AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **28** (2003), Pages 263–272

Noncrossing partitions with fixed points

A. Sapounakis P. Tsikouras

Department of Informatics
University of Pireaus
80, Karaoli and Dimitriou
18534 Pireaus, Greece.

Abstract

The noncrossing partitions with each of their blocks containing a given element are introduced and studied. The enumeration of these partitions is described through a polynomial of several variables which is proved to satisfy a recursive formula. It is shown that each variable increased by one is a factor of this polynomial.

1 Introduction

A partition $\pi = B_1/B_2/\dots/B_m$ of a totally ordered set X is called *noncrossing partition* (n.c.p.) if and only if there do not exist four elements $a < b < c < d$ of X such that $a, c \in B_i$, $b, d \in B_j$ and $i \neq j$. We denote by $NC(X)$ the set of all n.c.p. of X and by $NC(X, m)$ the set of all n.c.p. of X that contain exactly m blocks $B_1, B_2, \dots, B_m$. If $|X| = n$, since there is an obvious order preserving bijection between X and the set $[n] = \{1, 2, \dots, n\}$, we can equivalently deal with $[n]$ instead of X. In this case we will use the notations NC_n and $NC_n(m)$ respectively.

Many authors have worked on n.c.p.; see for example Kreweras [4] and Poupard [6], followed by Edelman [2], [3] and Prodinger [7] and more recently by Athanasiadis [1] and Simion [9].

It is well known that $|NC_n|$ equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, whereas $|NC_n(m)|$ equals the Narayana number $N(n, m) = \frac{1}{n}\binom{n}{m}\binom{m}{m-1}$.

In this work we introduce a particular class of n.c.p. More precisely, we say that a n.c.p. $\pi \in NC(X)$ is *a noncrossing partition with fixed points* the elements of $A \subseteq X$, if and only if every block of π contains exactly one element of A. The set of all these n.c.p. is denoted by $NC(X, A)$. Again if we deal with $[n]$ instead of X we use the notation $NC_n(A)$.

For an application of n.c.p. with fixed points, consider a distribution network for a product manufactured in various plants (the elements of A) of a firm. The product is distributed to particular destinations (the blocks of the partition) from each plant, covering the whole country. Obviously, the noncrossing condition for the various routes might be necessary in order to minimize the cost.

The n.c.p. with fixed points were used in [5] for the construction of the generalized nested sets. In that paper, the set of all n.c.p. of X, with fixed points the elements of A, had been determined by induction on the number $k = |X \backslash A|$.

This paper deals with the evaluation of the cardinal number $|NC_n(A)|$. We will see that this number is determined by the relevant positions of the elements of A. For this reason, we firstly consider the equivalence relation of the translation in the set of all nonempty subsets of $[n]$, defined as follows:

$$A_1 \simeq A_2 \text{ if and only if there exists } c \in [n] \text{ with } A_1 = (A_2 + c) \pmod{n}.$$

Furthermore, for $A \subseteq [n]$ we define a finite sequence $\varkappa_A = (x_i)$ of length $m = |A|$, where x_i, $i \in [m-1]$ is the number of elements of $[n] \backslash A$ lying between the ith and the $(i+1)$st element of A and x_m is the number of elements of $[n]$ that are either smaller or greater than every element of A.

It is easy to prove the following:

Proposition 1.1: *Let $A_1, A_2 \subseteq [n]$; then $A_1 \simeq A_2$ if and only if each one of $\varkappa_{A_1}$,$\varkappa_{A_2}$ is a cyclic permutation of the other.*

Now let $A_1 \simeq A_2$ and $c \in [n]$ such that $A_1 = (A_2 + c) \pmod{n}$. It is obvious that the mapping $\tau : NC_n(A_1) \to NC_n(A_2)$ with $\tau(\pi) = \{(B+c) \pmod{n};\, B \in \pi\}$ is a bijection, so that we obtain the following result:

