A Characterisation of $P_{n,q}$ among the Symmetric Designs using Elations*

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Abstract

Let D be a symmetric 2-design which has an automorphism group G such that every point of D is the centre for some non-identity elation in G and some point of D is the centre for two non-identity elations in G whose axes are distinct. In [4] a classification (into Classes A,B,C) of the pairs (D, G) is given, and the designs in Class A shown to be projective spaces. In this paper we investigate the designs in Class B. We show that $D \cong P_{n,q}$ and a certain subgroup of G has a factor group isomorphic to $PSL(n-s, q)$ for some $0 \leq s \leq n-3$. The designs in Class C are studied in a further paper [5].

§0 Introduction

We are interested in the classification of a specific class of designs among a larger class using automorphisms. For example, in 1959, Wagner [13] classified the finite Desarguesian projective planes among the finite projective planes by postulating the existence of certain elations. In 1963 Piper [11] weakened Wagner's postulates and generalised his results. More recently Kelly [8,9] has generalised the result concerning projective planes having two translation lines by characterising the symmetric designs having at least two translation blocks. He has also given necessary and sufficient conditions for a symmetric design having more than one translation block to be a projective space. Butler has generalised translation blocks to semi-translation blocks. He has classified symmetric designs with more than one semi-translation block [2,3]. We are interested in characterising the finite projective spaces $P_{n,q}$ among the symmetric designs by considering elations, since projective spaces have a large automorphism group containing many elations.

Let D be a symmetric 2-(v, k, λ) design. An automorphism of D which fixes a block x pointwise and a point X blockwise is called an elation, or an (X, x)-elation; and

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X is called a centre, x an axis for the elation. In this paper we reserve the term "elation" for such an automorphism which is not the identity.

Now let G be a subgroup of $\mathrm{Aut}(D)$. We denote by $\mathcal{E}l_G(X)$, $\mathcal{E}l_G(x)$ and $\mathcal{E}l_G(X,x)$ the subgroups of G generated respectively by all elations in G having centre X, all elations in G having axis x, and all the (X,x)-elations in G. The elation structure $E(G)$ of G has as points the points of D which are centres of elations in G, as blocks the blocks of D which are axes for elations in G, and P is incident with x in $E(G)$ if there exists a (P,x)-elation in G. We say that the pair (D,G) has Property A if every point of D is a point of $E(G)$, and some point of $E(G)$ is incident in $E(G)$ with more than one block of $E(G)$.

As in [4], we say that a block x of $E(G)$ is a maximal axis if the number $\mathcal{C}_G(x)$ of points of $E(G)$ incident with x in $E(G)$ is as large as possible, that is, $\mathcal{C}_G(x) \geq \mathcal{C}_G(y)$ for all blocks y of $E(G)$. Assuming that (D,G) has Property A, we use (as in [4]) the following notation:

T = the set of all maximal axes;

N = the subgroup of G generated by the elations in G whose axes are maximal;

S = the set of all points of D which are fixed by all the elements of N.

In [4] (Lemma 3.3 and Theorem 3.5) it was shown that if (D,G) has Property A and $\lambda > 1$ then (D,N) also has Property A, indeed every point of D is the centre of an elation in N whose axis belongs to T, and some point of D is the centre of two such elations with distinct axes. Furthermore it was shown that we have one of the following three situations:

Class A: $S = \emptyset$, T is all the blocks of D and $E(N) = D$.

Class B: $S \neq \emptyset$, every point not in S lies on at least two blocks in T, and whenever a point X lies on a block y in T then there exists an (X,y)-elation in N.

Class C: $S \neq \emptyset$ and every point not in S lies on exactly one block in $E(N)$.

It was shown in [4] that if (D,G) is in Class A then $D \cong \mathrm{P}_{n,q}$ and $N \cong \mathrm{PSL}(n+1,q)$ for some $n \geq 3$ and prime power q. In this paper we investigate the case where (D,G) is in Class B. Our aim is to prove the following theorem.

Theorem B Let D be a symmetric 2-(v,k,λ) design with $\lambda > 1$ and let G be a subgroup of $\mathrm{Aut}(D)$. Suppose that (D,G) has Property A and is in Class B. Let T denote the set of all maximal axes of (D,G), N the subgroup of G generated by the elations in G whose axes belong to T, and H the (normal) subgroup of N fixing all the blocks in T. Then

(a) $D \cong \mathrm{P}_{n,q}$ for some $n \geq 3$ and prime power q.

(b) $N/H \cong \mathrm{PSL}(n-s,q)$ for some s, $0 \leq s \leq n-3$.

§1 Assumed Results

We assume throughout the remainder of this paper that (D, G) satisfies the hypotheses of Theorem B, and that T, S, N and H have the meanings assigned to them in §0. We shall denote the complement of S in the point-set of D by $\overline{S}$, and the complement of T in the block-set of D by $\overline{T}$. By a $\mathrm{D}(n, q)$ we mean a symmetric 2-design which has the same parameters as the projective space $\mathrm{P}_{n,q}$. From [4] we have the following results.

