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Light subgraphs of order 4 in large maps of minimum degree 5 on compact 2-manifolds

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Abstract

A graph H is said to be light in a class $\mathcal{G}$ of graphs if at least one member of $\mathcal{G}$ contains a copy of H and there is an integer $w(H, \mathcal{G})$ such that each member G of $\mathcal{G}$ with a subgraph isomorphic with H has also a copy H with degree sum $\sum_{x \in V(H)} \deg_G(x) \leq w(H, \mathcal{G})$.

In this paper we prove that all proper spanning subgraphs H of the complete graph K_4 are light in the class $\mathcal{G}$ of large graphs of minimum degree 5 embedded in a compact 2-manifold. The exact values of $w(H, \mathcal{G})$ are determined as well. Moreover, we have proved that K_4 itself is not light in this family $\mathcal{G}$.

1 Introduction

An orientable compact 2-manifold $\mathbb{S}_g$ or orientable surface $\mathbb{S}_g$ (see [17]) of genus g is obtained from the sphere by adding g handles. A non-orientable compact 2-manifold $\mathbb{N}_q$ or non-orientable surface $\mathbb{N}_q$ of genus q is obtained from the sphere by adding q crosscaps. The Euler characteristic of a surface is defined by

$$\chi(\mathbb{S}_g) = 2 - 2g \text{ and } \chi(\mathbb{N}_q) = 2 - q.$$

Compact 2-manifolds are called surfaces for short throughout the paper. We shall consider graphs without loops and multiple edges. Multigraphs may have loops or multiple edges.

If a graph G is embedded in a surface $\mathbb{M}$ then the connected components of $\mathbb{M} - G$ are called the *faces* of G. If each face is an open disc then the embedding is called a 2-*cell embedding.* If each vertex has degree ≥ 3 and each vertex of degree h is incident with h different faces then G is called a map in $\mathbb{M}$. If, in addition, G is 3-connected and the embedding has "representativity" at least three, then G is called a *polyhedral map* in $\mathbb{M}$, see e.g. Robertson and Vitray [18] or Mohar [15]. Let us recall that the representativity $rep(G, M)$ (or face width) of a (2-cell) embedded graph G into a manifold $\mathbb{M}$ is equal to the smallest number k such that $\mathbb{M}$ contains a noncontractible closed curve that intersects the graph G in k points. We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The *facial walk* of a face α in a 2-cell embedding is the shortest closed walk induced by all the edges incident with α. The *degree* or *size* of a face α of a 2-cell embedding is the length of its facial walk. Vertices and faces of degree i are called i*-vertices* and i*-faces*, respectively. A (2-cell) 3-face is also said to be a triangle. The number of i-faces and j-vertices in a map is denoted by f_i and n_j, respectively. For a map G let $V(G), E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively.

For a 2-cell embedding G in a manifold $\mathbb{M}$ the famous Euler's formula states

$$|V(G)| - |E(G)| + |F(G)| = \chi(\mathbb{M}).$$

The degree of a vertex v in G is denoted by $\deg_G(v)$ or $\deg(v)$ if G is known from the context. Correspondingly, $\deg_G(\alpha)$ and $\deg(\alpha)$ denote the size of a face α. A path and a cycle on k distinct vertices is defined to be the k*-path* and the k*-cycle*, respectively. A k*-path* is always denoted by P_k. The *length* of a path or a cycle is the number of its edges.

It is a consequence of Euler's formula that each planar graph contains a vertex of degree at most 5. It is well known that any graph embedded in a surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \neq 2$ has minimum degree

$$\delta(G) \leq \left\lfloor \frac{5 + \sqrt{49 - 24\chi(\mathbb{M})}}{2} \right\rfloor.$$

(For a proof see e.g. Sachs [19], p. 227.)
For two graphs H and G we write $G \cong H$ if the graphs H and G are isomorphic.

A beautiful theorem of Kotzig [13, 14] states that every 3-connected planar graph contains an edge with degree sum of its endvertices being at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example Ivančo [5] has proved that every polyhedral map on $\mathbb{S}_g$ contains an edge with degree sum of their end vertices being at most $2g + 13$ if $0 \leq g \leq 3$ and at most $4g + 7$, if $g \geq 4$. For other results in this area see e.g. our surveys [9] and [10].

The *weight* of a subgraph H of a graph G is the sum of the degrees (in G) of its vertices. Let $\mathcal{G}$ be a class of graphs and let H be a connected graph such that infinitely many members of $\mathcal{G}$ contain a subgraph isomorphic to H. Then we define $w(H, \mathcal{G})$ to be the smallest integer w such that each graph $G \in \mathcal{G}$ which contains a subgraph isomorphic to H has a subgraph isomorphic to H of weight at most w. If $w(H, \mathcal{G})$ exists then H is called *light* in $\mathcal{G}$, otherwise H is *heavy* in $\mathcal{G}$. For brevity, we write $w(H)$ if $\mathcal{G}$ is known from the context.

Fabrici and Jendrol' [3] showed that all paths are light in the class of all 3-connected planar graphs. They further showed that no other connected graphs are light in the class of all 3-connected planar graphs. Fabrici, Hexel, Jendrol' and Walther [2] proved that the situation remains unchanged if the minimum degree is raised to four, i.e., in this class of graphs only the paths are light. Mohar [16] showed that the same is true for 4-connected planar graphs.

The situation is the same on each compact 2-manifold $\mathbb{M}$ other than the plane. We proved [8] that all paths are light in the class of all polyhedral maps on a compact 2-manifold $\mathbb{M}$, and each other connected graph is not light in this class. From the counterexamples of Fabrici, Hexel, Jendrol' and Walther [2] and Mohar [16] we can derive counterexamples which show that only the paths are light in the class of all polyhedral maps on $\mathbb{M}$ of minimum degree 4; moreover, only the paths are light in the class of all 4-connected polyhedral maps on $\mathbb{M}$. There is a dramatic change when passing to the class of all polyhedral maps on $\mathbb{M}$ of minimum degree 5. In this class not only the paths but also some cycles, stars, etc. are light. In [11] we surveyed the weights known in this class.

For more information we refer the reader to our survey [10] on light subgraphs in plane graphs and in graphs embedded in the projective plane and to our survey [9] on light subgraphs in graphs on surfaces $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$.

First we briefly recall the results of our paper [11] on light subgraphs of order ≤ 3.

Each 2-cell embedding G on a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ with vertex number $n \geq 6|\chi(\mathbb{M})|$ contains a vertex of degree at most 6. If G has enough vertices of degree > 6, i.e., $n > 6|\chi(\mathbb{M})|$, then G contains even a vertex of degree at most 5. A graph G on $\mathbb{M}$ is said to be *large* if it has a large number of vertices or a large positive charge. The *positive charge* $ch(G)$ of a graph G is defined to be $ch(G) := \sum_{\deg(v) \geq 7} (\deg_G(v) - 6)$. (Here and in the sequel the sum in $\sum_{\deg(v) \geq 7} (\deg_G(v) - 6)$ is taken over all k-vertices $v \in V(G)$ with $k \geq 7$.) Here we will study large 2-cell embeddings of minimum degree 5 in surfaces $\mathbb{M}$.

Our theorems are formulated and proved for the wider class of 2-cell embeddings of graphs of minimum degree ≥ 5 in surfaces . We remark that each connected graph has a 2-cell embedding in a compact 2-manifold. Thus this restriction is not essential in the class of all graphs embedded in $\mathbb{M}$.

All connected graphs of order ≤ 3 are light.

Theorem 1 (Jendrol' and Voss [11]). *Let G be a 2-cell embedding of a graph of minimum degree ≥ 5 in a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$. Then G contains a triangle of weight at most* 18, *if*

(a) $\sum_{\deg(v)\geq 7} (\deg_G(v) - 6) > 24|\chi(\mathbb{M})|$, *or*

(b) *the number of vertices of* G *is* $n > 83|\chi(\mathbb{M})|$, *or*

(c) G *is a triangulation with more than* $18|\chi(\mathbb{M})|$ 5*-vertices.*

The bound 18 *is tight.*

For the connected subgraphs of a 3-cycle Theorem 1 implies immediately:

Corollary 2. *The graphs of Theorem 1 contain paths* P_1, P_2, P_3 *of weights* $w(P_1) \leq 6$, $w(P_2) \leq 12$, $w(P_3) \leq 18$. *All bounds are tight, if the number of vertices* $n > 83|\chi(\mathbb{M})|$.

If the positive charge of the graphs of Theorem 1 is large enough then the upper bound 18 can be lowered to 17 for the path P_3 on three vertices.

Theorem 3. *Let* G *be a* 2*-cell embedding of a graph of minimum degree* ≥ 5 *in a surface* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$. *Then* G *contains a* 3*-path* P_3 *of weight at most* 17*, if its positive charge*

$$ch(G) = \sum_{\deg(v)\geq 7} (\deg_G(v) - 6) > 16|\chi(\mathbb{M})|.$$

Moreover, if G *has more than* $10|\chi(\mathbb{M})|$ 5*-vertices then* G *again contains a* 3*-path of weight at most* 17*. The bound* 17 *is tight.*

Note in the second assertion of Theorem 3 the graph G is not necessarily a triangulation (compare with Theorem 1).

By a slight extention of the proof of Theorem 3 in [11] a proof of Corollary 4 can be obtained.

Corollary 4. *The graphs of Theorem 5 contain paths* P_1 *and* P_2 *of weight* $w(P_1) \leq 5$ *and* $w(P_2) \leq 11$. *These bounds are tight.*

In the following we deal with subgraphs of order 4. A *diamond* D_4 is a graph obtained from the complete graph K_4 by deleting one edge and embedding it in a surface so as to form two (2-cell) 3-faces with one common edge. A *kite* T_4 is a subgraph of D_4 obtained from D_4 by deleting an edge incident with a 2-vertex of D_4.

The kite T_4 is light.

Theorem 5. *Let* G *be a* 2*-cell embedding of a graph of minimum degree* ≥ 5 *in a surface* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$. G *has a kite* T_4 *of weight* $w(T_4) \leq 24$, *if*

(a) $\sum_{\deg(v)\geq 7} (\deg(v) - 6) > 36|\chi(\mathbb{M})|$, *or if*

(b) *the number of vertices is* $n > 149|\chi(\mathbb{M})|$.

Moreover, the bound 24 *is sharp, if the number of vertices* $n > 149|\chi(\mathbb{M})|$.

For the connected subgraphs of a kite Theorem 5 implies:

Corollary 6. *The graphs of Theorem 5 contain a 4-path* P_4 *and a 3-star* $K_{1,3}$ *of weight* $w(P_4) \leq 24$ *and* $w(K_{1,3}) \leq 24$*, respectively. All these bounds are tight if the number of vertices* $n > 149|\chi(\mathbb{M})|$.

If the positive charge of the graphs of Theorem 5 is large enough then the upper bound 24 can be lowered to 23.

Theorem 7. *Let* G *be a 2-cell embedding of a graph of minimum degree* ≥ 5 *in a surface* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$. *The graph* G *has a kite* T_4 *of weight* $w(T_4) \leq 23$*, if*

$$\sum_{\deg(v)\geq 7} (\deg_G(v) - 6) > 150|\chi(\mathbb{M})|.$$

For connected spanning subgraphs of a kite Theorem 7 implies:

Theorem 8. *The graphs of Theorem 7 contain a 4-path* P_4 *and a 3-star* $K_{1,3}$ *of weight* $w(P_4) \leq 23$ *and* $w(K_{1,3}) \leq 23$*, respectively. All these bounds are tight if the positive charge* $> 150|\chi(\mathbb{M})|$.

The result about kites cannot be derived from the corresponding results relating to diamonds. There are polyhedral maps of $\mathbb{M}$ in which each diamond has weight ≥ 27.

