\Delta(G)$  is even and, if there exists a path which joins two vertices  $x$  and  $y$  with maximum degree and all the vertices on this path other than  $x$  and  $y$  have degree  $\geq \Delta(G)-1$ , then  $p_2(G) > \frac{\Delta(G)}{2}$ ; the reason for this can be obtained directly from decomposing this subgraph. In what follows we will call such a path a critical path.

Proposition 2.5. Let  $G$  be a graph with  $\Delta(G)$  being an even number. Then  $p_2(G) \geq \frac{\Delta(G)}{2} + 1$  provided that  $G$  has a critical path.

### 3. $p_2$ -decompositions of trees and nets.

A tree is a connected acyclic graph.

Proposition 3.1. Let  $T$  be a tree with maximum degree  $\Delta$ . Then we have  $p_2(T) = \frac{\Delta}{2} + 1$  if  $\Delta$  is even and  $T$  has a critical path. For the other cases  $p_2(T) = \lceil \frac{\Delta}{2} \rceil$ .

Proof. If  $\Delta$  is odd, by choosing a vertex  $x$  with maximum degree as a root and use a straight forward decomposition we can obtain  $\lceil \frac{\Delta}{2} \rceil$  path forests of the type we need. By Proposition 2.4, we conclude that  $p_2(T) = \lceil \frac{\Delta}{2} \rceil$ . It is easy to see that we can always construct a supergraph  $T^*$  of  $T$  such that  $\Delta(T^*) = \Delta + 1$ . Now if  $\Delta$  is even,  $\Delta(T^*)$  is odd. Hence by above argument  $\Delta/2 \leq p_2(T) \leq p_2(T^*) \leq \Delta/2 + 1$ , and we conclude that if  $T$  has a critical path then  $p_2(T) = \Delta/2 + 1$ . If  $T$  doesn't have a critical path, then  $p_2(T)$  can be shown to be  $\frac{\Delta}{2}$  by using a greedy algorithm to produce the required path forests.

Q.E.D.

The idea of a critical path can also be used to find  $p_2(G)$  in some other cases. For example, if  $G$  is the graph in Figure 3.1,  $\Delta(G) = 4$  and  $G$  has a critical path. Hence  $p_2(G) \geq 3$ .  $p_2(G) = 3$  can be shown easily as in Figure 3.1.

	1	2	3	
1	3	2	3	2
2	3	2	1	3
3	2	3	1	2
4	2	3	1	2
5	.	.	.	.
6	.	.	.	.
7	.	.	.	.

Figure 3.1.

#### 4. $p_2(K_n)$ .

A proper edge-coloring of a graph is an assignment of colors to its edges so that no two incident edges have the same color. If a graph  $G$  can be colored by no more than  $k$  colors, then this graph is called  $k$ -colorable and the number  $\chi'(G) = \min \{k: G \text{ is } k\text{-colorable}\}$  is the chromatic index of  $G$ . As mentioned in Section 1,  $\chi'(G)$  is the minimum number of matchings required to edge-decompose a graph  $G$ . Similarly,  $p_2(G)$  can be considered to be the minimum number of colors required to color the graph  $G$  so that no proper connected supergraph of  $K_{1,2}$  is induced by edges of one color; call such an edge-coloring a  $p_2$ -coloring. For example, the numbers we put on Figure 3.1 are actually the colors. If we focus on the  $p_2$ -colorings of  $K_n$  and  $K_{n,n}$ , then we can use an  $n \times n$  array to represent the coloring. It is well-known that a  $K_{n,n}$  with proper coloring can be represented by a latin square of order  $n$ . But, if we consider a  $p_2$ -coloring, it is slightly different from a latin square. Figure 4.1 is an example of  $K_{6,6}$  with  $p_2(K_{6,6}) = 5$ . As we have seen in this array,  $L = [\ell_{i,j}]$ , a number occurs in each row and each column at most twice and furthermore if  $\ell_{i,j} = \ell_{i',j'}$ ,  $i \neq i'$  and  $j \neq j'$ , then  $\ell_{i,j'} \neq \ell_{i,j}$  and  $\ell_{i',j} \neq \ell_{i,j}$ . The number  $p_2(K_n)$  can also be obtained in this way except that the array  $L = [\ell_{i,j}]$  is symmetric, i.e.,  $\ell_{i,j} = \ell_{j,i}$  for all  $i$  and  $j$ , and  $\ell_{i,i}$  is empty for each  $i = 1, 2, \dots, n$ . Figure 4.2 is an example showing that  $p_2(K_5) = 4$ .

$p_2(K_{6,6})$ :

1	1	2	2	3	3
3	3	1	1	2	2
2	4	3	5	1	4
2	5	3	4	1	5
4	2	5	3	4	1
5	2	4	3	5	1

Figure 4.1.

$p_2(K_5)$ :

//	1	2	2	1
1	//	3	4	2
2	3	//	1	3
2	4	1	//	4
1	2	3	4	//

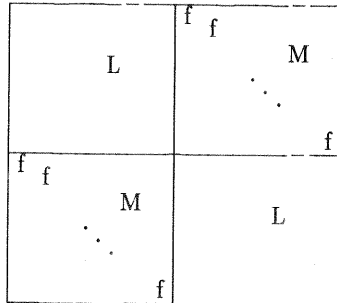
Figure 4.2.