Proposition 1.2: *If $A_1, A_2 \subseteq [n]$ with $A_1 \simeq A_2$ then $|NC_n(A_1)| = |NC_n(A_2)|$.*

For every $n \in \mathbb{N}^*$ and for every sequence $\varkappa = (x_i), i \in [m]$ in $\mathbb{N}$ with $m \leq n$ and with $\sum_{i=1}^{m} x_i = n - m$ there exists at least one set $A \subseteq [n]$ with $\varkappa_A = \varkappa$. Indeed, for the set $A = \{t_1, t_2, \ldots, t_m\}$ with $t_1 = 1$ and $t_{i+1} = t_i + x_i + 1$, $i \in [m-1]$ we have that $\varkappa_A = \varkappa$.

So, we can define a function f_m of m variables as follows:

$$f_m(x_1, x_2, \ldots, x_m) = |NC_n(A)|$$

where A is any subset of $[x_1 + x_2 + \ldots + x_m + m]$, with $\varkappa_A = (x_1, x_2, \ldots, x_m)$.

From the previous propositions it is clear that f_m is well defined and that $f_m(x_1, x_2, \ldots, x_m) = f_m(y_1, y_2, \ldots, y_m)$, whenever the sequence $(y_1, y_2, \ldots, y_m)$ is a cyclic permutation of $(x_1, x_2, \ldots, x_m)$.

For the evaluation of the formula of the function f_m, which counts $|NC_n(A)|$, it is more convenient to express the problem in the following equivalent form:

Let $X = [m] \cup Y$, where the elements of Y are distributed in the intervals $(i, i+1)$, $i \in [m-1]$ and $(m,+\infty)$, so that $|(i, i+1) \cap X| = x_i$, $\forall i \in [m-1]$ and $|(m,+\infty) \cap X| = x_m$. We want to determine the number $f_m(x_1, x_2, \ldots, x_m) = |NC(X, [m])|$.

In Section 2 we give a recursive formula for f_m, which is used to deduce the explicit formulae of f_m for certain values of m.

In Section 3 we show that $f_m(x_1, x_2, \ldots, x_m)$ is a polynomial which can be written as a product of $\prod_{r=1}^{m}(x_r + 1)$ and of a polynomial $P_m(x_1, x_2, \ldots, x_m)$ of degree $m-2$. Finally, we use the Stirling numbers of the first kind in order to determine the coefficients of ${x_i}^k$ in the polynomial $P_m(x_1, x_2, \ldots, x_m)$.

2 The recursive formula of $\mathbf{f_m}$

Proposition 2.1: *For every $x_1, x_2 \in \mathbb{N}$, we have*

$$f_2(x_1, x_2) = (x_1 + 1)(x_2 + 1).$$

Proof. Let $\pi \in NC(X, [2])$, with $\pi = B_1/B_2$. The block B_1 is equal either to $\{1\}$, or $[1, p] \cap X$, or $\{1\} \cup ([q,+\infty) \cap X)$, or $([1, p] \cup [q,+\infty)) \cap X$, where p (resp. q) is the max (resp. min) element of $(1, 2)$ (resp. $(2,+\infty)$) that belongs to B_1. So, B_1 (and hence π) is uniquely determined by the number of elements μ, ν that it contains in $(1, 2)$ and $(2,+\infty)$ respectively. But since $0 \le \mu \le x_1$, $0 \le \nu \le x_2$ we have $(x_1 + 1)(x_2 + 1)$ choices for these numbers giving $f_2(x_1, x_2) = (x_1 + 1)(x_2 + 1)$. ∎

We now consider the general case $m \ge 3$. Firstly, notice that obviously $f_m(0, 0, \ldots, 0) = 1$ and $f_m(1, 0, \ldots, 0) = m$.

We give a recursive formula for f_m, $m \ge 3$.

