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Edge proximity and matching extension in planar triangulations

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Abstract

Let G be a graph with at least $2(m+n+1)$ vertices. Then G is $E(m,n)$ if for each pair of disjoint matchings $M, N \subseteq E(G)$ of size m and n respectively, there exists a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. In the present paper we wish to study property $E(m,n)$ for the various values of integers m and n when the graphs in question are restricted to be planar. It is known that no planar graph is $E(3,0)$ or $E(2,1)$. In this paper we show that in planar even triangulations, matchings of size three satisfying certain proximity conditions can be extended to perfect matchings. We also determine precisely for which values of m and n, the property $E(m,n)$ holds when the graphs involved are even triangulations or near-triangulations of the plane.

1 Introduction

In this paper all graphs will be finite and, unless otherwise specified, simple as well. Let G be a graph with at least $2(m+n+1)$ vertices. Graph G is said to be $E(m,n)$ if for every pair of disjoint matchings $M, N \subseteq E(G)$ of size m and n respectively, there is a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. If G is $E(n,0)$, we say that G is n-extendable and it was this special case which led to the study of $E(m,n)$ in general. (For two surveys of work on n-extendable graphs, see [7, 8].) The present paper deals with the parameter $E(m,n)$ applied primarily to triangulations and near-triangulations of the plane. It was shown in [5] that no planar graph is

$E(3,0)$ and more recently in [1], this result was strengthened to show that no planar graph is even $E(2,1)$. In the present paper we determine precisely those values of m and n for which the property $E(m,n)$ holds for planar triangulations. In addition, we will present a first result dealing with the extension of matchings in which the edges are sufficiently far apart pairwise. Examples are also given to show that the results presented are best possible in several senses. For a general background on matching in graphs, the reader is referred to [3].

2 Doubly independent edge sets

Let G be a graph with with edge set $E(G)$. An independent set of edges $F = \{e_i = u_iv_i : i = 1, \ldots, k\} \subseteq E(G)$ is said to be *doubly independent* if for each $x \in \{u_i, v_i\}$ and $y \in \{u_j, v_j\}$, $1 \leq i < j \leq k$, $xy \notin E(G)$. (Note that the subgraph of G induced by the endvertices of a doubly independent set of edges is a matching. As such, a doubly indepedent set of edges is sometimes known as an *induced* matching.)

Theorem 2.1 *Let G be a 5-connected planar triangulation on an even number of vertices and let $F = \{e_1, e_2, e_3\}$ be a doubly independent subset of $E(G)$. Then there is a perfect matching in G containing F.*

Proof. Suppose to the contrary that there is no such perfect matching. We denote by $V(F)$ the set of six endvertices of edges in F. Then $G - V(F)$ contains a vertex cutset S such that $G - V(F) - S$ has at least $|S| + 2$ odd components, by Tutte's Theorem. Choose such an S to be of minimum size. Note that, since G is a 5-connected triangulation and F is doubly independent, $G - V(F)$ is connected. (Recall that a cutset in a triangulation induces at least one separating cycle.) Consequently, $S \neq \emptyset$ and $G - V(F) - S$ has at least three odd components. Let $K = V(F) \cup S$. If $G - K$ has t odd components and $|S| = s$, then by parity, $t \geq s + 2$. Since G is 5-connected, planar and even, G is $E(2,0)$ (cf.[6], [2]), so we may conclude that $t = s + 2$. Thus $|K| = k = s + 6 = t + 4$. Note also that each odd component in $G - K$ has at least 5 neighbours in K.

We fix a plane embedding of G with respect to which we consider the subgraph H induced by the vertices in K. Then H is a plane graph with t faces of size 5 or more. All other faces have size 3. We call faces of size 5 or more *holes*. Each hole in H is bounded by a set of vertices in K forming a separating cycle in G.