Result 1.1 ([4], Result 0.1) If x is a block in D then the group $\mathcal{E}l_G(x)$ acts semi-regularly on the points of D not incident with x. □

Result 1.2 ([4], Result 1.2) Let D^* be a 2-design and G^* a subgroup of $\mathrm{Aut}(D^*)$. Let $\alpha, \beta \in G^*$ and let (X, x), (Y, y) be flags with $X \notin y$ and $Y \notin x$ such that α has exactly one centre X and one axis x and β has exactly one centre Y and one axis y. Suppose further that α and β have the same prime order p. Then there exists $\gamma \in \langle \alpha, \beta \rangle \subseteq G^*$ such that γ maps X to Y, fixes $x \cap y$ pointwise, and fixes all the blocks containing the line XY. Furthermore, if D^* is symmetric then γ maps x to y. □

Result 1.3 ([4], Result 1.3) If $\alpha \in \mathcal{E}l_G(X, x)$, $\beta \in \mathcal{E}l_G(Y, y)$ with $Y \in x$, $X \notin y$ and $\alpha, \beta \neq 1$ then $[\alpha, \beta] \neq 1$ and $[\alpha, \beta] \in \mathcal{E}l_G(Y, x)$ (where $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$). □

Result 1.4 ([4], Lemma 1.4) If $\alpha \in \mathcal{E}l_G(A, x)$ and $\beta \in \mathcal{E}l_G(B, x)$, where $A \neq B$ and $\alpha, \beta \neq 1$, then α and β commute and have the same prime order. □

Result 1.5 ([4], Corollary 1.5) If $\mathcal{E}l_G(x)$ contains two elations with different centres, then $\mathcal{E}l_G(x)$ is an elementary p-group. □

Result 1.6 ([4], Lemma 1.6) If D^* is a $\mathrm{D}(n, p)$, with p prime, and there is a block x in D^* such that every point X on x is the centre of an (X, x)-elation, then $D^* \cong \mathrm{P}_{n,p}$. □

Result 1.7 ([4], Theorem 2.4) All the elations in G have the same prime order. □

Result 1.8 ([4], Lemma 3.3) The subgroup N of G acts transitively on each of $\overline{S}$ and T. □

Following Kelly [8], we say that a block x of a symmetric 2-design D^* is a translation block if the group of automorphisms of D^* fixing every point incident with x acts transitively on the set of all points of D^* which are not incident with x. In [8], Kelly describes several methods for obtaining one $\mathrm{D}(n, q)$ having translation blocks from another. One of these he calls substitution. For a description of this procedure, see either [8] or our proof of Theorem B in §3.

We shall need the following result of Kelly.

Result 1.9 (Kelly [9], Theorem 6.2) Let D^* be a $\mathrm{D}(n,q)$ with $n \geq 3$ and $q > 2$, let T^* be a set of translation blocks of D^*, and let $S^* = \bigcap_{y \in T^*} y$. Suppose that S^* is not equal to and does not contain the intersection of any two blocks of D^*, and that every block of D^* containing S^* is a translation block. Then either $D^* \cong \mathrm{P}_{n,q}$ or $D^*(S^*)$ is a $\mathrm{D}(s,q)$ for some s, $2 \leq s \leq n-3$, and D^* is isomorphic to a design obtained from $\mathrm{P}_{n,q}$ by substituting $D^*(S^*)$ for the design of points and hyperplanes of an s-dimensional subspace of $\mathrm{P}_{n,q}$.

Here $D^*(S^*)$ denotes the design on S^* whose blocks are the distinct sets $x \cap S^*$ as x ranges over the set of all blocks of D^* which do not contain S^*.

Finally we shall need the following powerful theorem of M. E. O'Nan.

Result 1.10 (O'Nan [10]) If N^* is a 2-transitive permutation group acting on a set Ω and, for some $a \in \Omega$, the group N^*_a contains an abelian normal subgroup A, with $A \neq 1$, such that A does not act semi-regularly on $\Omega \backslash \{a\}$, then $\mathrm{PSL}(m+1,q) \leq N^* \leq \mathrm{P\Gamma L}(m+1,q)$ for some $m \geq 2$ and prime power q. □

This theorem, together with related results and constructions, is discussed in [12], Chapter 6.

§2 The quotient design

We assume, throughout, the hypotheses of Theorem B. In this section we construct from (D,G) a 2-design D_1 and a homomorphism κ: $N \longrightarrow \mathrm{Aut}(D_1)$, and then proceed to show (using the theorem of O'Nan, Result 1.10) that, for some $m \geq 2$ and prime power q, $D_1 \cong \mathrm{P}_{m,q}$ and $N^\kappa \cong \mathrm{PSL}(m+1,q)$.

In analogy with the construction of quotient spaces from projective spaces, we define the quotient design D_1 of (D,G) as follows. The points of D_1 are just the elements of T (considered as subsets of the point-set of D). The blocks of D_1 are the distinct sets

$$X' = \left(\bigcap \{y \in T \mid y \ni X\}\right) \Big\backslash\, S$$

as X ranges over the whole of $\overline{S}$. Incidence in D_1 is just set inclusion. When we wish to emphasise that an element y of T is being considered as a point in D_1 (rather than as block of D) we shall denote it by $[y]$. Similarly, when we wish to emphasise that the set X' (consisting of various points of D) is being considered as a block in D_1, we shall denote it by $[X']$.

Lemma 2.1 The sets X' are a constant size j.

Proof The group N fixes S pointwise, is transitive on $\overline{S}$ (Result 1.8), and maps T onto itself. □

Lemma 2.2 D_1 is a 2-$(\underline{v}, \underline{k}, \underline{\lambda})$ design with $\underline{b} = \frac{v-|S|}{j}$ blocks, $\underline{v} = |T|$ points, $\underline{r} = \frac{k-|S|}{j}$ blocks on a point and $\underline{\lambda} = \frac{\lambda-|S|}{j}$ blocks on two points.

Proof As N is transitive on S, each block $[X']$ is incident with the same number $\underline{k}$ of points of D_1. Note that the blocks of D_1, as point sets of $\overline{S}$, partition $\overline{S}$, hence $\underline{b} = \frac{v-|S|}{j}$. Since N is transitive on T (Result 1.8), each point $[y]$ in D_1 is incident with a constant number $\underline{r} = \frac{k-|S|}{j}$ of blocks of D_1. Further D_1 is a 2-design since two points $[y_1]$, $[y_2]$ determine a constant number $\underline{\lambda} = \frac{\lambda-|S|}{j}$ of blocks of D_1. □

In order to examine the automorphisms of D_1 inherited from D, we define a mapping $\kappa: N \subseteq \mathrm{Aut}(D) \to \mathrm{Aut}(D_1)$.