We note here that the arrays in Figure 4.1 and Figure 4.2 provide an upper bound on  $p_2(K_{6,6})$  and  $p_2(K_5)$ , respectively, and that this upper bound equals the lower bound in Proposition 2.2 and 2.3, respectively.

Based on the relationships between  $K_n$  and  $K_{n,n}$  we have the following results.

- Proposition 4.1. (1)  $p_2(K_n) \geq p_2(K_{n-1})$  and  $p_2(K_{n,n}) \geq p_2(K_{n-1,n-1})$ .  
 (2)  $p_2(K_n) + 1 \geq p_2(K_{n,n})$ .  
 (3)  $p_2(K_{2n}) \leq p_2(K_n) + p_2(K_{n,n})$ . (4)  $p_2(K_{i,i}) \leq i \cdot p_2(K_{n,n})$ . (5)  $p_2(K_{2n}) \leq 2 \cdot p_2(K_n) + 1$ .

Proof. Obviously, we have (1). (2) can be obtained by putting a number (new) in the diagonal of the array which corresponds to  $K_n$ . Since  $K_{2n} \setminus K_{n,n}$  is a disjoint union of two  $K_n$ 's, hence we have (3). (4) is a direct result of the direct product of the array corresponding to  $K_{n,n}$  and a latin square of order  $i$ . Finally, by Figure 4.3, we obtain (5). Q.E.D.



L: array for  $p_2$ -coloring of  $K_n$  based on  $\{1, 2, \dots, p_2(K_n)\}$   
M: array for  $p_2$ -coloring of  $k_n$  based on  $\{p_2(K_n)+1, \dots, 2 \cdot p_2(K_n)\}$   
 $f$ : new color.

Figure 4.3.

These inequalities will help us in finding the number  $p_2(G)$ . But we need some ingredients first. Let us start with the following definitions.

A Steiner triple system of order  $v$  (briefly STS( $v$ )) is a pair  $(S, t)$  where  $S$  is a  $v$ -set and  $t$  is a collection of 3-element subset (called triple) of  $S$  such that every 2-element subset of  $S$  occurs in exactly one triple of  $t$ . It is well-known that an STS( $v$ ) exists if and only if  $v \equiv 1$  or  $3 \pmod{6}$ . A parallel class of an STS( $v$ ) is a collection of  $v/3$  mutually disjoint triples. Clearly, a parallel class exists only if  $v \equiv 3 \pmod{6}$ . An STS( $v$ ) is called resolvable provided that the collection of all triples can be partitioned into parallel classes. In [7], it was shown that a resolvable STS( $v$ ) (briefly RSTS( $v$ )) exists for each  $v \equiv 3 \pmod{6}$ . This is equivalent to the fact that  $K_{6k+3}$  can be decomposed into  $3k+1$  collections of  $2k+1$  mutually disjoint triangles. Thus we can prove the following

Proposition 4.2.  $p_2(K_{12t+3}) = 9t + 2$  and  $p_2(K_{12t+9}) = 9t + 6$ .

Proof. Let  $(S, t)$  be an RSTS( $6k+3$ ). First, we consider  $k = 2t$ . Then there are  $6t+1$  parallel classes. The idea of the proof is as follows: we start with two parallel classes; from each class, if we take one edge away from each triangle, then we obtain a path forest of the type we need, so is the other class. Now, if we can suitably combine the edges we took away, we obtain one more path forest of the type we want. If the above process can be done, then  $p_2(K_{12t+3}) \leq 3 \cdot (3t) + 2$ . By

Proposition 2.2,  $p_2(K_{12t+3}) \geq \lceil \frac{3}{4}(3 \cdot (4t-1) - 1) \rceil = \lceil \frac{3}{4}(12t+2) \rceil = 9t + 2$ . Hence we have the proof. Thus the problem left is how to combine the deleting edges.