Proposition 2.2: *For every sequence $(x_1, x_2, \ldots, x_m)$ of natural numbers, with $m \ge 3$ and $x_1 \ne 0$ the following relation holds:*

$$f_m(x_1, x_2, \ldots, x_m) =$$
$$(x_1 + 1) f_m(0, x_2, \ldots, x_m) + \sum_{t=1}^{x_1} \sum_{k=2}^{m-1} f_k(t-1, x_2, \ldots, x_k) f_{m-k+1}(0, x_{k+1}, \ldots, x_m).$$

Proof. Let $r = \min(1, 2) \cap X$. We partition the set $NC(X, [m])$ into the sets T_i, $i \in [m]$, so that each partition in T_i contains r and i in the same block. Obviously,

$$|T_1| = f_m(x_1 - 1, x_2, \ldots, x_m) \text{ and } |T_2| = f_m(0, x_2, \ldots, x_m).$$

For $k \geq 3$, to each $\pi \in T_k$ correspond two uniquely determined partitions π_1, π_2 where π_1 is a n.c.p. of $(r, k] \cap X$ with fixed points $2, 3, \ldots, k$ and π_2 is a n.c.p. of $\{1\} \cup ([k,_{+\infty}) \cap X)$ with fixed points $1, k, k+1, \ldots, m$. Thus,

$$|T_k| = f_{k-1}(x_2, x_3, \ldots, x_{k-1}, x_1 - 1) f_{m-k+2}(0, x_k, x_{k+1}, \ldots, x_m).$$

Hence, we have $f_m(x_1, x_2, \ldots, x_m) = \sum_{k=1}^{m} |T_k| = f_m(x_1 - 1, x_2, \ldots, x_m) +$
$f_m(0, x_2, \ldots, x_m) + \sum_{k=3}^{m} f_{k-1}(x_1 - 1, x_2, x_3, \ldots, x_{k-1}) f_{m-k+2}(0, x_k, x_{k+1}, \ldots, x_m)$,
giving the relation $f_m(x_1, x_2, \ldots, x_m) - f_m(x_1 - 1, x_2, \ldots, x_m) =$
$f_m(0, x_2, \ldots, x_m) + \sum_{k=2}^{m-1} f_k(x_1 - 1, x_2, \ldots, x_k) f_{m-k+1}(0, x_{k+1}, \ldots, x_m)$.

If we apply the above relation for every $t \in [x_1]$ and then add, we get
$\sum_{t=1}^{x_1} (f_m(t, x_2, \ldots, x_m) - f_m(t-1, x_2, \ldots, x_m)) =$
$x_1 f_m(0, x_2, \ldots, x_m) + \sum_{t=1}^{x_1} \sum_{k=2}^{m-1} f_k(t-1, x_2, \ldots, x_k) f_{m-k+1}(0, x_{k+1}, \ldots, x_m)$,
which gives the required result. ∎

In the next proposition, we will give a recursive relation for f_m, using the values of f_k with $k < m$. For this, we firstly introduce a new function g_m on m variables (which uses f_k with $k < m$ only). Namely, for $m \geq 3$ let

$$g_m(x_1, x_2, \ldots, x_m) = \sum_{t=0}^{x_1} \sum_{k=2}^{m-1} f_k(t-1, x_2, \ldots, x_k) f_{m-k+1}(0, x_{k+1}, \ldots, x_m),$$

where we assume that $f_m(-1, x_2, x_3, \ldots, x_k) = 0$, $\forall k \geq 2$.

Notice that $g_m(0, x_2, \ldots, x_m) = 0$.

Proposition 2.3: *For every sequence $(x_1, x_2, \ldots, x_m)$ of natural numbers, with $m \geq 3$, we have*

$$f_m(x_1, x_2, \ldots, x_m) = \prod_{\nu=1}^{m} (x_\nu + 1) + \sum_{\lambda=1}^{m} \Big(\prod_{\nu=0}^{\lambda-1} (x_\nu + 1)\Big) g_m(x_\lambda, x_{\lambda+1}, \ldots, x_{\lambda+m-1})$$

where $x_0 = 0$ and $x_k = 0$, $\forall k > m$.