Lemma 2.3 If $\alpha \in N$, define the action of α^κ in D_1 by: for $y \in T$, $[y]^{\alpha^\kappa} = [y^\alpha]$; for X' ($X \in \overline{S}$), $[X']^{\alpha^\kappa} = [(X^\alpha)']$. Then κ is a group homomorphism, $\kappa: N \to \mathrm{Aut}(D_1)$.

Proof We show that the mapping above is well defined. If $A, B \in X'$ with $[(A^\alpha)'] \neq [(B^\alpha)']$ then the set of blocks in T containing A^α is different from the set of blocks in T containing B^α. However $A, B \in X'$, so A and B and hence A^α and B^α are incident with the same set of blocks of T, contradicting $[(A^\alpha)'] \neq [(B^\alpha)']$.

If $\alpha \in N$, then α^κ is an automorphism of D_1, since $\alpha \in \mathrm{Aut}(D)$. Further, by considering the image of a point $[y]$ of D_1 under $(\alpha\beta)^\kappa$, we can show that κ is an homomorphism. □

Although elations are only defined for symmetric designs, if α is an elation of D, α^κ has also 'elation-like' properties, as seen in the following lemma.

Lemma 2.4 If $x \in T$, $X \in x \backslash S$, and α is an (X, x)-elation in N, then the following hold (a) α^κ has exactly one axis, and the set of fixed points of α^κ is $\{[y] \mid y \in T \text{ and } y \ni X\}$ (b) α^κ has exactly one centre, and the set of fixed blocks of α^κ is $\{[Y'] \mid Y \in x \backslash S\}$ (c) If α has order p and $X \in \overline{S}$, then α^κ has order p.

Proof (a),(b) α fixes the block x pointwise in D, hence the point $[x]$ blockwise in D_1. Let $y \in T$ be any block through X. So $y^\alpha = y$ and so α^κ fixes the points $[y]$, $y \ni X$. We claim that these are exactly the fixed points and blocks of α^κ. Firstly, α fixes no other blocks in T. Secondly, if we suppose $[Y']^{\alpha^\kappa} = [Y']$, for $Y \notin x$ then $Y'^\alpha = Y'$ as sets, so there exists $y \supseteq Y'$ with $y \in T$ such that $y \not\ni X$ (since $X \notin Y'$). Hence α fixes a set of size $\geq \lambda + |Y'| > \lambda$ in the block y, so y is a fixed block. This contradicts $y \not\ni X$. Thus α^κ fixes exactly the blocks $[Y']$, $Y \in x \backslash S$.

(c) This follows since κ is a group homomorphism, so $|\alpha^\kappa| = 1$ or p. Since α does not fix all the blocks of T, α^κ is not the identity in D_1, hence it has order p. □

Denote by $\mathcal{E}l'_N([y])$ the image under κ of $\mathcal{E}l_N(y)$. Since $\mathcal{E}l_N(y)$ is an elementary p-group, so is $\mathcal{E}l'_N([y])$. The next lemma determines the properties of N^κ.

Lemma 2.5 (a) For each flag $([x],[X'])$ in D_1, there exists $\gamma \in N^{\kappa}$ with exactly one centre $[x]$ and exactly one axis $[X']$, (b) N^{κ} is 2-transitive on the points of D_1, (c) for any point $[x]$, $\mathcal{E}l'_N([x])$ is a normal subgroup of $N^{\kappa}{}_{[x]}$.

Proof (a) Follows from Lemma 2.4 since there exists an (X,x)-elation for each $x \in T$, $X \in x$.

(b) As N is transitive on T, N^{κ} is transitive on the points of D_1. Let $[x]$ be any point in D_1. We show that $N^{\kappa}{}_{[x]}$ is transitive on the remaining points of D_1. Let $[y]$, $[z]$ be any two distinct points of D_1, distinct from $[x]$.

If $[x]$, $[y]$, $[z]$ are not collinear, then there exists a block $[Y'] \ni [y]$, $[Y'] \not\ni [z]$ and a block $[Z'] \ni [z]$, $[Z'] \not\ni [y]$. By (a) and Result 1.2 there exists an automorphism $\gamma \in N^{\kappa}$ with γ fixing $[x]$ and mapping $[y]$ to $[z]$.

If $[x]$, $[y]$ and $[z]$ are all collinear, let $[w] \notin [x][y]$. So $[x]$, $[y]$, $[w]$ are not collinear, and $[x]$, $[z]$ and $[w]$ are not collinear. A repeat of the above argument gives the required automorphism.

(c) follows from the fact that κ is a group homomorphism and $\mathcal{E}l_N(x)$ is normal in N_x. □

Lemma 2.6 Let D^* be a symmetric 2-design and let G^* be a subgroup of $\mathrm{Aut}(D^*)$. Suppose that there exist points X, Y and blocks x, y with $X, Y \in x$, $X \notin y$ and $Y \in y$, such that $\mathcal{E}l_{G^*}(X,x)$, $\mathcal{E}l_{G^*}(Y,y)$ and $\mathcal{E}l_{G^*}(Y,x)$ are non-trivial. Then $\mathcal{E}l_{G^*}(X,x)$ and $\mathcal{E}l_{G^*}(Y,y)$ are abelian groups.