Let two parallel classes be denoted by  $\left\{ \{a_i, b_i, c_i\} : i = 1, 2, \dots, 4t+1 \right\}$  and  $\left\{ \{d_i, e_i, f_i\} : i = 1, 2, \dots, 4t+1 \right\}$ . Construct a bipartite graph  $G = (A, B)$  with  $A = \{u_1, u_2, \dots, u_{4t+1}\}$ ,  $B = \{v_1, v_2, \dots, v_{4t+1}\}$  and  $\{u_i, v_j\}$  is an edge of  $G$  provided that  $\{a_i, b_i, c_i\} \cap \{d_j, e_j, f_j\} \neq \emptyset$ . Since  $(S, t)$  is a resolvable Steiner triple system, we conclude that  $G$  is a regular bipartite graph with of degree 3. Hence  $G$  has a complete matching  $M : \{u_1, v_{i_1}\}, \{u_2, v_{i_2}\}, \dots, \{u_{4t+1}, v_{i_{4t+1}}\}$ . Without loss of generality, let  $a_j = d_{i_j}$ ,  $j = 1, 2, \dots, 4t+1$ . Consider the graph  $G \setminus M$ , it is a graph with each vertex of degree two, and  $G \setminus M$  is a union of even cycles. Now we are ready to determine which edges should be taken away from the triangles  $\{a_i, b_i, c_i\}, \{d_i, e_i, f_i\}, i=1, 2, \dots, 4t+1$ . Since  $a_j = d_{i_j}$ , for each  $j=1, 2, \dots, 4t+1$ , we will take  $\{a_j, x_j\}, \{d_{i_j}, y_j\}$  away from the parallel classes  $\{ \{a_i, b_i, c_i\} \}$  and  $\{ \{d_i, e_i, f_i\} \}$  respectively such that all  $x_j$ 's and  $y_j$ 's are all distinct. In order to do that, we start with  $b_1 (=x_1)$ , if  $v_{i_1}$  is one the two vertices of  $G$  which is adjacent to  $u_1$  and  $e_{i_1} = b_1$ , then let  $f_{i_1}$  be  $y_1$ . By considering the cycle of  $G \setminus M$ ,  $(u_1, v_{i_1}, u_{i_h}, v_{i_h}, \dots, u_1)$ ,  $u_{i_h} = \{a_{i_h}, f_{i_h}, c_{i_h}\}$ . We can let  $c_{i_h} = x_2$ , and  $f_{i_h} = y_2$ . Since these deleted edges actually induced a path forest of the type we need, we are done. For the case  $k = 2t+1$ , it is similar and  $p_2(K_{12t+9}) \leq 3 \cdot \left(\frac{6t+4}{2}\right) = 9t+6$ . Q.E.D.

We note here that this proposition can be obtained by using Lemma 4 in [9]. Also, it has been shown by J.D. Horton in [5] that  $p_2(K_{12t+9}) = 9t+6$ .

By direct counting we obtain the following corollary.

Corollary 4.3.  $p_2(K_{12t+2}) = 9t+2$ ,  $t \geq 1$ , and  $p_2(K_{12t+7}) = p_2(K_{12t+8}) = 9t+6$ .



Proof.  $9t+2 = p_2(K_{12t+3}) \geq p_2(K_{12t+2}) \geq 9t+2$ .  $9t+6 = p_2(K_{12t+9}) \geq p_2(K_{12t+8}) \geq p_2(K_{12t+7}) \geq 9t+6$ .  
Q.E.D.

We can use the construction of a  $(12t+6) \times (12t+6)$  array to show the following

Proposition 4.4.  $p_2(K_{12t+6}) = 9t+4$ .

Proof. By Proposition 4.2, we have  $p_2(K_{6t+3}) = 3t+1 + \lceil \frac{3t+1}{2} \rceil$ . Let L and M be two arrays which correspond to  $p_2$ -colorings of  $K_{6t+3}$  which are based on the colors  $\{1, 2, \dots, p_2(K_{6t+3})\}$  and  $\{p_2(K_{6t+3})+1, p_2(K_{6t+3})+2, \dots, 2p_2(K_{6t+3})\}$  respectively.

Now if t is odd, then  $p_2(K_{12t+6}) \geq \lceil \frac{3}{4}(12t+5) \rceil = 9t+4$  and  $p_2(K_{12t+6}) \leq 2 \cdot p_2(K_{6t+3}) + 1 = 2 \cdot (3t+1 + \frac{3t+1}{2}) + 1 = 9t + 4$  (proposition 4.1). Hence we have  $p_2(K_{12t+6}) = 9t+4$ .

If t is even, then  $3t+1$  is odd. This implies that if the construction in Proposition 4.2 is used, we have one matching in the decomposition of  $K_{6t+3}$  which has  $2t+1$  edges. Of course, as in the proof of Proposition 4.2 we can combine two matching  $M_1$  and  $M_2$  from two parallel classes together to obtain a path forest of the type we need. Let this path forest be denoted by  $a_1-b_1-c_1, a_2-b_2-c_2, \dots, a_{2t+1}-b_{2t+1}-c_{2t+1}$  where  $\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_{2t+1}, b_{2t+1}\}$  are in the matching  $M_1$  and  $\{b_1, c_1\}, \{b_2, c_2\}, \dots, \{b_{2t+1}, c_{2t+1}\}$  are in the matching  $M_2$ . Now in L and M of Figure 4.3, we take the edges of  $M_1$  and  $M_2$  out respectively, i.e. the following cells of the array in Figure 4.3 are empty:  
 $(a_i, b_i), (b_i, a_i), (b_i, 6t+3+c_i), (c_i, b_i+6t+3), (b_i+6t+3, c_i), (c_i+6t+3, b_i), (a_i+6t+3, b_i+6t+3)$  and  $(b_i+6t+3, c_i+6t+3)$ .  $i = 1, 2, \dots, 2t+1$ . By letting all of the these  $8 \cdot (2t+1)$  cells be filled with a common number f, then we have an array which is

corresponding to  $K_{12t+3}$ . Since the induced graph of the edges corresponding to  $f$  is a path forest of the type we need. Thus  $p_2(K_{12t+6}) \leq 2[p_2(K_{6t+3})-1] + 2 = 2 \cdot (3t+1 + \lceil \frac{3t+1}{2} \rceil) = 6t+2+3t+2 = 9t+4$ . By Proposition 2.2 we conclude the proof.