Let us assume first that H is connected. We form a connected spanning subgraph H' of H as follows. Successively delete edges lying in two triangles. From the resulting graph in which no two triangles share an edge, delete one edge arbitrarily from each triangle. From the resulting triangle-free graph, successively delete any edge lying in two quadrilaterals. Finally, at this stage delete precisely one edge from each remaining quadrilateral. Note that in the above process, at no stage are two holes merged into a single face. Thus H' is connected on k vertices, has at least t faces and has minimum face size at least 5. Now, suppose H' has e edges and f faces. By Euler's formula we have

$$k - e + f = 2.$$

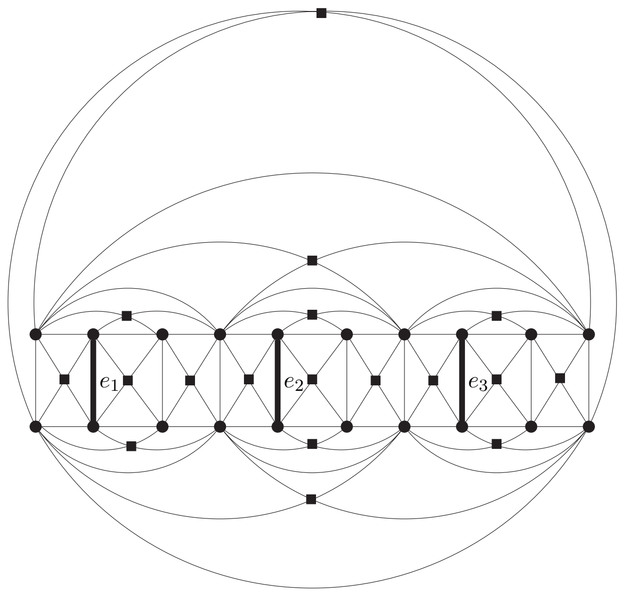


Figure 2.1

Since each face has size at least 5 and there are at least t faces this yields

$$2k - 5f + 2f = 2k - 3f \geq 4$$

or

$$2k = 2t + 8 \geq 3f + 4 \geq 3t + 4$$

and so

$$t \leq 4.$$

If H is not connected, performing a similar analysis on each component and allowing for the common infinite face in H', we conclude that in all cases $t = s{+}2 \leq 4$.

Thus, to complete the proof we need only show that $s \geq 3$ (and hence $t \geq 5$). Hence suppose $s \leq 2$.

As we noted earlier, $S \neq \emptyset$ so there are at least three odd components in $G - K$. Each such odd component has at least 5 neighbours in K forming a separating cycle in G. Since F is doubly independent, $s = 2$ and each separating cycle in H is either a 5-cycle or a 6-cycle and must contain at least one edge from F. Moreover, if we denote S by $\{u, v\}$, then $t = 4$ and both u and v lie in each of the four separating cycles in K which surround the four odd components respectively in $G - K$ and each such separating cycle also contains at least one edge in F. Now, if there is such a separating cycle, C, of length 6, then C must contain two edges from F and both vertices u and v. We cannot have u adjacent to v as this would yield a separating 4-cycle in G. In order that u might lie in the separating cycles around four odd

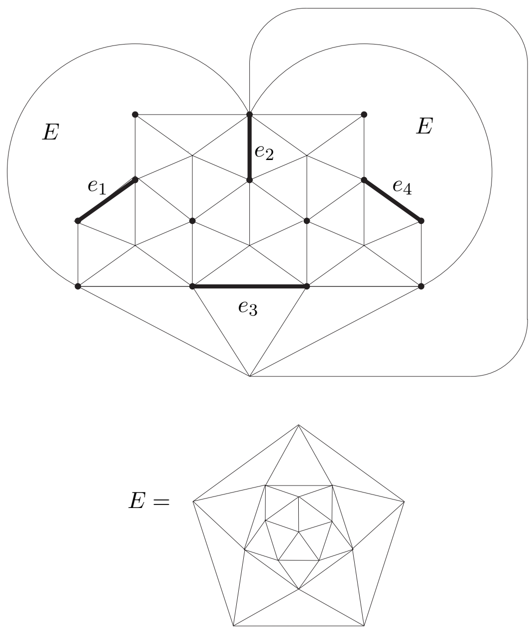


Figure 2.2

components, it must have at least two neighbours in K outside of C. These must be both ends of the one remaining edge of F outside C. Similarly, v must be adjacent to both ends of this third edge in F. Consequently, this third edge in F belongs to two triangles and hence cannot lie in any of the four separating cycles around odd components of $G-K$. Hence each of the four separating cycles must contain at least one of the remaining two edges from F. But the remaining pair of edges in F both lie in the same separating 6-cycle, so together they can lie in at most three of the required separating cycles and we have a contradiction. Thus we may assume that all separating cycles in G around odd components of $G-K$ are of length 5.