Proof. We use induction on the number s of nonzero elements of the sequence (x_i).

Obviously, for $s = 0$ the result holds. Suppose that the result holds for every sequence with s nonzero elements and suppose that (x_i) has got $s+1$ nonzero elements. Without loss of generality suppose that $x_1 \neq 0$.

The sequence (y_i), $i \in [m]$, with $y_i = x_{i+1}$, has got s nonzero elements; so, by induction hypothesis, we have:

$$f_m(y_1, y_2, \dots, y_m) = \prod_{\nu=1}^{m}(y_\nu + 1) + \sum_{\lambda=1}^{m}(\prod_{\nu=0}^{\lambda-1}(y_\nu + 1))g_m(y_\lambda, y_{\lambda+1}, \dots, y_{\lambda+m-1})$$

with $y_0 = 0$ and $y_k = 0$, $\forall k > m$.

Thus, $f_m(x_2, x_3, \dots, x_m, 0) =$
$\prod_{\nu=1}^{m}(x_{\nu+1} + 1) + g_m(x_2, x_3, \dots, x_{m+1}) + \sum_{\lambda=2}^{m}(\prod_{\nu=1}^{\lambda-1}(x_{\nu+1} + 1))g_m(x_{\lambda+1}, x_{\lambda+2}, \dots, x_{\lambda+m}) =$
$\prod_{\nu=2}^{m}(x_\nu + 1) + g_m(x_2, x_3, \dots, x_{m+1}) + \sum_{\lambda=2}^{m}(\prod_{\nu=2}^{\lambda}(x_\nu + 1))g_m(x_{\lambda+1}, x_{\lambda+2}, \dots, x_{\lambda+m}).$ (1)

But, from Proposition 2.2 and the definition of g_m we have :

$$f_m(x_1, x_2, \dots, x_m) = (x_1 + 1)f_m(x_2, x_3, \dots, x_m, 0) + g_m(x_1, x_2, \dots, x_m)$$

and so, substituting $f_m(x_2, x_3, \dots, x_m, 0)$ from (1), we finally get the required result.
∎

We use Proposition 2.3 in order to find the formulae for f_3, f_4 and f_5. We present the proof for f_3 only. The proofs of the other two propositions are quite more complicated, but since they follow a similar line of argument, they are omitted.

Proposition 2.4: *For every $x_1, x_2, x_3 \in \mathbb{N}$ we have*

$$f_3(x_1, x_2, x_3) = \prod_{\nu=1}^{3}(x_\nu + 1)(\tfrac{x_1+x_2+x_3}{2} + 1).$$

Proof. Suppose $x_1x_2x_3 \neq 0$. We first determine the function g_3. We have (using Proposition 2.1) that $g_3(x_1, x_2, x_3) = \sum_{t=1}^{x_1} f_2(t-1, x_2)f_2(x_3, 0) = (x_2+1)(x_3+1)\sum_{t=1}^{x_1} t =$ $(x_1 + 1)(x_2 + 1)(x_3 + 1)\frac{x_1}{2}$.

Thus we get $f_3(x_1, x_2, x_3) =$
$\prod_{\nu=1}^{3}(x_\nu + 1) + g_3(x_1, x_2, x_3) + (x_1 + 1)g_3(x_2, x_3, 0) + (x_1 + 1)(x_2 + 1)g_3(x_3, 0, 0) =$
$\prod_{\nu=1}^{3}(x_\nu + 1)(\frac{x_1+x_2+x_3}{2} + 1).$

It is easy to check that the formula holds for $x_1x_2x_3 = 0$, too. ∎

In the following two propositions the variables x_4, x_1 and x_5, x_1 respectively are considered as consecutive variables.