Proof This proof uses techniques from [6]. For $\alpha \in \mathcal{E}l_{G^*}(X,x)$ and $\beta \in \mathcal{E}l_{G^*}(Y,y)$ define the maps f_α and f_β as follows (as in [6]):

$$f_\alpha : \mathcal{E}l_{G^*}(Y,y) \to \mathcal{E}l_{G^*}(Y,x) \qquad f_\alpha : \beta \mapsto [\alpha,\beta]$$
$$f_\beta : \mathcal{E}l_{G^*}(X,x) \to \mathcal{E}l_{G^*}(Y,x) \qquad f_\beta : \alpha \mapsto [\alpha,\beta]$$

If $|\mathcal{E}l_{G^*}(X,x)| = 2$ then $\mathcal{E}l_{G^*}(X,x)$ is abelian. Otherwise, suppose α_1, α_2 are distinct non-identity elements of $\mathcal{E}l_{G^*}(X,x)$. Let $\gamma = [\alpha_2,\beta]$ for some $\beta \in \mathcal{E}l_{G^*}(Y,y)$, $\beta \neq 1$. By Result 1.3, $\gamma \in \mathcal{E}l_{G^*}(Y,x)$. By definition $\gamma = \alpha_2^{-1}\beta^{-1}\alpha_2\beta$ and so $\alpha_2\gamma = \beta^{-1}\alpha_2\beta = \alpha_2^{\beta}$, an elation with centre $X^{\beta} \neq X, Y$. So α_1, γ and $\alpha_2\gamma$ are elations with distinct centres and hence commute (Result 1.4). Thus $(\alpha_1\alpha_2)\gamma = \alpha_1(\alpha_2\gamma) = (\alpha_2\gamma)\alpha_1 = (\alpha_2\alpha_1)\gamma$. Hence $\alpha_1\alpha_2 = \alpha_2\alpha_1$ as required. As $\mathcal{E}l_{G^*}(X,x)$ is a group, it is an abelian group.

To show that $\mathcal{E}l_{G^*}(Y,y)$ is an abelian group, use the dual of the result just proved for $\mathcal{E}l_{G^*}(X,x)$. □

Lemma 2.7 For each $z \in T$, $\mathcal{E}l_N(z)$ is an elementary abelian p-group.

Proof By Result 1.5 we only need to show the abelian property. By Result 1.4 elations with distinct centres commute, so it remains to show that elations with the same centre commute.

Let $z \in T$ and $X \in z \backslash S$. Then $|\mathcal{E}l_N(X,z)| \neq 1$. Choose $Y \in S$. Then there exists $y \in T$ with $Y \in y$ and $X \notin y$. Note that $\mathcal{E}l_N(Y,z)$, $\mathcal{E}l_N(Y,y)$ are non-trivial. Applying Lemma 2.6, we have $\mathcal{E}l_N(X,z)$ is abelian.

Let $z \in T$ and $Y \in S$. Choose $X \in \overline{S} \backslash z$ Then $|\mathcal{E}l_N(X,x)| \neq 1$ for some $x \in T$. Also $\mathcal{E}l_N(Y,z)$ and $\mathcal{E}l_N(Y,x)$ are non-trivial. Again by Lemma 2.6, $\mathcal{E}l_N(Y,z)$ is abelian. □

Theorem A Suppose that (D,G) satisfies the hypotheses of Theorem B, that N and H are defined as in the statement of Theorem B, and that D_1 is the quotient design of (D,G). Then

(a) $D_1 \cong \mathrm{P}_{m,q}$ for some $m \geq 2$ and prime power q.

(b) $N/H \cong \mathrm{PSL}(m+1,q)$.

Proof We apply the theorem of O'Nan, Result 1.10, to the group N^κ considered as a permutation group acting on the points of D_1. By Lemma 2.5, N^κ is 2-transitive on the points of D_1 and, for any point $[a]$ of D_1, $\mathcal{E}l'_N([a])$ is a normal subgroup of $N^\kappa_{[a]}$ and $\mathcal{E}l'_N([a]) \neq 1$. By Lemma 2.7, $\mathcal{E}l'_N([a])$ is abelian and by Lemma 2.4, $\mathcal{E}l'_N([a])$ does not act semi-regularly on $T \backslash \{[a]\}$. So we may apply Result 1.10, with $N^* = N^\kappa$, $\Omega = T$ and $A = \mathcal{E}l'_N([a])$.

Examining the proof of Result 1.10 (see [10] or [12]) we see that N^* is shown to act as a group of automorphisms of a 2-design $D(\Gamma)$ isomorphic to the design of points and lines of a projective space $\mathrm{P}_{m,q}$, for some $m \geq 2$ and prime power q. The points of $D(\Gamma)$ are just the elements of Ω, while the blocks of $D(\Gamma)$ are the subsets of Ω of the form $F_\Omega(A^\sigma_y)$, where $\sigma \in N^*$ and $y \in \Omega \backslash \{a^\sigma\}$ (see [12], p306). In our situation, the points of $D(\Gamma)$ are the elements of T and the blocks are the distinct subsets of T of the form $\Gamma(x,y) = \{t \in T \mid t \text{ is fixed by every element of } \mathcal{E}l'_N([x])_{[y]}\}$, where x and y range over T with $x \neq y$.

In the proof of Result 1.10 it is also shown (see [12], p311) that if $x = a^\sigma$ then A^σ consists of all the elations with centre x in the projective geometry $D(\Gamma)$. The fixed points of any non-trivial element of A^σ therefore form a hyperplane in this projective geometry.

In our situation, suppose $[X']$ is any block of D_1. Choose a point $[x]$ of D_1 which is incident with $[X']$, and choose a non-trivial element α of $\mathcal{E}l_N(X,x)$. Then $\alpha^\kappa \in \mathcal{E}l'_N([x])$ and so the elements of T, i.e. points of $D(\Gamma)$, fixed by α^κ form a hyperplane in $D(\Gamma)$. By Lemma 2.4, the elements of T fixed by α^κ are just those which are incident (as blocks of D) with the point X of D. So the points of D_1 incident with the block $[X']$ of D_1 are precisely the points of a hyperplane in the projective geometry $D(\Gamma)$.

It follows from the definition of "blocks" in D_1 that the hyperplanes corresponding to two distinct blocks of D_1 must themselves be distinct. However D_1 has at least as many blocks as points (since it is a 2-design), whereas $D(\Gamma)$ has equally many points and hyperplanes (as it is a projective geometry). So every hyperplane of $D(\Gamma)$ corresponds to a block in D_1.