Theorem 9. *Let* G *be a 2-cell embedding of a graph of minimum degree* ≥ 5 *in a surface* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$. *Then the map* G *has a diamond* D_4 *of weight* $w(D_4) \leq 27$*, if*

(a) $\sum_{\deg(v)\geq 7} (\deg(v) - 6) > 186|\chi(\mathbb{M})|$, *or if*

(b) *the number of vertices* $n > 335|\chi(\mathbb{M})|$.

Moreover, if G *has no* 12*-vertices, or no* 12*- and* 11*-vertices, or no* 12*-,* 11*- and* 10*-vertices, then* G *has a diamond* D_4 *of weight* $w(D_4) \leq 26$*, or* $w(D_4) \leq 25$*, or* $w(D_4) \leq 24$*, respectively.*

The previous assertion can be generalized.

Theorem 10. *Let* G *be a 2-cell embedding of a graph of minimum degree* ≥ 5 *in a compact 2-manifold* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$.
Let c_0, c_1, c_2 *be real numbers. Let*

$$\sigma(G) := \sum_{\deg(v)\geq 7} (\deg(v) - 6) - 186|\chi(\mathbb{M})| \quad \text{and}$$

$n^* := n - 335|\chi(\mathbb{M})|$. *The map* G *has a diamond* D_4 *with weight* $w(D_4)$:

(i) $w(D_4) \leq 27$, *if* $\sigma(G) > 0$ *or* $n^* > 0$.

(ii) $w(D_4) \leq 26$, *if* $v_{12} \leq c_2$ *and* $\sigma(G) > 14c_2$ *or* $n^* > 75c_2$.

(iii) $w(D_4) \leq 25$, *if* $n_{12} \leq c_2$ & $n_{11} \leq c_1$ *and*

$$\sigma(G) > 14c_2 + 28c_1 \text{ or } n^* > 75c_2 + 49c_1.$$

(iv) $w(D_4) \leq 24$, *if* $n_{12} \leq c_2$ & $n_{11} \leq c_1$ & $n_{10} \leq c_0$ *and*

$$\sigma(G) > 14c_2 + 28c_1 + 42c_0 \text{ or } n^* > 75c_2 + 49c_1 + 22c_0.$$

If $\mathbb{M} = \mathbb{S}_0$ is the plane then by Jendrol' and Madaras [6] the precise bound for both the 4-path P_4 and the 3-star $K_{1,3}$ is 23. Borodin and Woodall [1] proved that 25 is the precise bound for the 4-cycle in triangulations of the plane. The proofs of Jendrol' and Madaras [6] and Borodin and Woodall [1] for the plane can be extended to a proof for the projective plane; thus the bound 23 for the 4-path P_4 and the 3-star $K_{1,3}$ and the bound 25 for the 4-cycle in triangulations is also valid for the projective plane. For methods of extending see e.g. Sanders [20]. We conjecture:

Conjecture. *Let G be a graph of minimum degree ≥ 5 embedded in the plane or the projective plane. Then G contains a diamond D_4 of weight $w(D_4) \leq 25$. Moreover, if G does not have 10-vertices then G has a diamond D_4 of weight $w(D_4) \leq 24$. If true both bounds are tight.*

If $\mathbb{M}$ is the torus $\mathbb{S}_1$ or the Klein bottle $\mathbb{N}_2$, i.e., $\chi(\mathbb{M}) \leq 0$, then condition (b) of Theorem 5 and condition (b) of Theorem 9 give no restrictions to the number of vertices of G. Hence Theorem 5, Corollary 6, and Theorem 9 are valid for all graphs G of minimum degree ≥ 5 embedded in the torus $\mathbb{S}_1$ or in the Klein bottle $\mathbb{N}_2$. Thus 24 is the precise bound for the weight of the kite T_4, the 4-path P_4, and the 3-star $K_{1,3}$, and 27 is the precise bound for the diamond D_4. Moreover, $w(D_4) \leq 26$, or 25, or 24, if G has no 12-vertices, or no 12-and 11-vertices, or no 12-, 11-, and 10-vertices, respectively. But Theorem 7 and Corollary 8 imply: if these graphs contain at least one vertex of degree ≥ 7 then 23 is the precise bound for T_4, P_4 and $K_{1,3}$. Now it follows easily from Euler's formula that the bound 24 for D_4, P_4, or $K_{1,3}$ are only attained at the triangulations of the torus and the Klein bottle in which each vertex has degree 6.

In the paper [12] we will deal with the cycles of lengths 5. Particularly, we will prove:

Let G be a graph of minimum degree ≥ 5 embedded in a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$. The graph G has a 5-cycle C_5^* with one diagonal of weight $w(C_5^*) \leq 32$, if

(a) $\sum_{\deg(v)\geq 7} (\deg(v) - 6) > 396|\chi(\mathbb{M})|$, or

(b) the number of vertices $n > 1859|\chi(\mathbb{M})|$.

Moreover, if G has no 12-vertices then G has a 5-cycle C_5 of weight $w(C_5) \leq 31$.

Jendrol', Madaras, Soták, and Tuza [7] proved that no plane connected graph H with maximum degree $\triangle(H) \geq 6$ or with a block having ≥ 11 vertices is light in the class of all polyhedral plane maps of minimum degree ≥ 5. Moreover, they proved that all cycles $C_s, s \geq 11$, and all stars $K_{1,r}, r \geq 5$, are not light in this class. Hexel [4] investigated this question in the families of all 4- or 5-connected plane maps.

Here we present the smallest plane connected graph which is not light in the class of all polyhedral maps of minimum degree ≥ 5 embedded in $\mathbb{M}$, when $\mathbb{M}$ differs from the plane or the projective plane.

Theorem 11. *Let $\mathbb{M}$ be a surface. The complete graph K_4 on four vertices is not light in the class of all polyhedral maps of $\mathbb{M}$.*

Finally we consider multigraphs H of minimum degree ≥ 5 embedded in $\mathbb{M}$ such that the minimum face size is at least 3. The weight of a face α of H is the degree sum of its vertices, where a vertex v is counted l times if v appears precisely l times on the boundary of α.

The proofs of Theorem 10 can be used to establish the following result.

Theorem 12. *Let H be a multigraph of minimum degree ≥ 5 embedded in a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ so that the minimum face size ≥ 3. Then H contains a 2-cell 4-face F_4^* with one diagonal of weight $w(F_4^*) \leq 27$, if*

(a) $\sum_{\deg(v) \geq 7} (\deg(v) - 6) > 186|\chi(\mathbb{M})|$, *or if*

(b) *the number of vertices $n > 335|\chi(\mathbb{M})|$.*

Moreover, if G has no 12-vertices or no 12- and 11-vertices, or no 12-, 11- and 10-vertices then G has a 2-cell 4-face F_4^* with one diagonal of weight $w(F_4^*) \leq 26$, or $w(F_4^*) \leq 25$, or $w(F_4^*) \leq 24$, respectively.

Note, there are multigraphs with these properties without a 3-cycle. Thus the existence of a light triangle does not imply the existence of a light 3-cycle.

2 Proof of Theorem 5 — the upper bound for kites

Let G be a counterexample with n vertices having the largest possible number of edges. Than each kite T_4 has a weight ≥ 25.

We assign the charges $c(v) = \deg_G(v) - 6$ and $c(\alpha) = 2\deg_G(\alpha) - 6$ to each vertex v and to each face α of G, respectively. Then Euler's formula implies

$$(1) \qquad \sum_{v \in V} c(v) + \sum_{\alpha \in F} c(\alpha) = \sum_{v \in V} (\deg_G(v) - 6) + \sum_{\alpha \in F} (2\deg_G(\alpha) - 6) = -6\chi(\mathbb{M}) = 6|\chi(\mathbb{M})|,$$

because $\chi(\mathbb{M}) \leq 0$.

We redistribute the charges according the following rules:

Rules of Discharging.

Let ε be a real number with $0 \leq \varepsilon \leq \frac{1}{15}$.

Rule R1. Let α be a face of size $\deg_G(\alpha) \geq 4$.
(a) If α is a 4-face incident with a vertex of degree ≥ 6 or $\deg_G(\alpha) \geq 5$ then α sends the charge $\frac{2}{3} - \varepsilon$ to each 5-neighbour.
(b) If α is a 4-face incident only with 5-vertices then α sends the charge $\frac{1}{3} + \varepsilon$ to each 5-neighbour. In this case we say that α has property $\mathcal{E}$.

Rule R2. Suppose u is a 5-vertex adjacent with a vertex v of degree ≥ 7 such that the edge uv belongs to two 3-faces.
(a) If $\deg_G(v) = 7$ then v sends $\frac{1}{4}$ to u.
(b) If $\deg_G(v) \geq 8$ then v sends $\frac{1}{3} + \varepsilon$ to u.

We remark that in Rule **R1** the charge $\frac{1}{3} + \varepsilon < \frac{2}{3} - \varepsilon$.

$$c^*(v) \geq \begin{cases} 0, \ \textit{if} \ \deg_G(v) \in \{5,6\}, \\ \frac{1}{4}, \frac{2}{3} - 4\varepsilon, \frac{5}{3} - 4\varepsilon \ \textit{if} \ \deg_G(v) = 7,8,9, \ \textit{respectively}, \\ \left(\frac{2}{3} - \varepsilon\right) d - 6 \geq \frac{2}{3} - 10\varepsilon, \ \textit{if} \ \deg_G(v) = d \geq 10 \end{cases} \tag{2}$$

$$c^*(\alpha) \geq \begin{cases} 0, \ \textit{if} \ \deg_G(\alpha) = 3 \\ 3\varepsilon, \ \textit{if} \ \deg_G(\alpha) = 4 \\ \left(\frac{4}{3} + \varepsilon\right) r - 6 \geq \frac{2}{3} + 5\varepsilon, \ \textit{if} \ \deg_G(\alpha) = r \geq 5. \end{cases}$$

Proof of (2)**.** Let v be an arbitrary vertex of degree d and let α be an r-face of G. Let H denote the subgraph of G induced by v and its neighbours. There are several cases.

1. $d = 5$
If u is a neighbour of v then $u, u^+, u^{++}, u^{+++} = u^{--}, u^-, u$ denote the neighbours of u in this cyclic order around v.
1.1. Let v be incident only with triangles.
Each kite of H contains a vertex of degree ≥ 7. Hence v has at least three neighbours of a degree ≥ 7, say x, y, and z.
If x, y, x have degrees ≥ 8 then x, y, z send a total charge $\geq 3\left(\frac{1}{3} + \varepsilon\right) = 1 + 3\varepsilon \geq 1$ to v.
Next, without loss of generality, let x be a 7-vertex. Since $H \setminus \{y, z\}$ is a kite the vertex v has a neighbour w of degree $\geq 7, w \notin \{x, y, z\}$. Therefore, x, y, z and w send a total charge $\geq 4 \times \frac{1}{4} = 1$ to v.

1.2. Let v be incident with at least two faces of size ≥ 4. If v is incident with three faces of size ≥ 4 then they send a total charge $\geq 3\left(\frac{1}{3} + \varepsilon\right) = 1 + 3\varepsilon \geq 1$ to v. If v is incident with two faces of size ≥ 4, one of them is no 4-face with $\mathcal{E}$, then these two faces send a total charge $\geq \left(\frac{2}{3} - \varepsilon\right) + \left(\frac{1}{3} + \varepsilon\right) = 1$ to v.

Next let v be incident with precisely two 4-faces with $\mathcal{E}$, say α and β. There is a neighbour $\bar{u}$ of v, $\bar{u} \in H \setminus \{\alpha, \beta\}$, forming with three 5-vertices of $\alpha \bigcup \beta$ a kite of weight ≥ 25. Hence $\bar{u}$ has a degree ≥ 10 sending a charge $\geq \frac{1}{3} + \varepsilon$ to v. Thus v receives $\left(\frac{1}{3} + \varepsilon\right) + 2\left(\frac{1}{3} + \varepsilon\right) = 1 + 3e \geq 1$ from its neighbourhood.