Proposition 2.5: *For every $x_1, x_2, x_3, x_4 \in \mathbb{N}$ we have*

$$f_4(x_1, x_2, x_3, x_4) = \prod_{\nu=1}^{4}(x_\nu + 1)(\tfrac{1}{6}\sum_{i=1}^{4} {x_i}^2 + \tfrac{1}{4}\sum_{\substack{1\le i,j\le 4\\ i\neq j}} a_{ij}x_ix_j + \tfrac{5}{6}\sum_{i=1}^{4} x_i + 1),$$

where

$$a_{ij} = \begin{cases} 1, & \textit{if } x_i,\ x_j \textit{ are consecutive variables} \\ 2, & \textit{otherwise.} \end{cases}$$

Proposition 2.6: *For every* $x_1, x_2, x_3, x_4, x_5 \in N$ *we have* $f_5(x_1, x_2, x_3, x_4, x_5) = \prod_{\nu=1}^{5}(x_\nu+1)[\frac{1}{24}\sum_{i=1}^{5} {x_i}^3 + \frac{1}{12}\sum_{\substack{1\le i,j\le 5 \\ i\ne j}} a_{ij}({x_i}^2x_j + x_i{x_j}^2) + \frac{1}{8}\sum_{\substack{1\le i,j,k\le 5 \\ i\ne j\ne k\ne i}} b_{ijk}x_ix_jx_k + \frac{1}{12}\sum_{\substack{1\le i,j\le 5 \\ i\ne j}} c_{ij}x_ix_j$ $+ \frac{3}{8}\sum_{i=1}^{5} {x_i}^2 + \frac{13}{12}\sum_{i=1}^{5} x_i + 1]$, *where*

$$a_{ij} = \begin{cases} 1, & \textit{if } x_i,\ x_j \textit{ are consecutive variables} \\ 2, & \textit{otherwise} \end{cases}$$

$$b_{ijk} = \begin{cases} 1, & \textit{if } x_i,\ x_j,\ x_k \textit{ are consecutive variables} \\ 3, & \textit{otherwise} \end{cases}$$

$$c_{ij} = \begin{cases} 7, & \textit{if } x_i,\ x_j \textit{ are consecutive variables} \\ 11, & \textit{otherwise.} \end{cases}$$

3 Properties of $\mathbf{f_m}$

In the previous section we have seen that the functions f_m for $2 \le m \le 5$ are polynomials, with positive rational coefficients and with the product $\prod_{\nu=1}^{m}(x_\nu + 1)$ as a factor. We will now prove that this is true for every $m \in \mathbb{N}$, $m \ge 2$.

Proposition 3.1: *For every* $m \ge 2$*, there exists a polynomial* $P_m(x_1, x_2, \ldots, x_m)$ *of degree* $m - 2$*, with positive rational coefficients, such that*

$$f_m(x_1, x_2, \ldots, x_m) = \prod_{\nu=1}^{m}(x_\nu + 1)P_m(x_1, x_2, \ldots, x_m)$$

for every sequence (x_i)*,* $i \in [m]$*, in* $\mathbb{N}$*.*

Proof. We will use induction on m.

For $m = 2$ the result holds with $P_2(x_1, x_2) = 1$, because of Proposition 2.1. Suppose now that $m \ge 3$ and that the result is correct for every $k \in N^*$, with $2 \le k \le m - 1$, so that there exist polynomials P_k of degree $k - 2$ with positive rational coefficients such that:

$$f_k(x_1, x_2, \ldots, x_k) = \prod_{\nu=1}^{k}(x_\nu + 1)P_k(x_1, x_2, \ldots, x_k)$$

for every sequence (x_i), $i \in [k]$, in $\mathbb{N}$.

In order to prove the result for m, we firstly show that

$$\sum_{t=1}^{x_1} tP_k(t-1, x_2, \ldots, x_k) = (x_1 + 1)q_k(x_1, x_2, \ldots, x_k)$$

where q_k is a polynomial of degree $k-1$ with positive rational coefficients.