We conclude that $D_1 \cong \mathrm{P}_{m,q}$ and $\mathrm{PSL}(m+1,q) \leq N^\kappa \leq \mathrm{P\Gamma L}(m+1,q)$. But N^κ is generated by elations, so $N^\kappa \leq \mathrm{PSL}(m+1,q)$. Thus $N^\kappa = \mathrm{PSL}(m+1,q)$. This completes the proof of Theorem A. □

§3 Translation Blocks

We are still assuming the hypotheses of Theorem B. In this section we first show (using Theorem A) that every block in T is a translation block. We then proceed, using the work of Kelly (Result 1.9), to complete the proof of Theorem B.

By Theorem A, $D_1 \cong \mathrm{P}_{m,q}$ for some $m \geq 2$ and prime power q. For any n, let us denote by $Q(n)$ the number $\frac{q^n-1}{q-1}$.

Lemma 3.1 We have the following relations holding between the numbers m, q, j (see Lemma 2.1), $|S|$, $|T|$ and the parameters v, k, λ of the design D:

(a) $|T| = \mathrm{Q}(m+1)$

(b) $v = |S|q^{m+1} + \mathrm{Q}(m+1)$.

(c) $k = |S|q^m + \mathrm{Q}(m)$.

(d) $\lambda = |S|q^{m-1} + \mathrm{Q}(m-1)$.

(e) $|S| = \frac{j-1}{q-1}$.

(f) $v - |S| = j\mathrm{Q}(m+1)$.

(g) $k - |S| = j\mathrm{Q}(m)$.

(h) $\lambda - |S| = j\mathrm{Q}(m-1)$.

Proof By Theorem A, $D_1 \cong \mathrm{P}_{m,q}$ and so the parameters of D_1 are $\underline{v} = \underline{b} = \mathrm{Q}(m+1)$, $\underline{r} = \mathrm{Q}(m)$, $\underline{\lambda} = \mathrm{Q}(m-1)$. Equating these to the expressions for $\underline{v}, \underline{b}, \underline{r}$ and $\underline{\lambda}$ in Lemma 2.2 gives (a), (f), (g) and (h) respectively.

We now prove (e). As D is a symmetric design, $\lambda(v-1) = k(k-1)$ and so $\lambda(v-k) = (k-\lambda)(k-1)$. By subtracting (g) from (f), and (h) from (g), we obtain $v - k = jq^m$ and $k - \lambda = jq^{m-1}$. Substituting the expressions for $\lambda, v-k, k-\lambda$ and $k-1$ in $\lambda(v-k) = (k-\lambda)(k-1)$ we obtain

$$\left(j\frac{q^{m-1}-1}{q-1} + |S|\right)(jq^m) = (jq^{m-1})\left(j\frac{q^m-1}{q-1} + |S| - 1\right).$$

Simplifying this expression gives $|S| = \frac{j-1}{q-1}$, which proves (e).

Rearranging the expression for $|S|$ in (e), we get $j = |S|(q-1)+1$. Substituting this expression for j in (f), (g) and (h) gives (b), (c) and (d). □

Lemma 3.2 If $X \in S$ and $Y \in \overline{S}$ then $|\mathcal{E}l_N(X)| = q^{m+1}$ and $|\mathcal{E}l_N(X)_Y| = q^m$.

Proof Let $X \in S$. Then $N_X = N$ and so N_X is transitive on T. Suppose $|\mathcal{E}l_N(X)| = p^a$, $|\mathcal{E}l_N(X)_Y| = p^{d(Y)}$. Since N fixes X and is transitive on $\overline{S}$, $d(Y)$ is a constant d for $Y \in \overline{S}$. Counting the pairs (α, Y) where $\alpha \neq 1$ is in $\mathcal{E}l_N(X)$ and $Y \in \overline{S}$ is fixed by α, we get

$$(p^a - 1)(k - |S|) = (p^d - 1)(v - |S|) \tag{1}$$

Using Lemma 3.1

$$(v - |S|, k - |S|) = (j\mathrm{Q}(m+1), j\mathrm{Q}(m)) = j(\mathrm{Q}(m+1), \mathrm{Q}(m)) = j.$$

So (1) gives

$$\underbrace{\left(\frac{k-|S|}{j}\right)}_{A}\underbrace{\left(\frac{p^a-1}{p^{(a,d)}-1}\right)}_{B} = \underbrace{\left(\frac{v-|S|}{j}\right)}_{C}\underbrace{\left(\frac{p^d-1}{p^{(a,d)}-1}\right)}_{D}$$

So $AB = CD$ and since $(A,C)=1$, $A \mid D$ and $C \mid B$, and since $(B,D)=1$, $B \mid C$ and $D \mid A$. Thus $A = D$ and $B = C$. Thus

$$\frac{p^a-1}{p^{(a,d)}-1} = \frac{p^{u(m+1)}-1}{p^u-1},$$

where $p^u = q$. It follows that $p^u = p^{(a,d)}$, $p^a = q^{m+1}$ and $p^d = q^m$. □

Under N, D has $|S|+1$ point orbits, namely the points of S and the orbit $\overline{S}$. As D is symmetric there are $|S|+1$ block orbits. Excluding the orbit T, let $\Lambda_1, \ldots, \Lambda_{|S|}$ be the block orbits. Let $\varphi_i = y \cap S$ for any $y \in \Lambda_i$; this is well defined since N fixes S pointwise.

Consider the structure D_2 whose points are the points of S and whose blocks are the subsets φ_i of S, for $i = 1, \ldots |S|$. We now show that the φ_i are a constant size.

Lemma 3.3 If $|S| > 1$ then D_2 is uniform with block size $\frac{|S|-1}{q}$.