The following corollary is a direct result of counting.

Corollary 4.5.  $p_2(K_{12t+5}) = 9t+4$ .

Proof.  $9t+4 = p_2(K_{12t+6}) \geq p_2(K_{12t+5}) \geq 9t+4$ . Q.E.D.

Before we prove next proposition, we need a special result. Since  $p_2(K_6) = 4$  and  $p_2(K_{6,6}) = 5$  (Figure 4.1.). We have  $p_2(K_{12}) = 9$ . (Proposition 4.1.) By a direct product of a symmetric latin square  $A$  of order  $t$  and a  $12 \times 12$  array, we obtain a  $12t \times 12t$  array  $M$ , and we are ready to prove the following

Proposition 4.6.  $p_2(K_{12t}) = 9t$ .

Proof. By Figure 4.4 we obtain  $p_2(K_{12,12}) = 9$ . Now in the upper triangular part of  $M$ , if the  $12 \times 12$  array corresponds to a number  $i$  in  $A$  we place the  $i$ th array corresponding to  $K_{12}$  if  $i$  is on the diagonal and  $i$ th array corresponding to  $K_{12,12}$  otherwise. For the lower triangular part we should place the transpose of the  $12 \times 12$  array in order to keep  $M$  symmetric. Then  $p_2(K_{12t}) \leq 9t$ . By Proposition 2.2 we have the equality. Q.E.D.

M =

1	1	2	2	3	3	8	7	5	4	6	9
8	4	4	5	5	6	6	7	9	3	2	1
4	6	7	7	8	8	9	9	5	3	2	1
4	6	9	1	1	2	2	3	3	8	7	5
3	2	1	8	4	4	5	5	6	6	7	9
3	2	1	4	6	7	7	8	8	9	9	5
8	7	5	4	6	9	1	1	2	2	3	3
6	7	9	3	2	1	8	4	4	5	5	6
9	9	5	3	2	1	4	6	7	7	8	8
2	3	3	8	7	5	4	6	9	1	1	2
5	5	6	6	7	9	3	2	1	8	4	4
7	8	8	9	9	5	3	2	1	4	6	7

Figure 4.4.

Corollary 4.7.  $p_2(K_{12t+11}) = 9t+9$ .

Proof. Similar to corollary 4.5.

Q.E.D.

So far, we can find  $p_2(K_n)$  for each  $n \equiv 0, 2, 3, 5, 6, 7, 8, 9, 11 \pmod{12}$ . We will need another technique to find  $p_2(K_n)$  when  $n \equiv 1, 4, 10 \pmod{12}$ .

Before we go any further, we need some more definitions. Let  $S$  is a  $v$ -set. A latin square of order  $v$  based on  $S$  is a  $v \times v$  array with entries from  $S$  such that in each row and each column every element of  $S$  occurs exactly once. For convenience, let  $S = \{1, 2, \dots, v\}$ . A latin square  $L = [\ell_{ij}]$  is said to be commutative provided that  $\ell_{ij} = \ell_{ji}$  for every  $1 \leq i, j \leq v$ . A latin square  $L = [\ell_{ij}]$  is idempotent if  $\ell_{ii} = i$  for each  $i \in S$ . It is well-known that an idempotent commutative latin square of order  $v$  exists if and only if  $v$  is odd. In case  $v = 2k$ , let  $H = \{\{1,2\}, \{3,4\}, \dots,$

$\{2k-1, 2k\}$ . The 2-element subsets are called holes. A latin square with holes  $H$  is a latin square such that for each hole  $h \in H$ , the subarray formed by  $h \times h$  is a subsquare based on  $h$ . Since all the holes are of size two, we also refer to the latin square as a latin square with  $2 \times 2$  holes  $H$ . Figure 4.5 is an example of a commutative latin square of order 8 with  $2 \times 2$  holes  $H$ . It was show by Fu that a commutative latin square of order  $2k$  with  $2 \times 2$  holes  $H$  (briefly CLSH( $2k$ )) exists for each  $k \geq 3$ . [4]

1	2	8	6	7	3	4	5
2	1	5	7	4	8	3	6
8	5	3	4	1	7	6	2
6	7	4	3	8	2	5	1
7	4	1	8	5	6	2	3
3	8	7	2	6	5	1	4
4	3	6	5	2	1	7	8
5	6	2	1	3	4	8	7

Figure 4.5.