Clearly, each edge in F can lie in at most two such separating cycles and, since $G-K$ has four odd components, if $s=2$, e_1, say, must lie in two separating cycles of length 5. But this forces a separating cycle of length 4 in G. This contradiction establishes that $s \geq 3$ and the result now follows. □

The above result is best possible in that we cannot weaken the connectivity hypothesis or strengthen the conclusion. In Figure 2.1 we have a 4-connected planar triangulation on an even number of vertices in which the indicated doubly independent set $\{e_1, e_2, e_3\}$ cannot be extended to a perfect matching. To see this note that after deleting the indicated edges and their endvertices, the remaining 14 round vertices form a Tutte set, that is, after deleting these 14 vertices there are 18 square vertices left as isolates (odd components). Moreover, we see that in the same graph the doubly independent set $\{e_1, e_2\}$ cannot be extended to a perfect matching. The graph in Figure 2.2 is a 5-connected planar triangulation on an even number of vertices in which the indicated edges form a doubly independent set of size four which cannot be extended to a perfect matching. (In Figure 2.2 note that it is understood that where two edges appear to meet there is vertex of $G - (V(F) \cup S)$.)

3 $E(m, n)$ and Regularity

In the plane there are two different types of regularity which may be considered: degree regularity and face regularity. For general graphs (i.e. not necessarily planar) Plesník [4] showed the following.

Theorem 3.1 *Let G be an r-regular $(r-1)$-edge-connected graph with an even number of vertices. Then if any $r-1$ edges are deleted from G, the resulting graph has a perfect matching.*

From this result we may conclude that an r-regular $(r-1)$-edge-connected graph with even order is $E(0, r-1)$ and $E(1, 0)$.

If we consider cubic graphs in the plane, the triangular prism ($K_2 \times K_3$) gives us an example of a 3-connected cubic planar graph which is neither $E(1, 1)$ nor $E(0, 3)$ so that we are unable to exceed the extendability properties guaranteed by Plesník, even when we add planarity to our hypotheses.

In [1], it was shown that 4-connected planar graphs are $E(1, 1)$ and 5-connected planar graphs are $E(1, 2)$. Both of these results were shown to be best possible via 4-regular, and 5-regular, examples respectively. Consequently, we cannot hope to gain by adding 4-regularity or 5-regularity to our hypotheses.

We next consider planar graphs in which all faces have the same size. Again, we have a limited number of cases to consider as each planar graph contains a face of size 3, 4 or 5. We first consider triangulations.

Theorem 3.2 *Let G be a 5-connected planar graph on an even number of vertices such that at most one face of G is not a triangle. Then G is $E(1, 3)$.*

Proof. Suppose to the contrary that G is such a graph and that we have an edge $e \in E(G)$ and three independent edges $\{f_1, f_2, f_3\} \subseteq E(G) - \{e\}$ such that there is no perfect matching in G which uses the edge e and which avoids the edges f_1, f_2 and f_3. That is, the graph $G' = G - V(e) - \{f_1, f_2, f_3\}$ contains no perfect matching. By Tutte's theorem there is a set $S \subseteq V(G')$ such that $G' - S$ has $\sigma \geq |S| + 2$ odd components. In fact, since G is $E(1, 2)$ by Corollary 3.2, we know that $\sigma = |S| + 2$.

Consider a graph G^* obtained from G via G' as follows. Contract to single vertices those subgraphs of G corresponding to odd components of $G' - S$ and delete those vertices of G corresponding to vertices in even components of $G' - S$. Suppress any multiple edges formed in this process (i.e. if on contracting a subgraph of G we get a pair of vertices joined by more than one edge, remove all but one of those edges). Now delete from this graph the edges e, f_1, f_2 and f_3 so that G^* is a bipartite planar graph with one part of the bipartition, B say, given by the vertices in $S \cup V(e)$ and the other part, say W, having $\sigma = |S| + 2$ vertices. (As we shall use this graph