Proof As $\overline{T} = \bigcup_{i=1}^{|S|} \Lambda_i$, we have (by Lemma 3.1)

$$\sum_{i=1}^{|S|} |\Lambda_i| = |S| q^{m+1} \tag{1}$$

Count the flags (P, y) where $P \in S$ and $y \in \overline{T}$, so $|S|(k - |T|) = \sum_{i=1}^{|S|} |\Lambda_i||\varphi_i|$. By Lemma 3.1,

$$\sum_{i=1}^{|S|} |\Lambda_i||\varphi_i| = q^m |S|(|S| - 1) \tag{2}$$

Count the 2-flags (P, Q, y) where $P, Q \in S$, $P \neq Q$ and $y \in \overline{T}$.

$$|S|(|S| - 1)(\lambda - |T|) = \sum_{i=1}^{|S|} |\Lambda_i||\varphi_i|(|\varphi_i| - 1)$$

$$\begin{aligned} |S|(|S| - 1)q^{m-1}(|S| - (q+1)) &= \sum_{i=1}^{|S|} |\Lambda_i||\varphi_i|^2 - \sum_{i=1}^{|S|} |\Lambda_i||\varphi_i| \\ &= \sum_{i=1}^{|S|} |\Lambda_i||\varphi_i|^2 - q^m |S|(|S| - 1) \end{aligned}$$

from (2), so

$$\sum_{i=1}^{|S|} |\Lambda_i||\varphi_i|^2 = |S|(|S| - 1)q^{m-1}(|S| - (q+1) + q) = |S|(|S| - 1)^2 q^{m-1} \tag{3}$$

We show that $|\varphi_i| = (|S| - 1)/q$ for $i = 1, \ldots, |S|$. Consider

$$\sum_{i=1}^{|S|} |\Lambda_i| \left(|\varphi_i| - \left(\frac{|S| - 1}{q} \right) \right)^2$$

$$= \sum_{i=1}^{|S|} |\Lambda_i||\varphi_i|^2 - 2\left(\frac{|S| - 1}{q} \right) \sum_{i=1}^{|S|} |\Lambda_i|||\varphi_i| + \left(\frac{|S| - 1}{q} \right)^2 \sum_{i=1}^{|S|} |\Lambda_i|.$$

So, from (3), (2) and (1)

$$= |S|(|S| - 1)^2 q^{m-1} - 2\left(\frac{|S| - 1}{q} \right) q^m |S|(|S| - 1) + \left(\frac{|S| - 1}{q} \right)^2 |S| q^{m+1}$$

$$= 0$$

Since the left hand side is a sum of squares, it follows that each term must be zero i.e. $|\varphi_i| = \frac{|S|-1}{q}$ for $i = 1, \ldots, |S|$. □

Lemma 3.4 Every block of D which contains S belongs to T.

Proof If $|S| > 1$ and $y \in \overline{T}$ then, by Lemma 3.3, y meets S in fewer than $|S|$ points and so y does not contain S. If $|S| = 1$ then N has only two block orbits: one must consists of the blocks which contain S and the other of the blocks which do not contain S.

Lemma 3.5

(a) $|\Lambda_i| = q^{m+1}$, $i = 1, \ldots, |S|$.

(b) For all $x \in T$, N_x is transitive on each Λ_i.

Proof For any given i, let $X \in S\backslash\varphi_i$. This is possible since if $|S| > 1$, $|\varphi_i| = \frac{|S|-1}{q}$, by Lemma 3.3, and if $|S| = 1$, $\varphi_i = 0$. Now consider the action of $\mathcal{E}l_N(X)$ on Λ_i. Since no block in Λ_i contains X, $\mathcal{E}l_N(X)$ acts semi-regularly on Λ_i (dual of Result 1.1). But $|\mathcal{E}l_N(X)| = q^{m+1}$, by Lemma 3.2. So $q^{m+1} \mid |\Lambda_i|$. However $\sum_{i=1}^{|S|} |\Lambda_i| = |S|q^{m+1}$, by Lemma 3.1. So $|\Lambda_i| = q^{m+1}$ and $\mathcal{E}l_N(X)$ is transitive on Λ_i. This proves both (a) and (b). □

Lemma 3.6 For all $x \in T$, N_x is transitive on the points of $x\backslash S$.

Proof Let $x \in T$ and $X_1, X_2 \in x\backslash S$. Choose $y_1, y_2 \in T$ with $X_1 \notin y_1$ and $X_2 \notin y_2$.

If $X_2 \notin y_1$ choose $Y_1 \in y_1\backslash x$ and then use N to map (X_1, x) to (Y_1, y_1) and then map (Y_1, y_1) to (X_2, x) (by Result 1.2).

If $X_1 \notin y_2$ we may similarly map (X_1, x) to (X_2, x) via (Y_2, y_2) where $Y_2 \in y_2\backslash x$.

Suppose now that $X_2 \in y_1$ and $X_1 \in y_2$. Then $y_1 \neq y_2$ so $y_1 \not\subseteq x \cup y_2$ and $y_2 \not\subseteq x \cup y_1$. We may therefore choose $Y_2 \in y_2\backslash(x \cup y_1)$ and $Y_1 \in y_1\backslash(x \cup y_2)$ and use N to map (X_1, x) to (X_2, x) via (Y_1, y_1) and (Y_2, y_2). □

Lemma 3.7 Every block in T is a translation block.