In what follows, we will show that  $p_2(K_{12t+4}) = 9t+3$ ,  $t \neq 4$ , and  $p_2(K_{12t+1})$ ,  $p_2(K_{12t+10})$  can be obtained similarly. To start with, we have  $p_2(K_{16}) = 12$  and  $p_2(K_{28}) = 21$ . See Figure 4.6 and Figure 4.7.

a	a	1	4	5	3	9	3	b	c	2	8	6	7	
a	b	b	2	2	5	6	4	4	3	c	7	1	8	9
a	b	c	c	3	3	6	7	5	2	2	1	9	4	8
1	b	c	a	a	4	4	7	8	6	3	5	9	2	3
1	2	c	a	b	b	5	5	8	9	7	3	3	4	6
4	2	3	a	b	c	c	6	6	9	1	5	7	7	8
5	5	3	4	b	c	a	a	7	7	1	9	2	8	6
3	6	6	4	5	c	a	b	b	8	8	9	1	2	7
9	4	7	7	5	6	a	b	c	c	9	2	8	3	1
3	4	5	8	8	6	7	b	c	a	a	1	2	9	5
b	3	2	6	9	9	7	8	c	a	b	4	4	5	1
c	c	2	3	7	1	1	8	9	a	b	6	6	5	4
2	7	1	5	3	5	9	9	2	1	4	6	b	b	a
8	1	9	9	3	7	2	1	8	2	4	6	b	a	c
6	8	4	2	4	7	8	2	3	9	5	5	b	a	c
7	9	8	3	6	8	6	7	1	5	1	4	a	c	c

=

$A_1$	$B_1$
$B_1^T$	$D$

Figure 4.6.

a	a	1	1	4	5	10	9	4	b	c	15	16	12	13	3	18	17	11	11	14	14	10	2	8	6	7
a	b	b	2	2	5	6	4	3	3	c	17	15	12	15	13	13	11	18	16	10	17	14	7	1	18	9
a	b	c	c	3	3	6	7	5	2	2	12	12	16	13	17	17	11	18	15	15	10	14	1	9	4	8
1	b	c	a	a	4	4	7	8	6	3	14	18	14	18	11	11	12	15	16	17	10	13	5	9	2	3
1	2	c	a	b	b	5	5	8	9	7	16	14	10	11	16	10	12	15	18	17	18	13	3	3	4	6
4	2	3	a	b	c	c	6	6	9	1	10	14	17	11	15	18	16	12	12	13	13	16	5	7	7	8
5	5	3	4	b	c	a	a	7	7	1	6	9	8	2	15	8	18	13	17	11	11	18	12	10	14	16
10	6	6	4	5	c	a	b	b	8	8	2	1	9	12	7	14	17	13	3	16	15	11	18	18	12	14
9	4	7	7	5	6	a	b	c	c	9	8	8	1	14	3	15	13	16	13	18	16	11	12	10	17	2
4	3	5	8	8	6	7	b	c	a	a	2	1	9	5	10	16	15	10	14	18	12	15	17	17	11	13
b	3	2	6	9	9	7	8	c	a	b	18	11	18	17	6	4	10	17	14	12	15	12	13	16	1	5
c	c	2	3	7	1	1	8	9	a	b	13	7	4	6	4	5	14	14	10	16	12	17	15	5	11	15
15	17	12	14	16	10	6	2	8	2	18	13	a	a	1	1	4	5	3	9	3	b	c	11	15	13	10
16	15	12	18	14	14	9	1	8	1	11	7	a	b	b	2	2	5	6	4	10	3	c	16	11	17	13
12	12	16	14	10	17	8	9	1	9	18	4	a	b	c	c	3	3	6	7	5	2	2	11	13	15	17
13	15	13	18	11	11	2	12	14	5	17	6	1	b	c	a	a	4	4	7	8	6	3	10	16	10	9
3	13	17	11	16	15	15	7	3	10	6	4	1	2	c	a	b	b	5	5	8	9	7	14	14	18	18
18	13	17	11	10	18	8	14	15	16	4	5	4	2	3	a	b	c	6	6	9	1	7	12	16	12	
17	11	11	12	12	16	18	17	13	15	10	14	5	5	3	4	b	c	a	a	7	7	1	9	2	8	6
11	18	18	15	15	12	13	13	16	10	17	14	3	6	6	4	5	c	a	b	b	8	8	9	1	2	7
11	16	15	16	18	12	17	3	13	14	14	10	9	4	7	7	5	6	a	b	c	c	9	2	8	3	1
14	10	15	17	17	13	11	16	18	18	12	16	3	10	5	8	8	6	7	b	c	a	a	1	2	9	4
14	17	10	10	18	12	11	15	16	12	15	12	b	3	2	6	9	9	7	8	c	a	b	4	4	5	1
10	14	14	13	13	16	18	11	11	15	12	17	c	c	2	3	7	1	1	8	9	a	b	6	6	5	4
2	7	1	5	3	5	12	18	12	17	13	15	11	16	11	10	14	7	9	9	2	1	4	6	a	a	b
8	1	9	9	3	7	10	18	10	17	16	5	15	11	13	16	14	12	2	1	8	2	4	6	a	b	c
6	8	4	2	4	7	14	12	17	11	1	11	13	17	15	10	18	16	8	2	3	9	5	5	a	b	c
7	9	8	3	6	8	16	14	2	13	5	15	10	13	17	9	18	12	6	7	1	4	1	4	b	c	c

Figure 4.7.