Indeed, since the function $xP_k(x-1, x_2, \ldots, x_k)$ is a polynomial of x with rational coefficients and degree $k-1$, it can be written in the form

$$xP_k(x-1, x_2, \ldots, x_k) = \sum_{i=1}^{k-1} a_{ik}(x_2, x_3, \ldots, x_k)F_i(x)$$

where $a_{ik}(x_2, x_3, \ldots, x_k)$, $i \in [k-1]$ are polynomials of degree $k-2$ with rational coefficients and $F_i(x) = x(x-1)\cdots(x-i+1)$ is the factorial polynomial.

If we cancel x in the above equality, we deduce that

$$P_k(x-1, x_2, \ldots, x_k) = \sum_{j=0}^{k-2} a_{j+1,k}(x_2, x_3, \ldots, x_k)F_j(x-1)$$

where $F_0(x) = 1$.

Since the coefficients of P_k in terms of $x-1, x_2, \ldots, x_k$ are positive, we conclude that $a_{ik}(x_2, x_3, \ldots, x_k)$ have positive coefficients for every $i \in [k-1]$.

Thus, using the difference operator Δ, we have :

$\sum_{t=1}^{x_1} tP_k(t-1, x_2, \ldots, x_k) = \sum_{t=1}^{x_1}\sum_{i=1}^{k-1} a_{ik}(x_2, x_3, \ldots, x_k)\frac{\Delta F_{i+1}(t)}{i+1} =$
$\sum_{i=1}^{k-1} \frac{a_{ik}(x_2,x_3,\ldots,x_k)}{i+1}(F_{i+1}(x_1+1) - F_{i+i}(1)) = \sum_{i=1}^{k-1} \frac{a_{ik}(x_2,x_3,\ldots,x_k)}{i+1}(x_1+1)F_i(x_1) =$
$(x_1+1)q_k(x_1, x_2, \ldots, x_k)$,
where $q_k = \sum_{i=1}^{k-1} \frac{a_{ik}(x_2,x_3,\ldots,x_k)}{i+1}F_i(x_1)$ is a polynomial of degree $k-1$ with positive rational coefficients.

Now, using the induction hypothesis, we have :

$g_m(x_1, x_2, \ldots, x_m) = \sum_{t=0}^{x_1}\sum_{k=2}^{m-1} f_k(t-1, x_2, \ldots, x_k)f_{m-k+1}(x_{k+1}, \ldots, x_m, 0) =$
$\sum_{t=0}^{x_1}\sum_{k=2}^{m-1}(t-1+1)\prod_{\nu=2}^{k}(x_\nu+1)P_k(t-1, x_2, \ldots, x_k)\prod_{\nu=k+1}^{m}(x_\nu+1)P_{m-k+1}(x_{k+1}, \ldots, x_m, 0) =$

$\prod_{\nu=2}^{m}(x_\nu+1)\sum_{k=2}^{m-1}P_{m-k+1}(x_{k+1},\dots,x_m,0)\sum_{t=1}^{x_1}tP_k(t-1,x_2,\dots,x_k)=$
$\prod_{\nu=2}^{m}(x_\nu+1)\sum_{k=2}^{m-1}P_{m-k+1}(x_{k+1},\dots,x_m,0)(x_1+1)q_k(x_1,x_2,\dots,x_k)=$
$\prod_{\nu=1}^{m}(x_\nu+1)\sum_{k=2}^{m-1}P_{m-k+1}(x_{k+1},\dots,x_m,0)q_k(x_1,x_2,\dots,x_k).$

But, for every $k\in\mathbb{N}$ with $2\le k\le m-1$, the polynomial P_{m-k+1} has degree $(m-k+1)-2$, whereas q_n has degree $k-1$; hence, the polynomial

$$r_m(x_1,x_2,\dots,x_m)=\sum_{k=2}^{m-1}P_{m-k+1}(x_{k+1},\dots,x_m,0)q_k(x_1,x_2,\dots,x_k)$$

has degree $m-2$, as well as positive rational coefficients.