Proof Suppose $x \in T$. Let $|\mathcal{E}l_N(x)| = p^a$. For $y \in T\backslash\{x\}$ let $|\mathcal{E}l_N(x)_y| = p^{d(y)}$, and for $y \in \overline{T}$, let $|\mathcal{E}l_N(x)_y| = p^{e(y)}$. Count (α, y) where $\alpha \neq 1$ in $\mathcal{E}l_N(x)$ fixes the block $y \neq x$.

$$(k-1)(p^a - 1) = \sum_{y \in T\backslash\{x\}} (p^{d(y)} - 1) + \sum_{y \in \overline{T}} (p^{e(y)} - 1) \tag{1}$$

Recall the group homomorphism $\kappa: N \to N^\kappa$. This results in the isomorphism:

$$\frac{\mathcal{E}l_N(x)}{\ker(\kappa) \cap \mathcal{E}l_N(x)} \cong \mathcal{E}l'_N([x]).$$

Since $|\mathcal{E}l_N(x)| = p^a$ and by Theorem A, $N^\kappa \cong \mathrm{PSL}(m+1, q)$ so $|\mathcal{E}l'_N([x])| = q^m$ (recall from §2 that $\mathcal{E}l'_N([x])$ is the image under κ of $\mathcal{E}l_N(x)$). If $|\ker(\kappa) \cap \mathcal{E}l_N(x)| = p^z$ then it follows that $p^a = q^m p^z$. For $y \in T\backslash\{x\}$,

$$\frac{\mathcal{E}l_N(x)_y}{\ker(\kappa) \cap \mathcal{E}l_N(x)_y} \cong (\mathcal{E}l'_N([x]))_{[y]}$$

and so $p^{d(y)}/p^z = q^{m-1}$, $p^{d(y)} = q^{m-1}p^z$, since $\ker(\kappa)\cap\mathcal{E}l_N(x) = \ker(\kappa)\cap(\mathcal{E}l_N(x)_y)$ for all $y \in T\backslash\{x\}$. So $p^{d(y)}$ is a constant p^d, say, for all $y \in T\backslash\{x\}$.

Let $y \in \overline{T}$. We show that $|y^{\mathcal{E}l_N(x)}| \le q$. For $\phi \in \mathcal{E}l_N(x)$, y^ϕ is a block not equal to x and containing $x \cap y$. So distinct elements of $y^{\mathcal{E}l_N(x)}$ intersect exactly in $x \cap y$. There are at most $\frac{v-k}{k-\lambda}$ blocks in $y^{\mathcal{E}l_N(x)}$, that is

$$\begin{aligned} |y^{\mathcal{E}l_N(x)}| &\le \frac{v-k}{k-\lambda} \\ &= \frac{jq^m}{jq^{m-1}} \quad \text{by Lemma 3.1} \\ &= q. \end{aligned}$$

By the orbit-stabilizer theorem, $|\mathcal{E}l_N(x)_y||y^{\mathcal{E}l_N(x)}| = |\mathcal{E}l_N(x)|$. As $|y^{\mathcal{E}l_N(x)}| \le q$ and $|y^{\mathcal{E}l_N(x)}| \mid |\mathcal{E}l_N(x)|$, we have $|y^{\mathcal{E}l_N(x)}| \mid q$. Hence

$$\frac{|\mathcal{E}l_N(x)|}{|\mathcal{E}l_N(x)_y|} = \frac{q^m p^z}{p^{e(y)}} \,\Big|\, q$$

and so $q^{m-1}p^z \mid p^{e(y)}$. Write $p^{e(y)} = q^{m-1}p^z p^{f(y)}$ where $p^{f(y)} \ge 1$. From (1)

$$\begin{aligned} &(k-1)(q^m p^z - 1) \\ &= (|T|-1)(p^z q^{m-1} - 1) + \left(\sum_{y\in\overline{T}} q^{m-1}p^z p^{f(y)}\right) - (v - |T|) \\ &= (|T|-1)(p^z q^{m-1} - 1) + q^{m-1}p^z \sum_{y\in\overline{T}}(p^{f(y)} - 1) + q^{m-1}p^z(v - |T|) - (v - |T|) \\ &= (|T|-1)(p^z q^{m-1} - 1) + (v - |T|)(q^{m-1}p^z - 1) + q^{m-1}p^z \sum_{y\in\overline{T}}(p^{f(y)} - 1) \\ &= (v-1)(q^{m-1}p^z - 1) + q^{m-1}p^z \sum_{y\in\overline{T}}(p^{f(y)} - 1) \end{aligned}$$

Hence

$$q^{m-1}p^z\left((k-1)q - (v-1)\right) - (k-1) + (v-1) = q^{m-1}p^z \sum_{y\in\overline{T}}(p^{f(y)} - 1)$$

However $(k-1)q - (v-1) = (k-1)q - \frac{1}{\lambda}k(k-1)$. We now show that $\frac{k-1}{\lambda} = q$. Using Lemma 3.1 we have $k = j\mathrm{Q}(m) + \frac{j-1}{q-1}$ and hence $k - 1 = j\mathrm{Q}(m) + \frac{j-q}{q-1}$. Again by Lemma 3.1, $\lambda = j\mathrm{Q}(m-1) + \frac{j-1}{q-1}$ and so

$$\frac{k-1}{\lambda} = \frac{j(q-1)\mathrm{Q}(m) + (j-q)}{j(q-1)\mathrm{Q}(m-1) + (j-1)}$$

which simplifies to q. Hence $(k-1)q-(v-1)=(k-1)q-kq=-q$ and $(v-1)-(k-1)=v-k=jq^m=Jq^mp^z$ for $J=j/p^z\geq 1$. So $q^{m-1}p^z(-q)+Jq^mp^z=q^{m-1}p^z\sum_{y\in\overline{T}}(p^{f(y)}-1)$ and so

$$q(J-1)=\sum_{y\in\overline{T}}(p^{f(y)}-1) \tag{2}$$

We now show $J\leq q-1$.