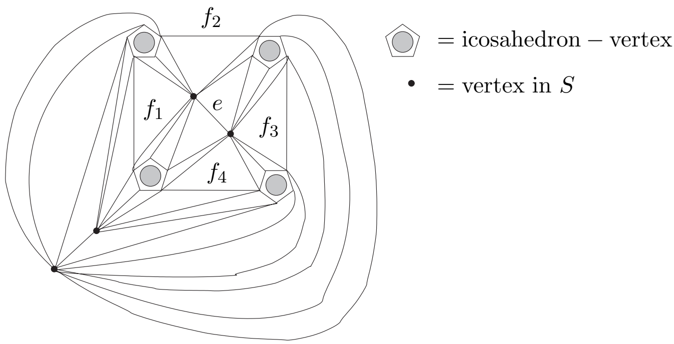


Figure 3.1

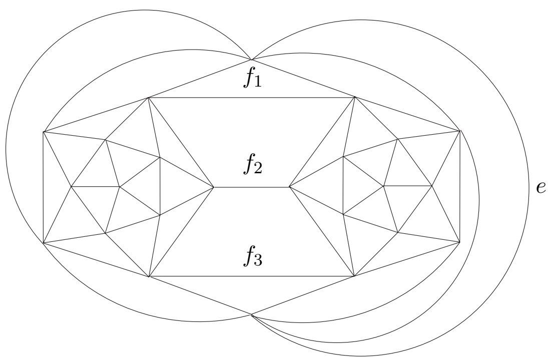


Figure 3.2

construction again in this paper, we name this graph, G^*, the *bipartite distillation of* G *based on* $\{e\}, \{f_1, f_2, f_3\}$ *and* S.)

Since G is 5-connected, there are at least 5 edges incident on each vertex of W in $G^* \cup \{f_1, f_2, f_3\}$ and hence there are at least $5\sigma - 6 = 5|S| + 4$ edges from vertices in W to vertices of B in G^*. However, G^* is bipartite and planar so that we have at most $2(\sigma + (|S| + 2) - 4 = 2(\sigma + |S|) = 4|S| + 4$ edges in total. Thus $|S| = 0$. This indicates that $G - V(e)$ has two odd components C_1 and C_2 joined by a matching formed by the edges f_1, f_2, f_3. But G has at most one non-triangular face so this structure is impossible and the result follows. □

Corollary 3.3 *Let* G *be a* 5*-connected planar triangulation on an even number of vertices. Then* G *is* $E(1, 3)$.

It should be noted that the conclusions of Theorem 3.2 and Corollary 3.3 cannot be strengthened in that the graph shown below in Figure 3.1 is a 5-connected planar even triangulation, but not $E(1, 4)$. (Deleting the endvertices of the edge e together with the edges f_1, f_2, f_3, f_4, leaves the remaining two dark vertices to act as a Tutte set.) On the other hand, the hypotheses of Theorem 3.2 cannot be weakened with respect to the number of non-triangular faces, for the graph displayed in Figure 3.2

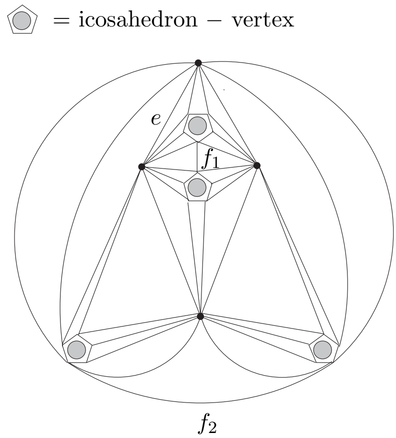


Figure 3.3

is 5-connected, planar and even with precisely two non-triangular faces, but is not $E(1,3)$, for there is clearly no perfect matching which contains edge e, but none of f_1, f_2 or f_3. (Deleting the endvertices of the edge e together with the edges f_1, f_2, f_3, leaves two odd components.)

Moreover, if we weaken the hypotheses to require only that our even planar triangulation be 4-connected, the graph in Figure 3.3 is such a graph which is not $E(1,3)$ (in fact, this graph is not even $E(1,2)$).

