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Some bicyclic antiautomorphisms of directed triple systems

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Abstract

A transitive triple, (a, b, c), is defined to be the set $\{(a, b), (b, c), (a, c)\}$ of ordered pairs. A directed triple system of order v, DTS(v), is a pair (D, β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D such that any ordered pair of distinct points of D is contained in precisely one transitive triple of β. An antiautomorphism of a directed triple system, (D, β), is a permutation of D which maps β to β^{-1}, where $\beta^{-1} = \{(c, b, a) \mid (a, b, c) \in \beta\}$. In this paper we give necessary and sufficient conditions for the existence of a directed triple system of order v admitting an antiautomorphism consisting of two cycles, where one cycle is twice the length of the other.

1 Introduction

A *Steiner triple system of order v*, STS(v), is a pair (S, β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any pair of distinct points of S is contained in precisely one block of β. Kirkman [6] showed that there is an STS(v) if and only if $v \equiv 1$ or $3 \pmod 6$ or $v = 0$.

An *automorphism* of (S, β) is a permutation of S which maps β to itself. An automorphism, α, of (S, β) is called *cyclic* if the permutation defined by α consists of a single cycle of length v. Peltesohn [10] proved that an STS(v) having a cyclic automorphism exists if and only if $v \equiv 1$ or $3 \pmod 6$ and $v \neq 9$. An automorphism, α, of (S, β) is called *bicyclic* if the permutation defined by α consists of two cycles. Calahan-Zijlstra and Gardner [1] have shown that there exists an STS(v) admitting a bicyclic automorphism having cycles of length M and N, with $1 < M \leq N$, if and only if $M \equiv 1$ or $3 \pmod 6$, $M \neq 9$, $M|N$, and $M + N \equiv 1$ or $3 \pmod 6$.

A *transitive triple*, (a, b, c), is defined to be the set $\{(a, b), (b, c), (a, c)\}$ of ordered pairs. A *directed triple system of order v*, DTS(v), is a pair (D, β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D, called *triples*, such that any ordered pair of distinct points of D is contained in precisely one element of β. Hung and Mendelsohn [4] have shown that necessary and sufficient conditions for the existence of a DTS(v) are that $v \equiv 0$ or 1 (mod 3).

For a DTS(v), (D, β), we define β^{-1} by $\beta^{-1} = \{(c, b, a) \mid (a, b, c) \in \beta\}$. Then (D, β^{-1}) is a DTS(v) and is called the *converse* of (D, β). A DTS(v) which is isomorphic to its converse is said to be *self-converse*. Kang, Chang, and Yang [5] have shown that a self-converse DTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) and $v \neq 6$. An *automorphism* of (D, β) is a permutation of D which maps β to itself. An *antiautomorphism* of (D, β) is a permutation of D which maps β to β^{-1}. Clearly, a DTS(v) is self-converse if and only if it admits an antiautomorphism.

An automorphism, α, on a DTS(v) is called *d-cyclic* if the permutation defined by α consists of a single cycle of length d and $v - d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a d-cyclic automorphism have been given by Micale and Pennisi [8]. An automorphism, α, on a DTS(v) is called *f-bicyclic* if the permutation defined by α consists of two cycles each of length $N = (v - f)/2$ and f fixed points. Micale and Pennisi [7] have given conditions for the existence of f-bicyclic directed triple systems.

An antiautomorphism, α, on a DTS(v) is called *d-cyclic* if the permutation defined by α consists of a single cycle of length d and $v - d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a d-cyclic antiautomorphism have been given by Carnes, Dye, and Reed [2]. We call an antiautomorphism, α, on a DTS(v) *f-bicyclic* if the permutation defined by α consists of two cycles each of length $N = (v - f)/2$ and f fixed points. A *bicyclic* antiautomorphism of a DTS(v) is an antiautomorphism, α, which consists of two cycles of length M and N respectively, where $v = M + N$. Carnes, Dye, and Reed [3] gave necessary and sufficient conditions for the case where $M = N$.

We now consider the case where $N > M$.