1.3. Let α be a 4-face only incident with 5-vertices. The face α sends $\geq \frac{1}{3} + \varepsilon$ to v. Let u be incident with α, then $(\alpha \cup H) \setminus \{u, u^-, u^{--}\}$ and $(\alpha \cup H) \setminus \{u^+, u^{++}, u^{+++}\}$ induce subgraphs of G containing a kite. Therefore the vertices u^- and u^{++} have a degree ≥ 10, each sending a charge $\geq \frac{1}{3} + \varepsilon$ to v. Hence v receives a charge $\geq 3\left(\frac{1}{3} + \varepsilon\right) = 1 + 3\varepsilon > 1$ from its neighbourhood.

2. $d = 6$. In this case $c^*(v) = c(v) = 0$.

By Rule **R2** each 5-neighbour u of v receiving a positive charge from a vertex v of degree ≥ 7 has the property that the edge vu is in two triangles. Such a 5-neighbour is also called a *receiving* 5-neighbour.

3. $7 \leq d \leq 9$.
Any two consecutive 5-neighbours u, u^+ of v receiving a positive charge from v form with v a triangle vuu^+v. A third 5-neighbour forms with vuu^+v a kite of weight ≤ 24, a contradiction! Thus v has at most ρ 5-neighbours receiving a positive charge from v, where $\rho = 3$, if $d = 7$, and $\rho = 4$, if $d \in \{8, 9\}$.
If $d = 7$ then v sends a total charge $\leq 3 \times \frac{1}{4} = \frac{3}{4}$ to its neighbourhood, and $c^*(v) \geq 1 - \frac{3}{4} = \frac{1}{4}$. If $d \in \{8, 9\}$ then v sends a total charge $\leq 4\left(\frac{1}{3} + \varepsilon\right)$ to its neighbourhood, and $c^*(v) \geq d - 6 - 4\left(\frac{1}{3} + \varepsilon\right)$. Hence $c^*(v) \geq \frac{2}{3} - 4\varepsilon$ or $c^*(v) \geq \frac{5}{3} - 4\varepsilon$, if $d = 8$ or $d = 9$, respectively.

4. $d \geq 10$.
The vertex v sends to each neighbour a charge $\leq \frac{1}{3} + \varepsilon$. Hence $c^*(v) \geq d - 6 - d\left(\frac{1}{3} + \varepsilon\right) = \left(\frac{2}{3} - \varepsilon\right) d - 6 \geq \frac{2}{3} - 10\varepsilon$.

5. $r = 3$. Obviously, $c^*(\alpha) = c(\alpha) = 0$.

6. $r = 4$. If α is only incident with 5-vertices then

$$c^*(\alpha) \geq c(\alpha) - 4 \times \left(\frac{1}{3} + \varepsilon\right) = \frac{2}{3} - 4\varepsilon \geq 3\varepsilon.$$

If α is incident with a vertex of degree ≥ 6 then

$$c^*(\alpha) = c(\alpha) - 3 \times \left(\frac{2}{3} - \varepsilon\right) = 3\varepsilon.$$

7. $r = 5$. By our rules:

$$c^*(\alpha) \geq 2r - 6 - r \times \left(\frac{2}{3} - \varepsilon\right) = \left(\frac{4}{3} + \varepsilon\right) r - 6 \geq \frac{2}{3} + 5\varepsilon$$

The proof of (2) is complete. □

The assertion (2) implies:

(3) $$\textit{If } \varepsilon = 0 \textit{ then } c^*(v) \geq \frac{1}{6}(\deg_G(v) - 6)$$

for all vertices v of degree $\geq 7, c^*(v) \geq 0$ for all vertices v of degrees 5 and 6, and $c^*(\alpha) \geq 0$ for all faces α.
For $\varepsilon = 0$ assertions (1) and (3) with $\sum_{\deg(v)\geq 7} (\deg_G(v) - 6) > 36|\chi(\mathbb{M})|$ imply

$$6|\chi(\mathbb{M})| = \sum_{v\in V(G)} c(v) + \sum_{\alpha\in F(G)} c(\alpha) = \sum_{v\in V(G)} c^*(v) + \sum_{\alpha\in F(G)} c^*(\alpha) \geq$$
$$\geq \sum_{v\in V(G),\deg(v)\geq 7} c^*(v) \geq \frac{1}{6} \sum_{\deg(v)\geq 7} (\deg_G(v) - 6) > 6|\chi(\mathbb{M})|.$$

This contradiction shows that there is no counterexample in case (a) of Theorem 5. Thus the proof of assertion (a) is complete.

Next the proof of (b) will be completed. Let $\varepsilon = \frac{1}{35}$. A triangle is said to be *light* if it is only incident with vertices of degrees ≤ 6. We discharge vertices and faces a second time according the following rules.

Rule R*1. If a vertex of degree ≥ 7 is incident with a triangle β then v sends $\frac{1}{50}$ to β.

Rule R*2. If a vertex v of degree ≥ 7 is incident with a triangle β which has a common edge with a light triangle γ then v sends $\frac{1}{150}$ to γ.

Rule R*3. If an r-face $\alpha, r \geq 4$, has a common edge with a light triangle β then α sends $\frac{1}{3.50}$ to β.

(4) $$\textit{With } \varepsilon = \frac{1}{35} \textit{ the new charges are:}$$
$$c^{**}(\alpha) \geq (r-2)\frac{1}{50} \textit{ for all faces } \alpha.$$
$$c^{**}(v) \geq 0 \textit{ for all vertices } v.$$

Proof of (4)**.** Let α be an r-face. Assertion (2) with $\varepsilon = \frac{1}{35}$ implies:
If $r = 3$ and α is incident with a vertex of degree ≥ 7 then by Rule **R*1** the face α receives $\frac{1}{50}$ from v.
If $r = 3$ and α is incident only with vertices of degrees ≤ 6 then α is a light face. The triangle α has a common edge with three faces β_1, β_2 and β_3. If β_1 has a size ≥ 4 then by Rule **R*3** the face β_1 sends $\frac{1}{150}$ to α. If β_1 is a triangle then $\alpha \cup \beta_1$ contains a kite of weight ≥ 25. Hence the unique vertex z_1 of $\beta_1 \setminus \alpha$ has a degree ≥ 7, and by Rule **R*2** the vertex z_1 sends $\frac{1}{150}$ to α. Consenquently, the face α receives

$\frac{1}{150}$ from each of its neighbouring faces. Thus α receives the total charge $\frac{1}{50}$, and $c^{**}(\alpha) \geq c^{*}(\alpha) + 3\frac{1}{150} = \frac{1}{50}$.
If $r = 4$ then $c^{**}(\alpha) \geq c^{*}(\alpha) - 4\frac{1}{150} = \frac{3}{35} - 4\frac{1}{150} = \frac{31}{525} \geq \frac{2}{50}$.
If $r \geq 5$ then $c^{**}(\alpha) \geq \left(\frac{4}{3} + \frac{1}{35}\right) r - 6 - r\frac{1}{150} = \frac{1423}{1050}r - 6 \geq \frac{1}{50}(r-2)$.

Let v be a d-vertex. Assertion (3) with $\varepsilon = \frac{1}{35}$ implies:
If $d \in \{5,6\}$ then $c^{**}(v) = c^{*}(v) \geq 0$.
If v is a vertex of degree ≥ 7 then v sends to each neighbouring face β a charge $\leq \frac{1}{50}$.
If β is a triangle and γ is a light triangle having with β a common edge (γ does not contain v) then v sends a charge $\leq \frac{1}{150}$ to γ. Hence $c^{**}(v) \geq c^{*}(v) - \frac{4}{150}d$.
If $d = 7, 8$, or 9 then $c^{*}(v) \geq \frac{1}{4}, \frac{2}{3} - \frac{4}{35}$, and $\frac{5}{3} - \frac{4}{35}$ and $c^{**}(v) \geq \frac{19}{300}, \frac{178}{525}$, and $\frac{689}{525}$, respectively. Hence $c^{**}(v) > 0$ for $d \in \{7,8,9\}$.
If $d \geq 10$ then $c^{*}(v) \geq \left(\frac{2}{3} - \frac{1}{35}\right) d - 6$ and

$$c^{**}(v) \geq \left(\left(\frac{2}{3} - \frac{1}{35}\right) d - 6\right) - d\frac{4}{3 \times 50} = \frac{107}{175}d - 6 \geq \frac{107}{175} \times 10 - 6 > 0.$$

□

Next each r-face $\alpha, r \geq 4$, will be subdivided into $r-2$ triangles by introducing $r-3$ diagonals. The charge $\geq \frac{r-2}{50}$ will be redistributed to the new triangles so that each new triangle has a charge $\geq \frac{1}{50}$. Thus a (semi-)triangulation $\mathcal{T}$ is obtained, each triangle β of $\mathcal{T}$ has a charge $c^{***}(\beta) \geq \frac{1}{50}$, and each vertex v has the charge $c^{***}(v) = c^{**}(v) \geq 0$ by assertion (4).

Let f denote the number of faces of $\mathcal{T}$. It is well known that $f = 2(n + |\chi(\mathbb{M})|)$, where n denotes the number of vertices of $\mathcal{T}$. Consequently, with $n > 149|\chi(\mathbb{M})|$ vertices we obtain

$$\begin{aligned}
6|\chi(\mathbb{M})| &= \sum_{v \in V(G)} c(v) + \sum_{\alpha \in F(G)} c(\alpha) = \\
&= \sum_{v \in V(G)} c^{***}(v) + \sum_{\beta \in F(\mathcal{T})} c^{***}(\beta) \geq \\
&= \sum_{\beta \in F(\mathcal{T})} c^{***}(\beta) \geq \frac{1}{50}f = \frac{1}{50}2(n + |\chi(\mathbb{M})|) \\
&> 6|\chi(\mathbb{M})|.
\end{aligned}$$

This contradiction proves the assertion (b) of Theorem 5.

3 Proof of Theorem 7 — the improved upper bound for kites

Let G be a counterexample with n vertices having the largest possible number of edges. Then each kite T_4 has a weight ≥ 24. We assign the charges $c(v) := \deg_G(v) - 6$ and $c(\alpha) := 2\deg_G(\alpha) - 6$ to each vertex v and to each face α of G, respectively.

From Section 2 we know that Euler's formula implies

$$(1) \qquad \sum_{v\in V} c(v) + \sum_{\alpha\in F} c(\alpha) = \\ = \sum_{v\in V}(\deg_G(v)-6) + \sum_{\alpha\in F}(2\deg_G(v)-6) = -6\chi(\mathbb{M}) = 6|\chi(\mathbb{M})|.$$

In a 2-cell embedding of a graph a face may meet a vertex several times, say t-times. In this case for brevity we say that v is incident with t faces.

We redistribute the charges according the rules **R1** and **R2**.

Rule R1. Let α be a face of G and u one of the 5-neighbours of α.

(a) If α has a size ≥ 6 or α is a 5-face incident with a vertex of degree ≥ 6 or α is a 4-face incident with two vertices of degree ≥ 6 then α sends 1 to u.

(b) If α is a 5-face incident only with 5-vertices then α sends $\frac{4}{5}$ to u.

(c) If α is a 4-face incident with precisely one vertex of degre ≥ 6 then α sends $\frac{2}{3}$ to u.

(d) If α is a 4-face incident only with 5-vertices then α sends $\frac{1}{2}$ to u.

If α meets u t-times then α sends the corresponding charge t-times.

Rule R2. Suppose u is a 5-vertex adjacent to a vertex v of degree ≥ 7 such that the edge uv belongs to two triangles.

(a) If $\deg(v) = 7$ and the two neighbours of v different from u, lying in the triangles incident with the edge uv have a degree ≥ 6 then v sends $\frac{1}{3}$ to u.