Theorem 3.4 *Let G be a 5-connected planar triangulation on an even number of vertices. Then G is $E(0,7)$.*

Proof. Let G be as in the hypothesis of the theorem. We then know that G is $E(1,3)$ by Corollary 3.2 and so $E(0,4)$ by Theorem 3.2 of [9]. But then by Theorem 2.7 of [9], G is also $E(0,3), E(0,2), E(0,1)$ and $E(0,0)$. Now let k be the smallest integer such that G is $E(0,k)$, but not $E(0,k+1)$. (If there is no such k, we are done.) So there exists a set of $k+1$ independent edges $F = \{f_1, \ldots, f_{k+1}\} \subseteq E(G)$ and a set $S \subseteq V(G)$ such that $G - F - S$ has $|S|+2$ odd components $C_1, \ldots, C_{|S|+2}$ and each f_i joins two different C_i's. As before, choose S to a smallest such set. Form G^*, the bipartite distillation as defined above, based upon $\emptyset, F$ and S, and denote by c_i the vertex resulting from the shrinking of odd component C_i, for $i = 1, \ldots, |S|+2$. Let $W = \{c_1, \ldots, c_{s+2}\}$.

Since G is 5-connected there are at least $5(|S|+2) - 2(k+1)$ edges in G^* incident with vertices in W. Graph G^* is planar and bipartite on $2|S|+2$ vertices, so $|E(G^*)| \leq 2(2|S|+2) - 4 = 4|S|$. That is to say,

$$|S| \leq 2k - 8. \tag{1}$$

If $k \geq 7$, we are done. Therefore, we may assume $k \leq 6$. Consequently, S is not a cutset in G. Form $\hat{G}$ from G^* by deleting S and reinserting the edges in F. Now,

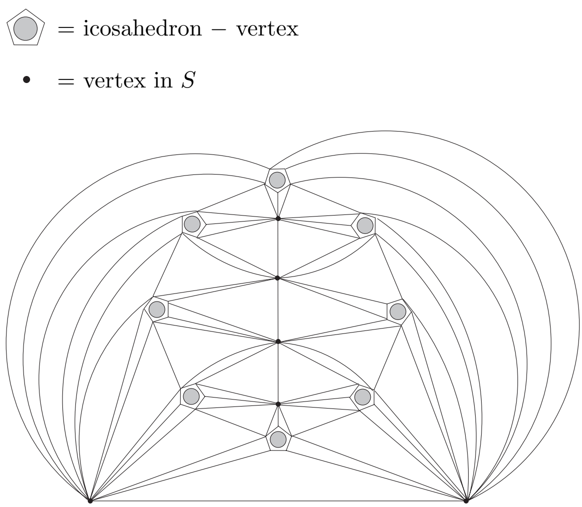


Figure 3.4

$\hat{G}$ is a connected planar multigraph (without loops) with $|S| + 2 \leq 2k - 6$ vertices, $k + 1$ edges and (say) $\hat{\phi}$ faces, so, by Euler's formula:

$$|S| + 2 - (k + 1) + \hat{\phi} = 2.$$

That is,

$$\hat{\phi} = k + 1 - |S|. \tag{2}$$

Since $\hat{G}$ is loopless, each face boundary of $\hat{G}$ contains at least two distinct edges of F. Now reinflate each of the c_i's in $\hat{G}$ to its corresponding C_i. Let the resulting subgraph of G, together with the f_i's, be denoted by H. Then H has at least $\hat{\phi}$ non-triangular faces, since the edges of F are disjoint. But since G is a triangulation, each of these non-triangular faces of H must contain a vertex of S in its interior. Thus $|S| \geq \hat{\phi}$. Thus

$$|S| \geq \frac{k+1}{2}. \tag{3}$$

From (1) and (3) we get

$$k + 1 \leq 2|S| \leq 4k - 16$$

and

$$k \geq 6, \qquad \text{since } k \text{ is an integer.}$$

Hence $k = 6$ and by (2), $\hat{\phi} = 7 - |S|$.

If $|S| < 4$, then $\hat{\phi} > |S|$ and, as before, G cannot be a triangulation. This contradiction implies that $|S| = 4$ and $\hat{\phi} = 3$.

Claim: $\hat{G}$ has no vertex of degree 1.