Denote by $\mathcal{E}l'_N([x],[Y'])$ the elation group in N^κ with centre $[x]$, axis $[Y']$, where $Y\in x\backslash S$. Consider the pre-image H (under the restriction of κ to $\mathcal{E}l_N(x)$) of $\mathcal{E}l'_N([x],[Y'])$. We have:

$$\frac{H}{\ker(\kappa)\cap H}\cong\mathcal{E}l'_N([x],[Y'])$$

Counting the elements of H, we get

$$|H|=j(p^h-1)+p^r+a_1$$

where a_1 is the number of elements in $H\backslash(\ker(\kappa)\cap\mathcal{E}l_N(x))$ with no centre, $p^h=|\mathcal{E}l_N(Y,x)|$ and $p^r=|\ker(\kappa)\cap H|$. (Note that $|\mathcal{E}l_N(X,x)|$ is constant as X ranges over Y', by Lemma 3.6.) So $j(p^h-1)+p^r+a_1=qp^r$ which implies that $j(p^h-1)+a_1=p^r(q-1)$. So $j\leq p^r(q-1)$ and hence

$$J=j/p^z\leq j/p^r\leq q-1. \tag{3}$$

(Note that $p^r\leq p^z$ because $H\subseteq\mathcal{E}l_N(x)$.) By Lemma 3.5(b), $p^{f(y)}=p^{f_i}$, a constant for all $y\in\Lambda_i$. It follows, using Lemma 3.5(a), that $q(J-1)=\sum_{i=1}^{|S|}q^{m+1}(p^{f_i}-1)$. Suppose for some Λ_i, $f_i\neq 0$, i.e. $p^{f_i}-1>0$. Then $q(J-1)\geq q^{m+1}$. But $J\leq q-1$ by (3), so $q(q-2)\geq q^{m+1}$, $q-2\geq q^m$ and $q(1-q^{m-1})\geq 2$. As $q\geq 2$ and $m\geq 2$, this is a contradiction. So $p^{f_i}=1$ for all i, and from (2), $J=1$.

Hence $v-k=jq^m=p^zq^m$ and $|\mathcal{E}l_N(x)|=p^a=p^zq^m$, i.e. x is a translation block. □

<u>Lemma 3.8</u> The design D is a $\mathrm{D}(n,q)$ for some $n\geq 3$.

Proof In the notation used in the proof of Lemma 3.7, we have $v-k=jq^m=p^zq^m$, and so $j=p^z$. By Lemma 3.1, $|S|=\frac{j-1}{q-1}$ and thus $p^z=q^t$ for some $t\geq 1$. By Lemma 3.1 again, $v=j\mathrm{Q}(m+1)+|S|=q^t\left(\frac{q^{m+1}-1}{q-1}\right)+\frac{q^t-1}{q-1}=\frac{q^{m+t+1}-1}{q-1}=\mathrm{Q}(m+t+1)$, and similarly $k=\mathrm{Q}(m+t)$ and $\lambda=\mathrm{Q}(m+t-1)$. Let $n=m+t$. As D is a symmetric 2-design, we have proved that D is a $\mathrm{D}(n,q)$. Note that $n\geq 3$ since $m\geq 2$. □

Proof of Theorem B By Lemmas 3.4, 3.7 and 3.8, D is a $D(n,q)$ for some $n \geq 3$ and prime power q, every block in T is a translation block, and every block of D which contains S belongs to T. Also, since (D,G) is in Class B, each point of $\overline{S}$ lies on at least two blocks in T, and therefore S is not equal to and does not contain the intersection of any two blocks of D. It follows, by Results 1.9 and 1.6, that either (i) $D \cong \mathrm{P}_{n,q}$ or (ii) $D_2 = D(S)$ is a $\mathrm{D}(s,q)$ for some s, $2 \leq s \leq n-3$, and D is isomorphic to a design D^* obtained from $\mathrm{P}_{n,q}$ by substituting the design D_2 for the design of points and hyperplanes of a suitable s-dimensional subspace U of $\mathrm{P}_{n,q}$.

Suppose that (ii) holds. Then the point-set of D (strictly speaking, D^*) is obtained from that of $\mathrm{P}_{n,q}$ by replacing the point-set of U by the point-set of D_2; and, if y is a block of $\mathrm{P}_{n,q}$ then the corresponding block y' in D is obtained by replacing $y \cap U$ by $(y \cap U)^\theta$ if $y \not\supset U$ (or by S if $y \supset U$), where θ is the fixed bijection from the block-set of U to the block-set of D_2 (as in [8], p239).

Now consider the map η from D to $\mathrm{P}_{n,q}$ defined as follows. Each block z of D is mapped by η to the (unique) block y of $\mathrm{P}_{n,q}$ such that $z = y'$. Each point of D which lies in $\overline{S}$ is mapped by η to itself. If $X \in S$ then there is a non-identity elation α in N with centre X and axis $x' \supset S$. Now α induces, in an obvious way, an automorphism β of $\mathrm{P}_{n,q}$ which fixes x pointwise but has no further fixed points. Since β is an axial automorphism of $\mathrm{P}_{n,q}$, it has a (unique) centre Y. If y' is any block of D which contains X, then α fixes y' and so β fixes y, that is y contains Y. So the blocks y of $\mathrm{P}_{n,q}$ such that $y' \ni X$ have a unique common point Y. Necessarily, $Y \in U$. For $X \in S$, we define $\eta(X)$ to be the point Y constructed in this way.

But then η is an isomorphism from D to $\mathrm{P}_{n,q}$ (as it maps concurrent blocks to concurrent blocks). So $D \cong \mathrm{P}_{n,q}$.

By Theorem A, $N/H \cong \mathrm{PSL}(m+1,q)$. But $n \geq m+1$ by the proof of Lemma 3.8, and $m \geq 2$. So $N/H \cong \mathrm{PSL}(n-s,q)$ for some s, $0 \leq s \leq n-3$. This completes the proof of Theorem B. □

Remark

The pairs (D,G) in Class C are examined in [5], where under extra conditions it is shown that D is a $\mathrm{D}(n,q)$ and that D has a subspace isormorphic to $\mathrm{P}_{s,q}$ for some s, $2 \leq s \leq n-2$. In particular, if $s = n-2$ then $D \cong \mathrm{P}_{n,q}$ or D is obtained from $\mathrm{P}_{n,q}$ by a process called K-alteration.

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