If $\deg(v) = 7$ and one neighbour w of v, $w \neq u$, lying in one triangle incident with the edge uv has degree 5 or 6 then v sends $\frac{1}{2}$ or 1 to u, respectively.

(b) If $\deg(v) = 8$ then v sends $\frac{1}{2}$ to u.

(c) If $\deg(v) = 9$ and one neighbour of v has a degree ≥ 6 then v sends $\frac{1}{3} + \frac{1}{24} = \frac{3}{8}$ to u. If $\deg(v) = 9$ and all neighbours of v have degree 5 then v sends $\frac{1}{3}$ to u.

(d) If $\deg(v) \geq 10$ then v sends $\frac{2}{5}$ to u.

We claim

(2) *After applying Rules* **R1** *and* **R2** *to* G, *the second charge* $c^*(v) \geq 0$ *and* $c^*(\alpha) \geq 0$ *for each vertex* $v \in V(G)$ *and each face* $\alpha \in F(G)$, *respectively. Moreover,*

$$c^*(v) \geq \frac{1}{24} \text{ for each 5-vertex of } G.$$

Proof of (2)**.**
Let v be an arbitrary vertex of degree d. Let H denote the subgraph of G induced by v and its neighbours. There are several cases.

Consider first 5-vertices, i.e. $d = 5$. If u is a neighbour of v then $u, u^+, u^{++}, u^{--}, u^-, u$ denote the neighbours of v in this cyclic order around v.
1. Let v be incident only with triangles. Each kite of H contains a vertex of degree ≥ 7. Hence v has at least three neighbours of a degree ≥ 7.
If v has a least 4 neighbours of degree ≥ 7 then they send a total charge $\geq 4 \times \frac{1}{3} = \frac{4}{3}$ to v, and $c^*(v) = -1 + \frac{4}{3} = \frac{1}{3}$. Next let v have precisely three vertices of degree ≥ 7. If v has an 8-neighbour or a neighbour of degree ≥ 10 then this vertex and the two other vertices of degree ≥ 7 send a total charge $\geq \frac{2}{5} + 2 \times \frac{1}{3} = 1 + \frac{1}{15}$ to v, and $c^*(v) \geq -1 + \left(1 + \frac{1}{15}\right) = \frac{1}{15}$. Next let v have precisely three 7- and 9-vertices.

1.1. If v has two 9-neighbours then one of these neighbours has a neighbour of degree ≥ 6 and by Rule **R2**(c) it sends $\frac{1}{3} + \frac{1}{24}$ to v. Thus the three neighbours of degree ≥ 7 send a total charge $\geq \left(\frac{1}{3} + \frac{1}{24}\right) + 2 \times \frac{1}{3} = 1 + \frac{1}{24}$ to v, and $c^*(v) \geq -1 + \left(1 + \frac{1}{24}\right) = \frac{1}{24}$.

1.2. The vertex v has precisely one 9-neighbour, say u. If $\deg_G(u^+) \geq 6$ or $\deg_G(u^-) \geq 6$ then by the above used arguments $c^*(v) \geq \frac{1}{24}$.
Next let $\deg_G(u^+) = \deg_G(u^-) = 5$. Then $\deg_G(u^{++}) = \deg_G(u^{--}) = 7$, and $vu^+u^{++}vu^-$ is a kite of weight 22, a contradiction!

1.3. The vertex v has no 9-neighbour. Therefore, v has three 7-neighbours and two neighbours of degree ≤ 6, say x and y. Since the vertices x, y, v and a 7-vertex form a kite of weight ≥ 24, the degrees $\deg_G(x) = \deg_G(y) = 6$. The vertex x is adjacent to a 7-neighbour of v, say z. By Rule **R2**(a) the vertex z sends 1 to v. Hence the three 7-neighbours of v send a total charge $\geq 1 + 2 \times \frac{1}{3}$ to v, and $c^*(v) \geq -1 + \left(1 + \frac{2}{3}\right) = \frac{2}{3}$.

2. Let v be incident with at least two faces of size ≥ 4.
If v is incident with three faces of size ≥ 4 then they send a total charge $\geq 3 \times \frac{1}{2} = 1 + \frac{1}{2}$ to v, and $c^*(v) \geq -1 + \left(1 + \frac{1}{2}\right) = \frac{1}{2}$.
If v is incident with two faces of size ≥ 4, one of them is no 4-face being only incident with 5-vertices, then these two faces send a total charge $\geq \frac{2}{3} + \frac{1}{2} = 1 + \frac{1}{6}$ to v, and $c^*(v) \geq -1 + \left(1 + \frac{1}{6}\right) = \frac{1}{6}$.
Next let v be incident with precisely two 4-faces being only incident with 5-vertices, say α and β. There is a neighbour u of $v, u \in H \setminus \{\alpha, \beta\}$, forming with three vertices $\alpha \cup \beta$ a kite of weight ≥ 24. Hence u has a degree ≥ 9 sending a charge $\geq \frac{1}{3}$ to v. Thus v receives a total charge $\geq \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{3}$ from u, α, and β, respectively. Thus $c^*(v) \geq -1 + \left(1 + \frac{1}{3}\right) = \frac{1}{3}$.

3. Let v be incident with precisely one face α of size ≥ 4. Let u and u^+ denote the neighbour of v such that vu and vu^+ are edges of α. We consider three subcases.

3.1. If $\deg_G(v) \geq 6$ and $\deg_G(u^+) \geq 6$ then by Rule **R1**a the face α sends the charge 1 to v. Since $vu^{++}u^{--}vu^-$ is a kite of weight ≥ 24 our of the vertices u^{++}, u^{--}, u^-

has degree ≥ 7 and sends a charge $\geq \frac{1}{3}$ to v. Hence v receives a charge $\geq 1+\frac{1}{3}$ from its neighbourhood, and $c^{+}(v) \geq -1+\left(1+\frac{1}{3}\right)=\frac{1}{3}$.

3.2. If $\deg_G(u)=5$ and $\deg_G(u+) \geq 6$ then by Rules **R1** the face α sends a charge $\geq \frac{2}{3}$ to v. If $\deg_G(u^{++}) \geq 8$ then by Rules **R2** vertex u^{++} sends a charge $\geq \frac{1}{3}+\frac{1}{24}$. Hence v receives a charge $\geq \frac{2}{3}+\left(\frac{1}{3}+\frac{1}{24}\right) = 1+\frac{1}{24}$ from its neighbourhood, and $c^{*}(v) \geq -1+\left(1+\frac{1}{24}\right) = \frac{1}{24}$. Next let $\deg_G(u^{++}) \leq 6$. Since $vu^{+}u^{++}vu^{--}$ and $vu^{+}u^{++}vu^{-}$ are kites of weight ≥ 24 the degrees $deg_G(u^{-}) \geq 8$ and $\deg_G(u^{--}) \geq 8$. Therefore by Rule **R2** each of these two vertices sends a charge $\geq \frac{1}{3}+\frac{1}{24}$ to v. Hence v receives a charge $\geq 2\left(\frac{1}{3}+\frac{1}{24}\right)+\frac{2}{3} = 1+\frac{5}{12}$ from its neighbourhood, and $c^{*}(v) \geq -1+\left(1+\frac{5}{12}\right)=\frac{5}{12}$.

3.3. Finally let $\deg_G(u^{++})=7$. Since $vu^{++}u^{--}vu$ and $vu^{-}uvu^{++}$ are kites of weight ≥ 24 the degrees $\deg_G(u^{--}) \geq 7$ and $\deg_G(u^{-}) \geq 7$. By Rule **R2** each of the three vertices u^{++}, u^{--}, and u^{-} sends a charge $\geq \frac{1}{3}$ to v. Hence v receives a charge $\geq 3 \times \frac{1}{3}+\frac{2}{3}=\frac{5}{3}$ from its neighbourhood, and $c^{*}(v) \geq -1+\frac{5}{3}=\frac{2}{3}$.

3.4. If $\deg_G(u)=\deg_G(u^{+})=5$ then by Rule **R1** the face α sends a charge $\geq \frac{1}{2}$ to v. Since $vu^{+}u^{++}vu$ and $vu^{-}uvu^{+}$ are kites of weight 24 the vertices u^{++} and u^{-} have a degree ≥ 9. By Rule **R2** the vertices u^{++} and u^{-} send $\frac{1}{3}$ to v, hence the vertex v receives a charge $\geq \frac{1}{2}+2\times\frac{1}{3} = 1+\frac{1}{6}$ from its neighbourhood, and $c^{*}(v) \geq -1+\left(1+\frac{1}{6}\right)=\frac{1}{6}$.

Consider d-vertices, $d \geq 6$. Obviously, $c^{*}(v)=c(v)=0$, if $\deg_G(v)=6$.

$d=7$ First let v be incident with a triangle $vuu^{+}v$ such that $\deg_G(u)=5$ and $\deg_G(u^{+}) \in \{5,6\}$. If w is a neighbour of $v, w \notin \{u,u^{+}\}$ then $vuu^{+}vw$ is a kite of weight ≥ 24, and $\deg_G(w) \geq 6$. Consequently, if u^{+} has degree 5 or 6 then v has precisely two or one 5-neighbours receiving a charge from v, respectively. By Rule **R2** the vertex v sends 1 to its neighbourhood in both cases, and $c^{*}(v) \geq 1-1=0$. Next let v be incident with no triangle $vuu^{+}v$ such that $\deg_G(u)=5$ and $\deg_G(u^{+}) \in \{5,6\}$. Therefore, v has no two consecutive 5-neighbours receiving a charge from v. Then v has at most three 5-neighbours receiving a charge from v, and by Rule **R2** the vertex v sends a charge $\leq 3\times\frac{1}{3}=1$ to its neighbourhood. Hence $c^{*}(v) \geq 1-1=0$.

$d=8$

If v has two consecutive 5-neighbours, say u and u^{+}, receiving a charge from v, then uu^{+} is an edge of G. If w is a neighbour of $v, w \notin \{u,u^{+}\}$ then $vuu^{+}vw$ is a kite of weight ≥ 24 and $\deg_G(w) \geq 6$. Consequently, v has precisely two 5-neighbours receiving a charge from v. If v has no two consecutive 5-neighbours receiving a charge from v then v sends a charge to at most four 5-neighbours. Hence in all cases v sends a charge $\leq 4\times\frac{1}{2}=2$ to its neighbourhood and $c^{*}(v) \geq 2-2=0$.

$d=9$

If v has a neighbour of degree ≥ 6 then v sends the charge $\frac{1}{3}+\frac{1}{24}$ to at most eight 5-neighbours. Hence v sends $\leq 8\left(\frac{1}{3}+\frac{1}{24}\right)=3$ to its neighbourhood. If v has only 5-neighbours then v sends the charge $\frac{1}{3}$ to at most nine 5-neighbours. Hence v sends $\leq 9\times\frac{1}{3}=3$ to its neighbourhood. Thus in all cases $c^{*}(v) \geq 3-3=0$.

$d \geq 10$

By Rule **R2** the vertex v sends a charge $\leq \frac{2}{5}$ to each neighbour. Hence v sends $\leq d \times \frac{2}{5}$ to its neighbourhood, and

$$c^*(v) \geq d - 6 - \frac{2}{5}d = \frac{3}{5}d - 6 \geq \frac{3}{5} \times 10 - 6 = 0.$$

It is obvious that the new charge $c^*(\alpha) \geq 0$ for all faces $\alpha \in F(G)$. This completes the proof of (2).