Proof of claim: Suppose v is a vertex of degree 1 in $\hat{G}$. Let ϕ_1, ϕ_2, ϕ_3 be the faces of $\hat{G}$ and suppose, without loss of generality, that v lies in the boundary of ϕ_1. Then

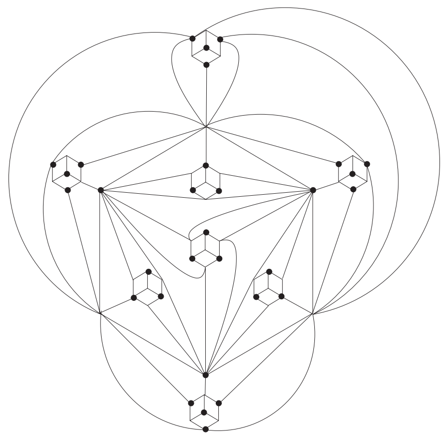


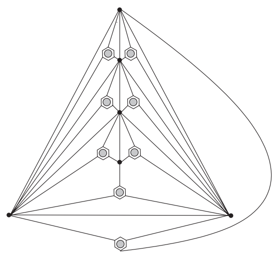
Figure 3.5

re-expand all vertices of $\hat{G}$ and let ϕ_i' be the face corresponding to ϕ_i in $\hat{G}$, $i = 1, 2, 3$ after this expansion. Since the odd component corresponding to v has at least four neighbours in S (by 5-connectivity), all four vertices in S are adjacent to this odd component and all must lie in the interior of ϕ_1'. But this means that ϕ_2' and ϕ_3' cannot be triangles in G and the claim follows.

Thus each vertex in $\hat{G}$ must have degree at least 2. But then, $\hat{G}$ must contain exactly four vertices of degree 2. Now G is a triangulation, so a vertex from S must lie in the interior of each face of $\hat{G}$. As there are four vertices in S and three faces in $\hat{G}$, there must be two vertices from S in the interior of one face, ϕ_1, say, and one each in the remaining two faces. Since G is 5-connected, each vertex of degree 2 in $\hat{G}$ when expanded to an odd subgraph of G must be adjacent to at least three vertices from S. Thus all such vertices must lie in the boundary of ϕ_1 and each must be adjacent to both vertices of S in the interior of ϕ_1. Clearly this cannot happen if G is planar. This contradiction completes the proof. □

Theorem 3.4 is seen to be sharp as the graph in Figure 3.4 is a 5-connected triangulation on an even number of vertices but it is not $E(0, 8)$. (The eight edges joining the eight "pentagonal clusters" cannot all be avoided by a perfect matching since deleting these yields a graph from which deleting the six dark vertices we get eight odd components.)

If all faces in a plane graph G are bounded by 4-cycles, then we say that G is a *quadrangulation*. Similarly, a *pentagonalization* is a plane graph in which all faces are bounded by 5-cycles. It is a straightforward consequence of Euler's formula that no quadrangulation or pentagonalization can be 4-connected. In Figures 3.5 and 3.6 we have examples of 3-connected quadrangulations and pentagonalizations which are not $E(0, 0)$. Note that the quadrangulation shown in Figure 3.5 is also bipartite with an equicardinal bipartition. Consequently, these higher degrees of face regularity cannot guarantee enhanced extendability properties without additional restrictions beyond connectivity.



= the 37 vertex graph obtained from two disjoint dodecahedra by identifying a path of length 2 in the infinite face boundary of each.

Figure 3.6

References

[1] R.E.L. Aldred and M.D. Plummer, On restricted matching extension in planar graphs, *Discrete Math.*, **231** (2001), 73–79.

[2] Dingjun Lou, N-extendability of graphs, *Ph.D. Thesis, University of Otago,* 1992.

[3] L. Lovász and M.D. Plummer, *Matching Theory*, Ann. Discrete Math. **29**, North-Holland, Amsterdam, 1986.

[4] J. Plesník, Connectivity of regular graphs and the existence of 1-factors, *Mat. Casopis Sloven. Akad. Vied.*, **22** (1972), 310–318.

[5] M.D. Plummer, A theorem on matchings in the plane, *Ann. Discrete Math.*, **41** (1989), 347–354.

[6] M.D. Plummer, Extending matchings in planar graphs IV, *Discrete Math.*, **109** (1992), 207–219.

[7] M.D. Plummer, Extending matchings in graphs: a survey, *Discrete Math.*, **127** (1994), 277–292.

[8] M.D. Plummer, Extending matchings in graphs: an update, *Congressus Numer.*, **116** (1996), 3–32.

[9] M.I. Porteous and R.E.L. Aldred, Matching extensions with prescribed and forbidden edges, *Australas. J. Combin.*, **13** (1996), 163–174.

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