If  $t$  is odd, let  $L = [\ell_{ij}]$  be an idempotent commutative latin square of order  $t$ . Now using  $A_1$  and  $B_1$  defined in Figure 4.6, construct a  $(12t+4) \times (12t+4)$  array as in Figure 4.8 where  $A_x(i,j) = A_1(i,j)$  if  $A_1(i,j) \in \{a,b,c\}$ ,  $A_x(i,j) = (x-1) \cdot 9 + A_1(i,j)$  if  $A_1(i,j) \notin \{a,b,c\}$ ,  $B_x = B_1 + (x-1) \cdot 9$  and  $C_x = M + (x-1) \cdot 9$ ,  $1 \leq x \leq t$ . It is a routine matter to check that  $p_2(K_{12t+4}) = 9t+3$  whenever  $t$  is odd.

$A_1$									$B_1$	
	$A_2$								$B_2$	
		$A_3$							$B_3$	
									$\vdots$	
				$A_i$		$C_{\ell_{ij}}$			$B_i$	
									$\vdots$	
									$\vdots$	
				$C_{\ell_{ij}}^T$		$A_j$			$B_j$	
									$\vdots$	
								$A_t$	$B_t$	
$B_1^T$	$B_2^T$	$B_3^T$	$\dots$	$B_i^T$	$\dots$	$\dots$	$B_j^T$	$\dots$	$B_t^T$	$D$

Figure 4.8.

If  $t$  is even, let  $L = [\ell_{ij}]$  be a commutative latin square of order  $t = 2k$  with  $2 \times 2$  holes. Then the corresponding array can be arranged as Figure 4.9, where  $A_1$  and  $B_1$  are now defined using Figure 4.7. Thus, we conclude that  $p_2(K_{12t+4}) = 9t+3$ ,  $t \neq 4$ . Since the proof of  $p_2(K_{12t+1}) = 9t+1$  and  $p_2(K_{12t+10}) = 9t+8$  is similar, we omit it here and put the arrays which correspond to  $p_2(K_{13}) = 10$ ,  $p_2(K_{25}) = 19$ ,  $p_2(K_{22}) = 17$  and  $p_2(K_{34}) = 26$  in the Appendix.

$A_1$									$B_1$
		$A_2$				$C_{\ell_{ij}}$			$B_2$
				$A_3$					$B_3$
									$\vdots$
		$C_{\ell_{ij}}^T$							
									$\vdots$
							$A_k$		$B_k$
$B_1^T$		$B_2^T$		$B_3^T$		$\dots$		$B_k^T$	$D$

Figure 4.9.

By the fact that a commutative latin square of order 4 with  $2 \times 2$  holes doesn't exist, we are not able to obtain  $p_2(K_{49})$ ,  $p_2(K_{52})$  and  $p_2(K_{58})$  now. But, we believe this is a matter of tedious work in constructions and we will not go through it.

Combining the above results we have the following

Theorem 4.8.  $p_2(K_{3v}) = \left\lfloor \frac{3(3v-1)}{4} \right\rfloor$ ,  $p_2(K_{3v+1}) = \left\lfloor \frac{3(3v+1)}{4} \right\rfloor$  and  $p_2(K_{3v+2}) = \left\lfloor \frac{(3v+2)(3v+1)}{2(2v+1)} \right\rfloor$  except possibly if  $3v+1 \in \{49, 52, 58\}$ .

## 5. Remarks.

We can also obtain the results on  $p_2(K_{n,n})$  by applying proposition 4.1 and direct constructions. Since the techniques are similar, we will not go any further in this direction. As a special case when  $n = 12t$ ,  $p_2(K_{12t,12t}) = 9t$  which can be obtained by the direct product of  $M$  (Figure 4.4.) and a latin square of order  $t$ . This result is equivalent to the existence of a  $P_3$ -factorization of a balanced complete bipartite graph with  $12t$  vertices in each part which has been shown by Ushio in [8].

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1 1 4 4 3 7 6 6 a a 8 9  
 1 2 2 5 3 6 a 7 7 9 8 4  
 1 2 3 3 4 6 7 9 8 8 a 5  
 4 2 3 1 1 a 7 8 9 6 6 5  
 4 5 3 1 5 a 9 8 6 7 9 2  
 3 3 4 1 5 8 8 a 6 9 7 2  
 7 6 6 a a 8 2 2 4 4 5 9  
 6 a 7 7 9 8 2 1 1 3 5 4  
 6 7 9 8 8 a 2 1 5 5 4 3  
 a 7 8 9 6 6 4 1 5 2 2 3  
 a 9 8 6 7 9 4 3 5 2 3 1  
 8 8 a 6 9 7 5 5 4 2 3 1  
 9 4 5 5 2 2 9 4 3 3 1 1