Let n_5 denote the number of 5-vertices of G. The assertion (1) implies

$$\sum_{\deg(v)\geq 7} (\deg_G(v) - 6) - n_5 + 2 \sum_{\alpha \in G(G)} (\deg_G(\alpha) - 3) = 6|\chi(\mathbb{M})|.$$

With $\sum_{\deg(v)\geq 7} (\deg_G(v) - 6) > 150|\chi(\mathbb{M})|$ this implies

$$v_5 \geq \sum_{\deg(v)\geq 7} (\deg_G(v) - 6) - 6|\chi(\mathbb{M})| > 144|\chi(\mathbb{M})|. \tag{3}$$

The assertions (1), (2), and (3) imply

$$\begin{aligned} 6|\chi(\mathbb{M})| &= \sum_{v \in V(G)} c(v) + \sum_{\alpha \in F(G)} c(\alpha) = \\ &= \sum_{v \in V(G)} c^*(v) + \sum_{\alpha \in F(G)} c^*(\alpha) \geq \\ &\geq \sum_{\deg(v)=5} c^*(v) \geq \frac{1}{24} n_5 > 6|\chi(\mathbb{M})|. \end{aligned}$$

This contradiction shows that there is no counterexample. Thus the proof of Theorem 7 is complete.

4 Proof of Theorem 10 — the upper bound for diamonds

Let G be a counterexample with n vertices. Then each diamond D_4 has a weight $\geq w^*$, where $w^* = 28, 27, 26,$ or 25 in the cases (i), (ii), (iii), or (iv), respectively.

We give a common proof with the hypothesis that the weight of each diamond D_4 is at least 25. The results we obtain are also valid in the other cases; only if it is necessary are the stronger bounds $28, 27,$ or 26, respectively, used.

We assign the charges $c(v) := \deg_G(v) - 6$ and $c(\alpha) := 2\deg_G(\alpha) - 6$ to each vertex v and to each face α of G, respectively. Then these charges fulfill the equality (1) of Section 2.

We redistribute the charges according the following rules.

Discharging faces

Rule R1. Let α be a face of size ≥ 4.

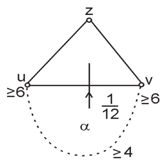

Fig. 1

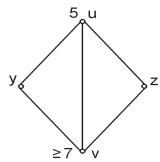

Fig. 2

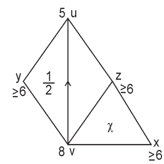

Fig. 3

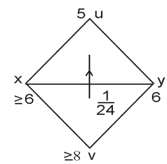

Fig. 4

(a) If α is a face of size ≥ 5 or α is a 4-face incident with a vertex of degree ≥ 6 then α sends $\frac{7}{12}$ to each 5-neighbour.

(b) If α is a 4-face only incident with 5-vertices then α sends $\frac{23}{48}$ to each 5-neighbour.

(c) Let β be a 3-face with boundary $uvzu$, where uv is also an edge of the boundary of α (Fig. 1). If $\deg_G(z) = 5$ and $\deg_G(u) \geq 6, \deg_G(v) \geq 6$ then α sends $\frac{1}{12}$ to z "across the edge uv".

Discharging vertices

Suppose u is a 5-vertex adjacent to a vertex v of degree ≥ 7 such that the edge vu belongs to two 3-faces, say $vuzv$ and $uvzu$ (Fig. 2).

Rule R2. Let $\deg_G(v) = 7$.

(a) If $\deg_G(y) = 5$ or $\deg_G(z) = 5$ then v sends $\frac{5}{24}$ to u.

(b) If $\deg_G(y) \geq 6$ and $\deg_G(z) \geq 6$ then v sends $\frac{1}{4}$ to u.

Rule R3. Let $\deg_G(v) = 8$.

(a) If $\deg_G(y) = 5$ or $\deg_G(z) = 5$ then v sends $\frac{7}{24}$ to u.

(b) If $\deg_G(y) \geq 6$ and $\deg_G(z) \geq 6$ then v sends $\frac{5}{12}$ to u.

(c) Let $vyzv$ bound a 3-face $\gamma, y \neq u$.

If $\deg_G(y) \geq b, \deg_G(z) \geq 6$ and $\deg_G(x) \geq 6$ then v sends to the vertex u the charge $\frac{1}{2}$ indead of $\frac{5}{12}$ (Fig. 3).

(Note: the four consecutive neighbours x, z, u, y form a path $xzuy$.)

Rule R4. Let $\deg_G(v) = 9$.

(a) If $\deg_G(y) = 5$ or $\deg_G(z) = 5$ then v sends $\frac{23}{48}$ to u.

(b) If $\deg_G(y) \geq 6$ and $\deg_G(z) \geq 6$ then v sends $\frac{25}{48}$ to u.

Rule R5. Let $\deg_G(v) \geq 10$.

(a) If $\deg_G(y) = 5$ or $\deg_G(z) = 5$ then v sends $\frac{25}{48}$ to u.

(b) If $\deg(y) \geq 6$ and $\deg_G(z) \geq 6$ then v sends $\frac{7}{12}$ to u.

Rule R6. Let v be incident with a triangle β which has a common edge xy with a triangle γ not containing v (Fig. 4). Let u denote the unique vertex of $\gamma \setminus \beta$.
If $\deg_G(v) \geq 8, \deg_G(u) = 5$, and $\deg_G(x) = \deg_G(y) = 6$ then v sends $\frac{1}{24}$ to u "across the edge xy".

Lemma 1. *The new charges c^* of vertices and faces are:*

$$c^*(v) \geq \begin{cases} 0, \ \textit{if} \ \deg_G(v) \in \{5,6\}, \\ \frac{1}{12}, \frac{1}{6}, \frac{1}{8}, \ \textit{if} \ \deg_G(v) = 7,8,9, \ \textit{respectively.} \\ \frac{23d}{48} - 6, \ \textit{if} \ \deg_G(v) = d \geq 10. \end{cases}$$

$$c^*(\alpha) \geq \begin{cases} \frac{1}{12}, \ \textit{if deg}_G(\alpha) = 4 \\ \frac{4r}{3} - 6 \geq \frac{2}{3}, \ \textit{if} \ \deg_G(\alpha) = r \geq 5. \end{cases}$$

Moreover

$$c^*(v) \geq \begin{cases} \frac{1}{3}, \ \textit{if} \ \deg_G(v) = 10 \ \textit{and} \ w^* \geq 26, \\ \frac{3}{4}, \ \textit{if} \ \deg_G(v) = 11 \ \textit{and} \ w^* \geq 27, \\ \frac{7}{6}, \ \textit{if} \ \deg_G(v) = 12 \ \textit{and} \ w^* \geq 28. \end{cases}$$

Proof of Lemma 1.
Let v be an arbitrary vertex of degree d. There are several cases.

$d = 5$
Let v have two objects of the following type:

(i) an r-face $\alpha, r \geq 4$, incident with v, and being no 4-face only incident with 5-vertices;

(ii) a neighbour $\bar{u}$ of degree ≥ 10, and the edge $\bar{u}v$ is incident with two 3-faces; or

(iii) a 4-face α incident only with 5-vertices.

The vertex v receives a charge ≥ 1 besides the case that v is incident with two 4-faces which have only 5-neighbours. In this case there is a neighbour u belonging to no 4-face. Since vu^-uu^+v is a diamond D_4, it contains a vertex x of degree ≥ 7, belonging to none of the two 4-faces. Therefore, by our rules the vertex x sends a charge $\geq \frac{5}{24}$ to v. Hence $c^*(v) \geq 2\left(\frac{1}{2} - \frac{1}{48}\right) + \frac{5}{24} > 1$.

Next let v be incident with at most one object.

A. The vertex v has none of the objects (i), (ii), or (iii). Hence all neighbour faces of v are triangles, the vertex v and its neighbourhood form a wheel H. Each neighbour has a degree ≤ 9. Obviously, at most one neighbour of v has degree 5.

A1. Let v have a neighbour u of degree 5. Then all other neighbours have a degree ≥ 6. H contains two diamonds D_4 and D_4' which meet only in the edge vu and its endvertices. Hence at least two neighbours of v have a degree ≥ 7. There are three subcases, namely, $0, 1$, or 2 neighbours of v have degree 6.

A1.1. No neighbour of v has degree 6.
Hence v has four neighbours of a degree ≥ 7. Then each of the two diamonds D_4 and D'_4 contains a vertex of degree ≥ 7 and a vertex of degree ≥ 8. Each vertex of degree ≥ 7 sends a charge $\geq \frac{5}{24}$ to v and each vertex of degree ≥ 8 sends a charge $\frac{7}{24}$ to v. Consequently, $c^*(v) \geq -1 + 2 \times \frac{5}{24} + 2 \times \frac{7}{24} = 0$.

A1.2. Precisely one neighbour has degree 6.

A1.2.1. The 6-neighbour of v belongs to a triangle containing u and v. Without loss of generality, let u^+ be the 6-neighbour of v. Since uu^+vu^-u and $uu^+u^{++}vu$ are diamonds D_4 of a weight ≥ 25 the vertices u^- and u^{++} are 9-vertices. By hypothesis $u^{+++} = u^{--}$ is a vertex of degree ≥ 7. It sends by Rule **R2** the charge $\frac{1}{4}$ to v. By Rule **R4** the vertices u^{++} and u^- send the charge $\frac{25}{48}$ and $\frac{23}{48}$ to v, respectively. Hence $c^*(v) \geq -1 + \frac{1}{4} + \frac{25}{48} + \frac{23}{48} > 0$.

A1.2.2. The 6-neighbour of v does not belong to a triangle containing u and v. Without loss of generality, let u^{++} be the 6-neighbour of v. Since $uu^+u^{++}vu$ and $uvu^{--}u^-u$ are diamonds D_4 the vertex u^+ is a 9-vertex and u^- and u^{--} have degrees ≥ 7, where one of them has a degree ≥ 8. The Rules **R4**, **R2** and **R3** imply: If $\deg_G(u^-) \geq 7$ and $\deg_G(u^{--}) \geq 8$ then u^+, u^-, and u^{--} send to v a charge $\geq \frac{23}{48}, \frac{5}{24}$, and $\geq \frac{5}{12}$, respectively. If $\deg_G(u^-) \geq 8$ and $\deg_G(u^{--}) \geq 7$ then u^+, u^-, and u^{--} send to v a charge $\geq \frac{23}{48}, \frac{7}{24}$, and $\frac{1}{4}$, respectively. Hence

$$c^*(v) \geq -1 + \frac{23}{48} + \frac{5}{24} + \frac{5}{12} = \frac{5}{48} > 0, \text{ or}$$
$$c^*(v) \geq -1 + \frac{23}{48} + \frac{7}{24} + \frac{1}{4} = \frac{1}{48} > 0.$$

A1.3. Precisely two neighbours of v have degree 6. Obviously, u and these two vertices are not consecutive neighbours of v.

A1.3.1. One 6-neighbour of v belongs to a triangle containing u and v. Without loss of generality, let u^+ and u^{--} be 6-neighbours of v. Since $uu^+u^{++}vu$ and uu^+vu^-u are diamonds D_4 the vertices u^- and u^{++} have degree 9.
By Rule **R4** the vertices u^- and u^{++} send the charges $\frac{23}{48}$ and $\frac{25}{48}$ to v, respectively. Hence $c^*(v) \geq -1 + \frac{23}{48} + \frac{25}{48} = 0$.

A1.3.2. No 6-neighbour of v belongs to a triangle containing u and v. Hence u^{++} and u^{--} are 6-vertices. Since $uu^+u^{++}vu$ and $uvu^{--}u^-u$ are diamonds the vertices u^+ and u^- have degree 9. Both of them send $\frac{23}{48}$ to v.
Let α denote the face containing the edge $u^{++}u^{--}$ and not containing v. If $\deg \alpha \geq 4$ then by Rule **R1**(c) this face sends $\frac{2}{24}$ to v, and $c^*(v) \geq -1 + 2 \times \frac{23}{48} + \frac{2}{24} \geq 0$.
Note let α be a triangle with vertex $y \notin \{u^{++}, u^{--}\}$. Since $vu^{++}yu^{--}v$ is a diamond D_4 the vertex y has a degree ≥ 8. By Rule **R6** the vertex a sends $\frac{1}{24}$ to v. Hence $c^*(v) \geq -1 + 2 \times \frac{23}{48} + \frac{1}{24} = 0$.