Finally, applying the recursive formula of Proposition 2.3 we obtain that $f_m(x_1,x_2,\dots,x_m)=\prod_{\nu=1}^{m}(x_\nu+1)+\sum_{\lambda=1}^{m}(\prod_{\nu=1}^{\lambda-1}(x_\nu+1))\prod_{\nu=\lambda}^{m}(x_\nu+1)r_m(x_\lambda,x_{\lambda+1},\dots,x_{\lambda+m-1})$ $=\prod_{\nu=1}^{m}(x_\nu+1)[1+\sum_{\lambda=1}^{m}r_m(x_\lambda,x_{\lambda+1},\dots,x_{\lambda+m-1})]=\prod_{\nu=1}^{m}(x_\nu+1)P_m(x_1,x_2,\dots,x_m)$, where the polynomial $P_m(x_1,x_2,\dots,x_m)=1+\sum_{\lambda=1}^{m}r_m(x_\lambda,x_{\lambda+1},\dots,x_{\lambda+m-1})$ has degree $m-2$ and positive rational coefficients. ∎

The final part of this work deals with the evaluation of the coefficient of ${x_i}^k$, for $i\in[m]$, $k=0,1,\dots,m-2$ in the polynomial P_m.

Notice that, by Proposition 3.1, if $(y_1,y_2,\dots,y_m)$ is a cyclic permutation of $(x_1,x_2,\dots,x_m)$, then $P_m(x_1,x_2,\dots,x_m)=P_m(y_1,y_2,\dots,y_m)$. Hence, since the coefficient of ${x_i}^k$ in $P_m(x_1,x_2,\dots,x_m)$ is equal to its coefficient in $P_m(0,\dots,0,x_i,0,\dots,0)$, for every $i\in[m]$ and $k=0,1,\dots,m-2$, it is enough to determine the coefficient of ${x_i}^k$ in $P_m(x_i,0,\dots,0)$.

For this we need the following result:

Proposition 3.2: *For every $m\in\mathbb{N}$ with $m\ge2$ and for every $x\in\mathbb{N}$ we have*

$$f_m(x,0,\dots,0)=\binom{x+m-1}{m-1}.$$

Proof. Here, we deal with the set $NC(X,[m])$ with $X=[m]\cup Y$, $Y\subseteq(1,2)$, $|Y|=x$.

For every n.c.p. $\pi=B_1/B_2/\dots/B_m$ of X, with $i\in B_i$ for every $i\in[m]$, let $n_i=|B_i|-1$. Then, the sequence (n_i), $i\in[m]$ is a nonnegative integer solution of the equation $t_1+t_2+\cdots+t_m=x$.

Conversely, if (n_i), $i\in[m]$ is a nonnegative integer solution of $t_1+t_2+\cdots+t_m=x$ we define recursively the blocks of a n.c.p. $\pi=B_1/B_2/\dots/B_m$ of X, with $i\in B_i$ for every $i \in [m]$ as follows: B_1 contains 1 as well as the first n_1 elements of $X \backslash [m]$; for $i = 2, 3, \ldots, m$, B_i contains i as well as the last n_i elements of $X \backslash ([m] \bigcup (\bigcup_{j=1}^{i-1} B_j))$.

We thus define a bijection between the set $NC(X, [m])$ and the set of all non-negative integer solutions of the equation $t_1 + t_2 + \cdots + t_m = x$. Since the cardinality of the second set is equal to $\binom{x+m-1}{m-1}$, (see [10]) we obtain that $f_m(x, 0, \ldots, 0) = \binom{x+m-1}{m-1}$ indeed. ∎

We now prove the following result:

Proposition 3.3: *For any $i \in [m]$, the coefficient of ${x_i}^k$, $0 \leq k \leq m-2$, in the polynomial $P_m(x_1, x_2, \ldots, x_m)$ is equal to*

$$a_{m,k} = \frac{(-1)^{m-k}}{(m-1)!} \sum_{p=k+2}^{m} s(m,p)$$

where $s(m,p)$ are the Stirling numbers of the first kind.