$$p_2(K_{13}) = 10$$

1 1 4 4 3 7 6 6 9 9 8 11 11 10 10 13 13 18 17 15 14 16 19 5  
 1 2 2 5 3 6 9 7 7 10 8 18 14 14 15 15 16 16 17 19 13 12 11 4  
 1 2 3 3 4 6 7 10 8 8 9 5 16 17 17 18 18 19 19 15 13 12 11 14  
 4 2 3 1 1 9 7 8 10 6 6 5 16 19 11 11 12 12 13 13 18 17 15 14  
 4 5 3 1 5 9 10 8 6 7 10 13 12 11 18 14 14 15 15 16 16 17 19 2  
 3 3 4 1 5 8 8 9 6 10 7 13 12 11 14 16 17 17 18 18 19 19 15 2  
 7 6 6 9 9 8 2 2 4 4 5 18 17 15 14 16 19 11 11 12 12 13 13 10  
 6 9 7 7 10 8 2 1 1 3 5 16 17 19 13 12 11 18 14 14 15 15 16 4  
 6 7 10 8 8 9 2 1 5 5 4 19 19 15 13 12 11 14 16 17 17 18 18 3  
 9 7 8 10 6 6 4 1 5 2 2 12 13 13 18 17 15 14 16 19 11 11 12 3  
 9 10 8 6 7 10 4 3 5 2 3 15 15 16 16 17 19 13 12 11 18 14 14 1  
 8 8 9 6 10 7 5 5 4 2 3 17 18 18 19 19 15 13 12 11 14 16 17 1  
 11 18 5 5 13 13 18 16 19 12 15 17 6 6 9 9 8 2 1 1 4 4 3 10  
 11 14 16 16 12 12 17 17 19 13 15 18 6 7 7 10 8 1 4 2 2 5 3 9  
 10 14 17 19 11 11 15 19 15 13 16 18 6 7 8 8 9 1 2 5 3 3 4 12  
 10 15 17 11 18 14 14 13 13 18 16 19 9 7 8 6 6 4 2 3 5 1 1 12  
 13 15 18 11 14 16 16 12 12 17 17 19 9 10 8 6 10 4 5 3 1 2 5 7  
 13 16 18 12 14 17 19 11 11 15 19 15 8 8 9 6 10 3 3 4 1 5 2 7  
 18 16 19 12 15 17 11 18 14 14 13 13 2 1 1 4 4 3 7 7 9 9 10 5  
 17 17 19 13 15 18 11 14 16 16 12 12 1 4 2 2 5 3 7 6 6 8 10 9  
 15 19 15 13 16 18 12 14 17 19 11 11 1 2 5 3 3 4 7 6 10 10 9 8  
 14 13 13 18 16 19 12 15 17 11 18 14 4 2 3 5 1 1 9 6 10 7 7 8  
 16 12 12 17 17 19 13 15 18 11 14 16 4 5 3 1 2 5 9 8 10 7 8 6  
 19 11 11 15 19 15 13 16 18 12 14 17 3 3 4 1 5 2 10 10 9 7 8 6  
 5 4 14 14 2 2 10 4 3 3 1 2 10 9 12 12 7 7 5 9 8 8 6 6

$$p_2(K_{25}) = 19$$

f h e f g b a d d c 1 8 3 3 8 6 9 2 5 7  
 f g h g h a d b b e c 1 4 2 2 7 7 9 3 5 8  
 h g 7 7 h f a b b e c c d 2 4 1 1 5 9 5 3 6 8  
 e h 7 1 g d b b c e a a 2 5 8 4 4 3 3 8 6 9  
 f g h 1 8 d e c c a b e 3 5 4 6 6 2 2 7 7 9  
 g h f g 8 c c d a e b 3 6 4 7 9 1 1 5 9 5  
 b a a d d c 9 e f g h 8 6 5 7 1 8 4 4 3 3  
 a d b b e c c 9 3 f g h 7 7 5 8 1 4 6 6 2 2  
 a b b e c c d e 3 g h f 5 9 6 8 2 4 7 9 1 1  
 d b c e a a f f g h g 4 3 6 9 2 5 7 1 8 4  
 d e c c a b e g g h h f 6 2 7 9 3 5 8 1 4 6  
 c c d a e b h h f g f 9 1 9 5 3 6 8 2 4 7  
 1 1 2 2 3 3 8 7 5 4 6 9 a a c e b d d f g  
 8 4 4 5 5 6 6 7 9 3 2 1 a b b c d e h f g  
 3 2 1 8 4 4 5 5 6 6 7 9 a b f c d e g h h  
 3 2 1 4 6 7 7 8 8 9 9 5 c b f a a h e d f  
 8 7 5 4 6 9 1 1 2 2 3 3 e c c a b h g d f  
 6 7 9 3 2 1 8 4 4 5 5 6 b d d a b f f e d  
 9 9 5 3 2 1 4 6 7 7 8 8 d e e h h f b b a c  
 2 3 3 8 7 5 4 6 9 1 1 2 d h g e g f b a c  
 5 5 6 6 7 9 3 2 1 8 4 4 f f h d d e b a c  
 7 8 8 9 9 5 3 2 1 4 6 7 g g h e f d a c c