A2. Let v have no neighbour of degree 5.
Then the vertex v has at least two neighbours of degree ≥ 7.

A2.1. If v has at least four neighbours of degree ≥ 7 then by Rule **R2**b the vertex v receives a total charge $\geq 4 \times \frac{1}{4}$ and $c^*(v) \geq 1 - 1 = 0$.

A2.2. If v has two 6-neighbours and three neighbours of degrees ≥ 7, two of them have a degree ≥ 8 then by Rule **R2** the vertex v receives a total charge $\geq \frac{1}{4} + 2 \times \frac{5}{2} = 1 + \frac{1}{12}$. Hence $c^*(v) \geq \frac{1}{12} > 0$.

A2.2.1. The only case open is when the vertex v has two 6-neighbours, two 7-neighbours and a neighbour of degree $7, 8$, or 9.

Let u be one of the 6-neighbours. If u^+ is a 6-neighbour, too, then the diamond $uu^+u^{++}vu$ or the diamond uu^+vu^-u has weight 24, a contradiction!

Without loss of generality let u^{++} be the second 6-neighbour. Since $uu^+u^{++}vu$ is a diamond the vertex u^+ has degree 8 or 9. If $\deg(u^+) = 9$ then u^+, u^{--}, and u^- send the charges $\frac{1}{2} + \frac{1}{48}, \frac{1}{4}$, and $\frac{1}{4}$ to v, respectively. Hence $c^*(v) \geq 0$.

Next let $\deg(u^+) = 8$. By Rule **R3**(b) the vertex u^+ sends $\frac{5}{12}$ to v. Let α denote the face containing the edge uu^+ and not containing v. If $\deg \alpha \geq 4$ then by Rule **R1**(c) this face sends $\frac{1}{12}$ to v, and $c^*(v) \geq -1 + 2 \times \frac{1}{4} + \frac{5}{12} + \frac{1}{12} = 0$.

Next let α be a triangle with vertex $x \notin \{u, u^+\}$. Since $vuxu^+v$ is a diamond D_4 the vertex x has a degree ≥ 6. By Rule **R3**(c) the vertex u^+ sends $\frac{1}{2}$ instead of $\frac{5}{12}$. Hence $c^*(v) \geq 0$ also in this subcase.

A2.3. If v has three 6-neighbours then they cannot be three consecutive neighbours of v because otherwise they and v would from a diamond of weight 23, a contradiction!

Without loss of generality let u, u^{++}, u^{--} be the 6-neighbours of v.

Since $vuu^+u^{++}v$ and $vu^{--}u^-uv$ are diamonds the degrees of u^+ and u^- are 8 or 9.

If u^+ has degree 9 then by Rule **R4**(b) the vertex u^+ sends $\frac{25}{48}$ to v. If u^+ has degree 8 then let α denote the face containing the edge uu^+ and not containing v. If $\deg \alpha \geq 4$ then by Rule **R1**(c) this face α sends $\frac{1}{12}$ to v; so by Rules **R1**(c) and **R3**(b) the vertex u^+ and the face α send a total charge $\frac{1}{12} + \frac{5}{12} = \frac{1}{2}$ to v. If $\deg \alpha = 3$ then let z be the vertex of α with $z \notin \{u, u^+\}$. Since $vuzu^+v$ is a diamond the vertex z has a degree ≥ 6. By Rule **R3**(c) the vertex u^+ sends $\frac{1}{2}$ to v. Consequently, in any case the vertex u^+ and the face α send the total charge $\frac{1}{2}$ to v.

The same is true for the vertex u^- and the face β containing the edge u^-u and not containing v. This implies u^+, α, u^-, β send a total charge ≥ 1 to v, and $c^*(v) \geq 0$.

B. The vertex v has precisely one of the objects (i), (ii), or (iii).

B1. Let the object be of type (iii), i.e., it is a 4-face α only incident with 5-vertices. The face α sends $\frac{23}{48}$ to v. Let u and u^+ be the neighbours of v such that the edges vu and vu^+ belong to α. Obviously, the vertices u^{++}, u^{--}, u^- have a degree ≥ 6. If u^{--} is a 9-vertex then by Rule **R4**(b) the vertex u^{--} sends $\frac{25}{48}$ to v, and $c^*(v) \geq -1 + \frac{23}{48} + \frac{25}{48} = 0$.

The cycles $vu^+u^{++}u^{--}v$ and $vu^{--}u^-uv$ are diamonds. If $\deg_G(u^{--}) = 6$ then $\deg_G(u^{++}) = \deg_G(u^-) = 9$ and by Rule **R4** each of the vertices u^{++} and u^- sends $\geq \frac{23}{24}$ to v. Hence u^{++}, u^-, and α send a total charge

$$\geq 2 \times \frac{23}{48} + \frac{23}{48} = 1 + \frac{7}{16} \text{ to } v, \text{ and } c^*(v) \geq \frac{7}{16} > 0.$$

Next let $\deg_G(u^{--}) \in \{7, 8\}$. The vertices u^{++}, u^{--}, u^- have degree ≥ 7. Hence they send a total charge $\geq 3 \times \frac{5}{24} = \frac{15}{24}$ to v, and together with the charge from α $c^*(v) \geq -1 + \frac{15}{24} + \frac{23}{48} > 0$.

B2. Next let v have precisely one of the objects (i), or (ii). Hence the vertex v is incident with precisely one face α of size ≥ 4 (not being of type (iii)) or is adjacent with precisely one vertex x of degree ≥ 10. The face α or the vertex x sends to v a charge $\geq \frac{7}{12}$ or $\geq \frac{25}{48}$, respectively.

B2.1. At least one neighbour u of v has degree 5. There is a diamond D_4 through u such that the two vertices of $D_4 \setminus \{u, v\}$ do not belong to α and do not include x. Since $w(D_4) \geq 25$ one of the two vertices of $D_4 \setminus \{u, v\}$ has degree 9 or one has degree 8 and the other has degree 7 or 8. Hence they send a total charge $\geq \frac{23}{48}$ to v, and v receives a total charge ≥ 1 from them and α or x, respectively.

B2.2. All neighbours of v have degree ≥ 6. The neighbours of v not beloying to α and different from x form with v a diamond D_4. Since $w(D_4) \geq 25$ one of the neighbours of v in D_4 has a degree ≥ 8 or two of the neighbours of v in D_4 have degree 7. In both cases these vertices send a total charge $\geq \frac{10}{24}$ to v. Since by Rules **R1**a and **R5**b the face α and the vertex x send to v the vertex v receives a charge ≥ 1 from its neighbourhood.

$d = 6$. Obviously, $c^*(v) = c(v) = 0$.

$d = 7$. No three consecutive 5-neighbours of v receive a charge from v. Hence at most four 5-neighbours of v receive a charge from v. If v sends a charge to at most three 5-neighbours then $c^*(v) \geq 1 - 3 \times \frac{1}{4} = \frac{1}{4} > \frac{1}{12}$.
If precisely four 5-neighbour of v receive a charge from v then at least two of them form a pair of consecutive neighbours of v, each of them receiving the charge $\frac{5}{24}$ by Rule **R2**(a). Hence

$$c^*(v) \geq 1 - 2 \times \frac{5}{24} - 2 \times \frac{1}{4} = \frac{2}{24} = \frac{1}{12}.$$

$d = 8$

No three consecutive 5-neighbours of v receive a charge from v. Hence at most five 5-neighbours of v receive a charge from v. If at most three 5-neighbours of v receive a charge from v then the Rules **R3**(c) and **R6** imply that $c^*(v) \geq 2 - 3 \times \frac{1}{2} - 8 \times \frac{1}{24} = \frac{1}{6}$. Next let precisely four 5-neighbours of v receive a charge from v. If two of them form a pair of consecutive vertices then these two 5-neighbours receive the charge $\frac{7}{24}$ from v by Rule **R3**(a). Each of the remaining two 5-neighbours receives a charge $\leq \frac{1}{2}$. The vertex v send $\frac{1}{24}$ over at most four edges by Rule **R6**, and $c^*(v) \geq 2 - 2 \times \frac{7}{24} - 2 \times \frac{1}{2} - 4 \times \frac{1}{24} = \frac{1}{4} > \frac{1}{6}$. If now two of the four 5-neighbours form a pair of consecutive vertices then each 5-neighbour receives the charge $\frac{5}{12}$ by Rule **R3**(b) (Rule **R3**(c) cannot be applied), and v does not send a charge over any edge. Hence $c^*(v) \geq 2 - 4 \times \frac{5}{12} = \frac{1}{3} > \frac{1}{6}$. If precisely five neighbours of v receive a charge from v then they form two pairs of consecutive vertices, each of which receives the charge $\frac{7}{24}$ from v. Rule **R6** cannot be applied. Hence $c^*(v) \geq 2 - 4 \times \frac{7}{24} - \frac{1}{2} = \frac{1}{3} > \frac{1}{6}$.

$d = 9$

No three consecutive 5-neighbours of v receive a charge from v. Hence at most six 5-neighbours of v receive a charge from v. If precisely ρ 5-neighbours of v receive a charge from v, $0 \leq \rho \leq 5$, then v sends $\frac{1}{24}$ across $\leq 9 - \rho$ edges and

$$c^*(v) \geq 3 - \rho \times \frac{25}{48} - (9 - \rho) \times \frac{1}{24} = 3 - \frac{23\rho + 18}{48} \geq \frac{11}{48} > \frac{1}{8}.$$

Next let six 5-neighbours of v receive a charge from v. Then v with its neighbours form a wheel, and there are three pairs of consecutive 5-neighbours on the cycle of the wheel. By Rule $\mathbf{R4}(a)$ the vertex v sends a total charge $\leq 6 \times \frac{23}{48}$ to its neighbourhood (the Rule **R6** cannot be applied). Hence $c^*(v) \geq 3 - 6 \times \frac{23}{48} = \frac{1}{8}$.

$d \geq 10, w^* \geq 25$

For this part of the proof we introduce the Rules $\mathbf{R5}(b)$ and **R6** in a new way. If $\deg u = 5, \deg u^- \geq 6, \deg u^+ \geq 6$ and vuu^+v and vu^+uv are triangles, then by Rule $\mathbf{R5}(b)$ the vertex v sends $\frac{1}{2} + \frac{2}{24}$ to u. Now we say: v directly sends $\frac{1}{2}$ to u, indirectly sends $\frac{1}{24}$ to u via u^- and indirectly sends $\frac{1}{24}$ to u via u^+. Altogether u receives again the charge $\frac{1}{2} + \frac{2}{24}$.

If according to Rule **R6** the vertex v sends across the edge uu^+ to a 5-vertex $z, z \neq v$, then $\deg_G u = \deg_G u^+ = 6$ and uzu^+u is a triangle. Now we say: v sends $\frac{1}{48}$ to z via u, and v send $\frac{1}{48}$ to z via u^+.

With this new intepretation the vertex v sends to each neighbour a charge $\leq \frac{1}{2} + \frac{1}{48}$. Hence the new charge

$$c^*(v) \geq c(v) - d\left(\frac{1}{2} + \frac{1}{48}\right) = d - 6 - d\left(\frac{1}{2} + \frac{1}{48}\right) = \frac{23d}{48} - 6.$$

Hence $c^*(v) \geq \frac{11}{48}$, if $d \geq 13$.

Better bounds are obtained, if $w^* > 25$. More precisely let

$$d = 10,\ w^* \geq 26 \ \text{ or } \ d = 11,\ w^* \geq 27 \ \text{ or } \ d = 12,\ w^* \geq 28.$$

In these three cases no three consecutive 5-neighbours of v receive a charge from v. A case analysis shows that the number of 5-neighbours receiving some charge from v is at most $d-4$. If precisely ρ 5-neighbours of v receive a charge from $v, 0 \leq \rho \leq d-4$, then v sends $\frac{1}{24}$ across $\leq d - \rho$ edges and

$$c^*(v) \geq d - 6 - \rho \times \frac{7}{12} - (d - \rho) \times \frac{1}{24} \geq \frac{5d - 46}{12}.$$

Next we prove our Lemma 1 for faces. For a face α let $r = \deg_G(\alpha)$: Obviously, $c^*(\alpha) = c(\alpha) = 0$ if α is a triangle.