Proof. Since $a_{m,k}$ is equal to the coefficient of x^k in the polynomial $P_m(x, 0, \ldots, 0)$ using Propositions 3.1 and 3.2 we get

$(x+1)P_m(x, 0, \ldots, 0) = f_m(x, 0, \ldots, 0)$, i.e. $(x+1) \sum_{k=0}^{m-2} a_{m,k} x^k = \binom{x+m-1}{m-1}$ and hence

$$\sum_{k=0}^{m-2} a_{m,k} x^k = \frac{(x+2)(x+3)\cdots(x+m-1)}{(m-1)!}. \tag{1}$$

Furthermore, using the relation

$$x(x+1)(x+2)\cdots(x+m-1) = \sum_{k=1}^{m} (-1)^{m-k} s(m,k) x^k$$

which is an immediate consequence of the definition of $s(m,k)$, we get that

$$(x+2)(x+3)\cdots(x+m-1) = \frac{\sum_{k=1}^{m} (-1)^{m-k} s(m,k) x^{k-1}}{x+1}. \tag{2}$$

It is easy to check that in the polynomial obtained from the division in (2), the coefficient of x^k, for $0 \leq k \leq m-2$, is equal to $(-1)^{m-k} \sum_{p=k+2}^{m} s(m,p)$.

Hence, (2) gives

$$(x+2)(x+3)\ldots(x+m-1) = \sum_{k=0}^{m-2} ((-1)^{m-k} \sum_{p=k+2}^{m} s(m,p)) x^k. \tag{3}$$

From (1), (3) we finally obtain:

$$a_{m,k} = \frac{(-1)^{m-k}}{(m-1)!} \sum_{p=k+2}^{m} s(m,p). \qquad \blacksquare$$

Notice that from the last result we easily obtain that $a_{m,0} = 1$ and $a_{m,m-2} = \frac{1}{(m-1)!}$.

Up to now, we do not know the value of every coefficient of $P_m(x_1, x_2, \ldots, x_m)$. Even the evaluation of the coefficient of ${x_i}^k {x_j}^l$, where $0 \leq k, l$ and $k + l \leq m - 2$, seems to be very complicated. For this, the basic step is to evaluate the formula of $f_m(x_1, x_2, \ldots, x_m)$ in the case where exactly two variables are nonzero, [8].

References

[1] C.A. Athanasiadis, On non crossing and non nesting partitions for classical reflection groups, *Electron. J. Combin.* **5** #42 (1998), 16pp.

[2] P.H. Edelman, Chain enumeration and non crossing partitions, *Discrete Math.* **31** (1980), 171–180.

[3] P.H. Edelman, Multichains, non crossing partitions and trees, *Discrete Math.* **40** (1981), 171–179.

[4] G. Kreweras, Sur les partitions non croisées d'un cycle, *Discrete Math.* **1** (1971), 335–350.

[5] A. Panayotopoulos, A. Sapounakis and P. Tsikouras, Matching generalized nested sets, *Australas. J. Combin.* **20** (1999), 25–36.

[6] Y. Poupard, Etude et denombrement paralleles des partitions non croisées d'un cycle et des decoupages d'un polygon convexe, *Discrete Math.* **2** (1971), 441–451.

[7] H. Prodinger, A correspondence between ordered trees and non crossing partitions, *Discrete Math.* **46** (1983), 205–206.

[8] A. Sapounakis and P. Tsikouras, On the enumeration of noncrossing partitions with fixed points, (in preparation).

[9] R. Simion, Combinatorial statistics on type-B analogues of non crossing partitions and restricted permutations, *Electron. J. Combin.* **7** #9 (2000), 27pp.

[10] I. Tomescu, *Introduction to Combinatorics*, Collet's Publishers Ltd., London, 1975.

(Received 23 July 2002)