$$p_2(K_{22}) = 17$$

11	h	e	f	g	b	a	a	d	d	c	8	7	5	4	6	9	1	1	2	2	3	3	16	10	12	15	17	18	14	14	13	13		
11	g	h	g	h	a	e	b	b	f	c	6	7	9	3	2	1	8	4	4	5	5	6	17	10	13	15	18	14	16	16	12	12		
h	g	12	h	f	a	b	e	c	c	d	9	9	5	3	2	1	4	6	7	7	8	8	10	15	13	16	18	14	17	10	11	11		
e	f	g	h	f	d	b	c	e	a	b	2	3	3	8	7	5	4	6	9	1	1	2	13	13	18	16	10	15	17	11	18	14		
g	h	f	g	13	c	c	c	d	a	e	7	8	8	9	9	5	3	2	1	4	6	7	11	11	15	10	15	16	18	12	14	17		
b	a	e	b	d	d	e	f	f	g	h	11	11	12	12	13	13	18	17	15	14	16	10	6	9	2	5	7	8	4	3	3			
a	b	e	c	e	c	d	e	d	2	h	14	16	17	17	18	18	10	10	15	13	12	11	9	5	3	6	8	4	7	9	1	2		
d	f	c	e	a	b	e	f	f	2	h	14	16	10	11	11	12	12	13	13	18	17	15	3	3	8	6	9	5	7	1	8	4		
d	c	d	a	a	b	e	h	h	f	g	3	3	13	12	11	18	14	14	15	16	17	2	2	7	7	9	5	8	1	4	6			
8	6	9	2	5	7	11	18	14	14	13	13	1	h	e	f	g	b	a	a	d	d	c	8	4	4	3	3	16	10	12	15	17		
7	7	9	3	5	8	11	14	16	16	12	12	1	g	h	g	h	a	e	b	b	f	c	4	6	6	2	2	17	10	13	15	18		
5	9	5	3	6	8	12	14	17	10	11	11	h	g	2	h	f	a	b	e	c	c	d	4	7	9	1	1	10	15	13	16	18		
4	3	3	8	6	9	12	15	17	11	18	14	e	h	2	f	g	d	b	e	c	a	a	5	7	1	8	4	13	13	18	16	10		
6	2	2	7	7	9	13	15	18	11	14	16	f	g	f	f	3	d	e	c	c	a	b	b	5	8	1	4	6	12	12	17	17	10	
9	1	1	5	9	5	13	16	18	12	14	17	g	h	f	g	3	c	e	d	a	e	e	6	8	2	4	7	11	11	15	10	15		
1	8	4	4	3	3	18	16	10	12	15	17	b	a	a	d	d	c	11	e	f	g	h	18	14	14	13	13	6	9	2	5	7		
1	4	6	6	2	2	17	17	10	13	15	18	a	e	b	b	e	c	d	e	d	12	h	f	14	17	10	11	11	9	5	3	6	8	
2	4	7	9	1	1	15	10	15	13	16	18	a	b	e	c	c	d	e	d	12	h	g	15	17	10	11	18	14	3	3	8	6	9	
2	5	7	1	8	4	14	13	13	18	16	10	d	b	c	e	a	a	f	12	h	g	15	17	10	11	11	9	5	3	6	8	9		
3	5	8	1	4	6	16	12	12	17	17	10	d	f	c	a	b	e	g	g	h	h	13	15	18	11	14	16	2	2	7	7	9		
3	6	8	2	4	7	10	11	11	15	10	15	c	c	d	a	b	e	h	h	f	g	13	16	18	12	14	17	1	1	5	9	5		
16	17	10	13	12	11	6	7	9	3	2	1	8	4	4	5	6	18	14	14	15	15	16	a	a	a	b	c	e	b	d	d	f	g	
10	10	15	13	12	11	9	9	5	3	2	1	4	6	7	7	8	8	14	16	17	17	18	18	a	b	b	c	e	h	e	f	g	h	
12	13	13	18	17	15	2	3	3	8	7	5	4	6	9	1	1	2	14	16	10	11	11	12	a	b	f	c	d	e	g	h	h	g	h
15	15	16	16	17	19	5	5	6	6	7	9	3	2	1	8	4	4	13	12	11	18	14	14	c	b	f	a	a	h	e	d	f	e	
17	18	18	10	10	15	7	8	8	9	9	5	3	2	1	4	6	7	13	12	11	14	16	17	e	c	c	a	b	h	f	g	e	d	e
18	14	14	15	15	16	8	4	4	5	5	6	16	17	10	13	12	11	6	7	9	3	2	1	b	h	d	a	b	f	f	g	e	d	a
14	16	17	17	18	18	4	6	7	7	8	8	10	10	15	13	12	11	9	5	3	2	1	d	e	e	h	h	f	b	b	a	c	c	
14	16	10	11	11	12	4	6	9	1	1	2	12	13	13	18	17	15	2	3	3	8	7	5	d	h	g	e	g	e	f	b	a	c	
13	12	11	18	14	14	3	2	1	8	4	4	15	15	16	16	17	10	5	5	6	6	7	9	f	f	h	d	e	f	b	a	c	c	
13	12	11	14	15	17	3	2	1	4	6	7	17	18	18	10	10	15	7	8	8	9	9	5	g	g	h	f	e	d	a	c	c	c	

$$p_2(K_{34}) = 26$$