$r = 4$: If α is a 4-face incident only with 5-vertices then α sends $\frac{23}{48}$ to each 5-neighbour, and $c^*(v) \geq 2 - 4 \times \frac{23}{48} = \frac{1}{12}$. Next let the 4-face α be bounded by a 4-cycle, where at least one vertex of α has a degree ≥ 6.

If α has precisely three 5-vertices then Rule **R1**(c) cannot be applied. Hence $c^*(v) \geq 2 - 3\frac{7}{12} = \frac{1}{4} > \frac{1}{12}$.

If α has at most two 5-vertices then

$$c^*(v) \geq 2 - 2 \times \frac{7}{12} - 4 \times \frac{1}{12} = \frac{1}{2} > \frac{1}{12}.$$

$r \geq 5$: By Rule **R1** we obtain

$$c^*(v) \geq 2(r-3) - r \times \frac{7}{12} - r \times \frac{1}{12} - \frac{4r}{3} - 6 \geq \frac{2}{3}.$$

This completes the proof of Lemma 1. □

Lemma 1 immediately implies

(1) $c^*(v) \geq \frac{1}{31}(d-6)$ *for each vertex* v *of degree* $d \geq 7$, $\deg_G(v) \notin \{10, 11, 12\}$. *Moreover,* $c^*(v) \geq \frac{1}{31}(d-6)$ *if* $d = 10, w^* \geq 26$, *or* $d = 11, w^* \geq 27$, *or* $d = 12$, $w^* \geq 28$.

By the hypothesis,

$$\sum_{\deg_G(v) \geq 7} (\deg_G(v) - 6) > 186|\chi(\mathbb{M})| + 42n_{10} + 28n_{11} + 14n_{12}. \tag{2}$$

Euler's formula, Lemma 1 and the assertions (1) and (2) imply

$$\begin{aligned}
6|\chi(\mathbb{M})| &= \sum_{v \in V}(\deg_G(v) - 6) + \sum_{\alpha \in F}(2\deg_G(\alpha) - 6) \\
&= \sum_{v \in V(G)} c(v) + \sum_{\alpha \in F(G)} c(\alpha) = \sum_{v \in V(G)} c^*(v) + \sum_{\alpha \in F(G)} c^*(\alpha) \\
&\geq \sum_{\deg_G(v) \geq 7} c^*(v) = \sum_{\substack{\deg_G(v) \geq 7 \\ \deg_G(v) \notin \{10,11,12\}}} c^*(v) + \sum_{\deg_G(v) \in \{10,11,12\}} c^*(v) \\
&\geq \frac{1}{31} \sum_{\substack{\deg_G(v) \geq 7 \\ \deg_G(v) \notin \{10,11,12\}}} (\deg_G(v) - 6) - \frac{58}{48}n_{10} - \frac{35}{48}n_{11} - \frac{12}{48}n_{12} \\
&= \frac{1}{31} \sum_{\deg_G(v) \geq 7} (\deg_G(v) - 6) - \frac{1}{31}(10-6)n_{10} - \frac{1}{31}(11-6)n_{11} \qquad (3) \\
&\quad - \frac{1}{31}(12-6)n_{12} - \frac{58}{48}n_{10} - \frac{35}{48}n_{11} - \frac{12}{48}n_{12} \\
&= \frac{1}{31} \sum_{\deg_G(v) \geq 7} (\deg_G(v) - 6) - \frac{1990}{31 \times 48}n_{10} - \frac{1325}{31 \times 45}n_{11} - \frac{660}{31 \times 48}n_{12} \\
&> \frac{1}{31}(186|\chi(\mathbb{M})| + 42n_{10} + 28n_{11} + 14n_{13}) \\
&\quad - \frac{1990}{31 \times 48}n_{10} - \frac{1325}{31 \times 48}n_{11} - \frac{660}{31 \times 48}n_{12} \geq 6|\chi(\mathbb{M})|.
\end{aligned}$$

This contradiction proves the assertion (i) of the theorem in the case that $w^* \geq 25$.
If $w^* \geq 26$ then in (2) and (3) the terms $\cdots v_{10}$ can be deleted.
If $w^* \geq 27$ then in (2) and (3) the terms $\cdots v_{10}, \cdots v_{11}$ can be deleted.
If $w^* \geq 23$ then in (2) and (3) the terms $\cdots v_{10}, \cdots v_{11}, \cdots v_{12}$ can be deleted.
This completes the proof of assertion (i) of our Theorem 10.

Next the proof of (ii)–(iv) will be finished. A triangle is said to be *light* if it is only incident with vertices of degrees ≤ 6. We discharge vertices and faces a second time according to the following rules.
Rule R*1. If a vertex v of degree ≥ 7 is incident with a triangle β then v sends $\frac{1}{112}$ to β.

Rule R*2. If a vertex v of degree ≥ 7 is incident with a triangle β which has a common edge with a light triangle γ then v sends $\frac{1}{3\times 112}$ to γ.

Rule R*3. If an r-face $\alpha, r \geq 4$, has a common edge with a light triangle β then α sends $\frac{1}{3\times 112}$ to β.

Lemma 2 *Let v be a d-vertex and α and r-face of G. Let c^{**} denote the new charge. Then:*
$c^{**}(\alpha) \geq (r-2) \times \frac{1}{112}$ *for all faces* α;
$c^{**}(v) \geq 0$ *for all vertices v of degree* $\deg_G(v) \geq 7, \deg_G(v) \notin \{10, 11, 12\}$;
$c^{**}(v) \geq -\frac{223}{168}, -\frac{289}{336}, -\frac{11}{28}$ *if* $\deg_G(v) = 10, 11, 12,$ *respectively.*
Moreover, $c^{**}(v) \geq 0$, *if* $d = 10, w^* \geq 26$ *or* $d = 11, w^* \geq 27$ *or* $d = 12, w^* \geq 28$.

Proof of Lemma 2. Let α be an r-face. Lemma 1 implies:
If $r = 3$ and α is incident with a vertex v of degree ≥ 7 then by Rule **R*1** the face α receives $\frac{1}{112}$ from v.
If $r = 3$ and α is incident only with vertices of degrees ≤ 6 then α is a light triangle, say $xyzx$. Let β denote the face incident with the edge $xy, \beta \neq \alpha$. If $\deg_G(\beta) \geq 4$ then by Rule **R*3** the face β sends $\frac{1}{3\times 112}$ to α. If $\deg_G(\beta) = 3$ then $\alpha \cup \beta$ is a diamond of weight ≥ 25. Hence the unique vertex $u \in \beta \setminus \alpha$ has a degree ≥ 7 and by Rule **R*2** the vertex u sends α the charge $\frac{1}{3\times 112}$. Thus α receives $\frac{1}{3\times 112}$ across each of its three bounding edges, and α receives the total charge $\frac{1}{112}$.
If $r = 4$ then $c^{**}(\alpha) \geq \frac{1}{12} - 4 \times \frac{1}{3\times 112} > \frac{2}{112}$.
If $r \geq 5$ then $c^{**}(\alpha) \geq \frac{4r}{3} - 6 - \frac{r}{3\times 112} = \frac{447}{336} \times r - 6 > \frac{1}{112} \times (r-2)$.

Let v be a d-vertex. Again we start with Lemma 1. It implies if $d \in \{5, 6\}$ then $c^{**}(v) = c^*(v) \geq 0$. Next let $d \geq 7$. By Rules **R*1** and **R*2** the vertex v sends a total charge $\leq d \times \frac{1}{112} + d \times \frac{1}{3\times 112} = d \times \frac{4}{3\times 112}$ to its neighbourhood. Hence $c^{**}(v) \geq c^*(v) - \frac{4}{3\times 112} \times d$.
If $7 \leq d \leq 9$ then $c^{**}(v) \geq 0, \frac{1}{14}, \frac{1}{56}$, if $d = 7, 8, 9$, respectively.
If $d \geq 10$ then

$$c^{**}(v) \geq \frac{23}{48} \times d - 6 - \frac{4}{3} \times d \times \frac{1}{112} = \frac{157}{336} \times d - 6.$$

If $d \geq 13$ then $c^{**}(v) \geq \frac{157}{336} \times 13 - 6 = \frac{25}{336} > 0$.

If $d = 10, 11, 12$ then $c^{**}(v) \geq -\frac{223}{168}, -\frac{289}{336}, -\frac{11}{28}$, respectively.
If $d = 10$ and $w^* \geq 26$ then $c^{**}(v) \geq \frac{1}{3} - 10 \times \frac{4}{3\times 112} = \frac{3}{14} > 0$.
If $d = 11$ and $w^* \geq 27$ then $c^{**}(v) \geq \frac{3}{4} - 11 \times \frac{4}{3\times 112} = \frac{13}{21} > 0$.
If $d = 12$ and $w^* \geq 28$ then $c^{**}(v) \geq \frac{7}{6} - 12 \times \frac{4}{3\times 112} = \frac{43}{42} > 0$.
Thus the proof of Lemma 2 is complete.

Next each r-face $\alpha, r \geq 4$, will be subdivided into $r-2$ triangles by introducing $r-3$ diagonals. The charge $c^{**}(\alpha) \geq \frac{r-2}{112}$ will be redistributed to the new triangles so that each new triangle has a charge $\geq \frac{1}{112}$. Thus a semitriangulation $\mathcal{T}$ is obtained, each triangle β of $\mathcal{T}$ has a charge $c^{***}(\beta) \geq \frac{1}{112}$, and each vertex v has the charge $c^{***}(v) = c^{**}(v) \geq 0$ of Lemma 2.

Let f denote the number of faces of $\mathcal{T}$. It is well known that $f = 2(n + |\chi(\mathbb{M})|$, where n denotes the number of vertices of $\mathcal{T}$. Consequently, with

$$(4) \qquad n > 335|\chi(\mathbb{M})| + 75v_{10} + 49v_{11} + 22v_{12}$$

we obtain from Lemma 2

$$\begin{aligned}
(5) \quad 6|\chi(\mathbb{M})| &= \sum_{v\in V(G)} c(v) + \sum_{\alpha\in F(G)} c(\alpha) \\
&= \sum_{\substack{\deg(v)\geq 7 \\ \deg(v)\notin\{10,11,12\}}} c^{***}(v) + \sum_{\deg(v)\in\{10,11,12\}} c^{***}(v) + \sum_{\beta\in F(T)} c^{***}(\beta) \\
&\geq \sum_{\deg(v)\in\{10,11,12\}} c^{***}(v) + \sum_{\beta\in F(T)} c^{***}(\beta) \\
&\geq -\frac{223}{168}n_{10} - \frac{289}{335}n_{11} - \frac{11}{28}n_{12} + \frac{1}{112} \times 2(n + |\chi(\mathbb{M})|) \\
&> 6|\chi(\mathbb{M})|.
\end{aligned}$$

This contradiction proves the assertion (b) of the theorem in the case that $w^* \geq 25$.
If $w^* \geq 26$ then in (4) and (5) the terms $\cdots v_{10}$ can be deleted.
If $w^* \geq 27$ then in (4) and (5) the terms $\cdots v_{10}$ and $\cdots v_{11}$ can be deleted.
If $w^* \geq 28$ then in (4) and (5) the terms $\cdots v_{10}, \cdots v_{11}, \cdots v_{12}$ can be deleted.
This completes the proof of our Theorem 10.
The proof of Theorem 10 is also valid for multigraphs. This gives a proof for Theorem 12.

5 The lower bounds

A. Classes of polyhedral graphs without a light K_4

Let G be a triangulation of $\mathbb{M}$. We say an icosahedron I is inserted into a triangle α of G if α and a triangle β of a copy of I are deleted, and the bounding 3-cycle of α is identified with the bounding 3-cycle of β. Let G' and $\mathcal{C}$ denote the graph so obtained, and a separating 3-cycle, respectively.

(1) *Each subgraph* $\mathcal{U}$ *of* G' *isomorphic to* K_4 *is also a subgraph of* G.

Proof of (1)**.** The icosahedron I does not contain a K_4. If there is a K_4 in G' which is not in G then this K_4 contains both a vertex of $G \setminus \mathcal{C}$ and a vertex of $I \setminus \mathcal{C}$. But these two vertices are not joined by an edge. This contradiction completes the proof of (1). □

Let $G_0 = \mathcal{U}$ denote an embedding of K_4 in the plane or a triangulation of surface $\mathbb{M}$ containing K_4 as a subgraph. Suppose the triangulation G_i has already been constructed. Then G_{i+1} is obtained by inserting an icosahedron into each 3-face of G_i. By (1) the graph G_{i+1} has precisely those subgraphs isomorphic to K_4 that the graph G_i has. Each vertex v of G_i has, in G_{i+1}, a degree $\deg_{G_{i+1}}(v) = 3 \times \deg_{G_i}(v)$. Consequently every vertex x of each copy of K_4 has in G_{i+1} a degree $\deg_{G_{i+1}}(x) = 3 \times 4^i$. Hence the weight of any copy of K_4 in G_i is $4 \times (3 \times 4^i)$, and K_4 is not light in $\{G_i \mid i \geq 0\}$.

B. Classes of polyhedral maps on the torus and the Klein bottle

Let $s \geq 3$ be an integer and $p_{s+1} \times p_{s+1}$ be the cartesian product of two paths of lengths s with vertex set

$$V = \{(i,j) \mid i,j \in \mathbb{Z}, 0 \leq i,j \leq s\}$$

and edge set

$$\{\{(i,j),(i,j+1)\} | 0 \leq i \leq s, 0 \leq j \leq s-1\} \cup \{\{(i,j),(i+1,j)\} \mid 0 \leq i \leq s-1, 0 \leq j \leq s\}.$$

Identifying opposite sides of the rectangle results in a toroidal quadrangulation $\mathfrak{R}_s$, and reversing the side $(s,0),(s,1),\cdots,(s,s)$ of this rectangle and then identifying opposite sides of the rectangle results in a quadrangulation $\mathcal{P}_s$ on the Klein bottle, respectively.

Next we distinguish two subcases:

B1. The lower bounds for D_4 and C_4 in maps of large order and large positive charge.

We put $s = 3t, t \in \mathbb{N}$. In $\mathfrak{R}_{3t}$ and $\mathcal{P}_{3t}$ the edge set

$$\{\{(i,j),(i+1,j-1)\} \mid 0 \leq i \leq 3t-1, 2 \leq j \leq 3t\}$$

is added, and each vertex of the set

$$\{(i, i+3t) \mid 0 \leq i \leq 3t, 0 \leq l \leq t\}$$

is deleted, together with its incident edges (all indices modulo $3t$). Thus we arrive at a 3-valent sixangulation $\mathcal{T}_{3t}$ of the torus and a 3-valent sixangulation $\mathcal{Q}_{3t}$ of the

Klein bottle, respectively. Let V denote the vertex set of $\mathcal{T}_{3t}$ or $\mathcal{Q}_{3t}$, respectively.

$$\begin{aligned}\text{Let } A &:= \{(1+3l, 3m) \mid 0 \le l, m \le 3t\},\\ B &:= \{(2+3l, 1+3m) \mid 0 \le l, m \le 3t\}, \text{ and}\\ C &:= \{(3l, 2+3m) \mid 0 \le l \le 3t-1, 0 \le m \le 3t\}.\end{aligned}$$

Obviously, A, B, C are pairwise vertex disjoint, and each of the three sets A, B, and C contains precisely one vertex from each 6-face of $\mathcal{T}_{3t}$ or $\mathcal{Q}_{3t}$, respectively.

We successively replace each vertex of V, or of $V \setminus A$, or of $V \setminus (A \cup B)$, or of $V \setminus (A \cup B \cup C)$ by a triangle such that the new triangle meets no old vertex. By this procedure 3-valent graphs $\mathcal{T}_{3t}^0, \mathcal{T}_{3t}^A, \mathcal{T}_{3t}^{AB}, \mathcal{T}_{3t}^{ABC}$ of the torus and $\mathcal{Q}_{3t}^0, \mathcal{Q}_{3t}^A, \mathcal{Q}_{3t}^{AB}, \mathcal{Q}_{3t}^{ABC}$ of the Klein bottle are obtained, respectively; all new faces are triangles, and each old face has size 12, 11, 10 or 9, respectively. Next in each 12-, 11-, 10-, or 9-face α a new vertex is inserted and joined with all vertices of the boundary of α. In the triangulations so obtained, $\tilde{\mathcal{T}}_{3t}^0, \tilde{\mathcal{T}}_{3t}^A, \tilde{\mathcal{T}}_{3t}^{AB}, \tilde{\mathcal{T}}_{3t}^{ABC}$ of the torus and $\tilde{\mathcal{Q}}_{3t}^0, \tilde{\mathcal{Q}}_{3t}^A, \tilde{\mathcal{Q}}_{3t}^{AB}, \tilde{\mathcal{Q}}_{3t}^{ABC}$ of the Klein bottle the new vertices have degree 12, 11, 10, or 9, respectively, and each other vertex has degree 5.

If the number of vertices is large enough then for any real constant c the constructed triangulations have a positive charge $> c|\chi(\mathbb{M})|$.

By construction, each 4-cycle of $\tilde{\mathcal{T}}_{3t}^0, \tilde{\mathcal{T}}_{3t}^A, \tilde{\mathcal{T}}_{3t}^{AB}, \tilde{\mathcal{T}}_{3t}^{ABC}$ and $\tilde{\mathcal{Q}}_{3t}^0, \tilde{\mathcal{Q}}_{3t}^A, \tilde{\mathcal{Q}}_{3t}^{AB}, \tilde{\mathcal{Q}}_{3t}^{ABC}$ contains a vertex of degree 12, 11, 10, or 9, respectively. Hence the weight of each 4-cycle is $\ge 27, \ge 26, \ge 25$, or ≥ 24, respectively. This shows that the bounds of Theorem 9 for the torus and the Klein bottle are tight.

B2. The lower bounds for $T_4, K_{1,3}$, and P_4 in maps of large order and large positive charge.

Finally, we choose s so that in $\mathcal{T}_s$ and $\mathcal{Q}_s$ at least $c|\chi(\mathbb{M})|$ edges can be found with the property that any two of these edges have a distance at least 5. We switch these edges, i.e., if e is one of these edges then we delete this edge and in the 4-face α obtained, we add a new edge joining the vertices of α not incident with the deleted edge e. The endvertices of e are 5-vertices and the two newly joined vertices are 7-vertices. Hence the obtained map $\tilde{\mathcal{T}}_s$ or $\tilde{\mathcal{Q}}_s$, respectively, has a positive charge $\ge 2c|\chi(\mathbb{M})|$.

Each connected subgraph of order $m, 1 \le m \le 4$, has a weight $\ge 6m - 1$. Hence each $P_4, K_{1,3}$ and T_4 has a weight $w(P_4) \ge 25, w(K_{1,3}) \ge 23$, and $w(T_4) \ge 23$. This shows the validity of the lower bounds in Theorem 7 and Corollary 8 in maps of large positive charge on the torus and the Klein bottle.

Since $\mathcal{T}_s$ and $\mathcal{Q}_s$ are 6-regular graphs each connected subgraph U of order m has weight $6m$. Hence each $P_4, K_{1,3}$ and T_4 has a weight $w(P_4) \ge 27, w(K_{1,3}) \ge 24$, and $w(T_4) \ge 24$. This shows the lower bounds in Theorem 5 and Corollary 6 in maps of larger order on the torus and the Klein bottle. We remark that the positive charge of $\mathcal{T}_s$ and $\mathcal{Q}_s$ is zero.

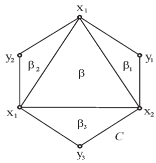

Fig. 5

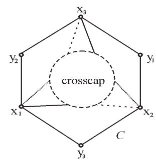

Fig. 6

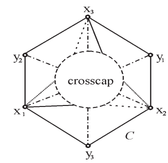

Fig. 7

C. Polyhedral maps on compact 2-manifolds of Euler characteristic $\chi(\mathbb{M}) \leq -2$.

Let $\mathcal{H}$ be one of the toroidal triangulation $\tilde{\Re}_{4t}$ of section $B1, \mathcal{T}_{3t}^0, \mathcal{T}_{3t}^A, \mathcal{T}_{3t}^{AB}$, or $\mathcal{T}_{3t}^{ABC}$ of section $B2$, or $\tilde{\mathcal{T}}_s$, or $\mathcal{T}_s$ of section $B2$.

The required polyhedral map on an orientable compact 2-manifold $\mathbb{S}_g$ of genus $g \geq 2$ will be constructed from the toroidal triangulation $\mathcal{H}$.

We choose $2g-3$ triangles of $\mathcal{H}$ so that any two of them have a distance ≥ 5 in $\mathcal{H}$ (this is possible if the number of vertices is large enough). In $\mathcal{H}$ from each of these triangles β we delete the interior part so that the bounding 3-cycle of β bounds now a hole of the torus. We join repeatedly two holes of $\mathcal{H}$ by a handle, and $g-1$ handles are added to the torus in this way.

The handles are triangulated in the following way: if $[x_1x_2x_3x_1]$ and $[y_1y_2y_3y_1]$ are the bounding cycles of some handle which are around the handle in the same cyclic order then add the cycle $[x_1y_1x_2y_2x_3y_3x_1]$. The polyhedral triangulations of $\mathbb{S}_g$ thus obtained also fulfil the degree requirements of sections $B1, B2$, or $B3$, respectively.

Let $\mathcal{K}$ be one of the triangulations of the Klein bottle $\mathcal{P}_{4t}$ of section $B1$, $\mathcal{Q}_{3t}^0$, $\mathcal{Q}_{3t}^A$, $\mathcal{Q}_{3t}^{AB}$, or $\mathcal{Q}_{3t}^{ABC}$ of section $B2$, or $\tilde{\mathcal{Q}}_s$, or $\mathcal{Q}_s$ of section $B3$.

The required polyhedral map on an unorientable compact 2-manifold $\mathbb{N}_q$ of genus $q \geq 3$ will be constructed from the triangulation $\mathcal{K}$ of the Klein bottle.

We choose $q-2$ triangles of $\mathcal{K}$ so that any two of them have a distance ≥ 5 in $\mathcal{K}$.

Let β be one of these triangles with bounding cycle $[x_1x_2x_3x_1]$ and $\beta_1, \beta_2, \beta_3$ the three neighbouring triangles in $\mathcal{K}$ with bounding cyles $[y_1x_3x_2], [y_2x_1x_3]$ and $[y_3x_2x_1]$ (see Fig. 5). In $\mathcal{K}$ we delete the separating edges x_1x_2, x_2x_3 and x_3x_1. A greater face F with bounding 6-cycle $\mathcal{C} = [x_1y_3x_2y_1x_3y_2x_1]$ is obtained (for the notation see Fig. 5).

In F a crosscap is placed and the edges x_1x_2, x_2x_3, and x_3x_1 are again added so that the interior of $\mathcal{C}$ is subdivided into three quadrangles (see Fig. 6). These quadrangles are subdivided by the edges $x_iy_i, i = 1, 2, 3$ (see Fig. 7).

The polyhedral triangulations of $\mathbb{N}_q$ so obtained fulfil the degree requirements of sections $B1, B2$, or $B3$, respectively.

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