2 Preliminaries

If K is the length of a cycle, $K \in \{M, N\}$, we let the cycles be $(0_i, 1_i, 2_i, \ldots, (K-1)_i)$, $i \in \{0, 1\}$. Let $\Delta = \{0, 1, 2, \ldots, (K-1)\}$. We shall use all additions modulo K in the triples. For $a_i, b_j, c_k \in D$, i, j, $k \in \{0, 1\}$, $(a_i, b_j, c_k) \in \beta$, let the *orbit* of (a_i, b_j, c_k) be $\{((a+t)_i, (b+t)_j, (c+t)_k) \mid t \in \Delta, t \text{ even}\} \cup \{((c+t)_k, (b+t)_j, (a+t)_i) \mid t \in \Delta, t \text{ odd}\}$. Clearly, the orbits of the elements of β yield a partition of β.

We say that a collection of triples, $\bar{\beta}$, is a collection of *base triples* of a DTS(v) under α if the orbits of the triples of $\bar{\beta}$ produce β and exactly one triple of each orbit occurs in $\bar{\beta}$. Also, we say that the *reverse* of the triple (a, b, c) is the triple (c, b, a).

Let (S, β') be an STS(v). Let $\beta = \{(a, b, c), (c, b, a) \mid \{a, b, c\} \in \beta'\}$. Then (S, β) is called the *corresponding* DTS(v), and the identity map on the point set is an antiautomorphism. This yields a self-converse DTS(v) for $v \equiv 1$ or 3 (mod 6). For $v \equiv 1$ (mod 6) the cyclic STS(v) from Peltesohn's constructions [10] has no orbits of length less than v, hence the corresponding DTS(v) admits a cyclic antiautomorphism.

In the constructions of the base triples, we also use the following systems. An

(A, n)-*system* is a collection of ordered pairs (a_r, b_r) for $r = 1, 2, \ldots, n$ that partition the set $\{1, 2, \ldots, 2n\}$, such that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$. Skolem [11] showed that an (A, n)-system exists if and only if $n \equiv 0$ or $1 \pmod 4$. A (B, n)-*system* is a collection of ordered pairs (a_r, b_r) for $r = 1, 2, \ldots, n$ that partition the set $\{1, 2, \ldots, 2n-1, 2n+1\}$, such that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$. O'Keefe [9] showed that a (B, n)-system exists if and only if $n \equiv 2$ or $3 \pmod 4$. In either case, the triples used in the constructions will be of the form $(0_i, (a_r + n)_i, (b_r + n)_i)$ for $r = 1, 2, \ldots, n$, where $i = 0$ in the cycle of length M and $i = 1$ in the cycle of length N.

3 Necessary conditions

The types of cyclic triples possible are:
1) Type 1: (x_0, y_0, z_0) where x_0, y_0, z_0 are in the cycle of length M;
2) Type 2: (x_1, y_1, z_1) where x_1, y_1, z_1 are in the cycle of length N;
3) Type 3: (x_0, y_1, z_1) or (y_1, x_0, z_1) or (y_1, z_1, x_0) where x_0 is in the cycle of length M and y_1, z_1 are in the cycle of length N;
4) Type 4: (x_0, y_0, z_1) or (x_0, z_1, y_0) or (z_1, x_0, y_0) where x_0, y_0 are in the cycle of length M and z_1 is in the cycle of length N.

Lemma 3.1 *If a DTS(v) admits a bicyclic antiautomorphism with cycles of length M and N, where $1 < M < N$, and a type 4 triple occurs, then M is odd and $N = 2M$.*

Proof: If the triple (x_0, y_0, z_1) occurs and M is even, then $\alpha^M((x_0, y_0, z_1)) = (x_0, y_0, (z + M)_1)$. But then $z + M = z$, so that $N|M$, a contradiction.

If the triple (x_0, y_0, z_1) occurs and M is odd, then $\alpha^{2M}((x_0, y_0, z_1)) = (x_0, y_0, (z + 2M)_1)$. But then $z + 2M = z$, so that $N = 2M$.

The proof is similar for the other type 4 triples. □

Lemma 3.2 *Let (D, β) be a DTS(v) admitting a bicyclic antiautomorphism, where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles. Then If M is even, $M \equiv 4 \pmod{12}$.*

Proof: Let $a_0, b_0, c_0 \in D$. Then $\alpha^M(a_0, b_0, c_0) = (a_0, b_0, c_0)$. Also if two points of a triple are in the cycle of length M, then all three must be, otherwise there would be two distinct triples with a common ordered pair after applying α^M, a contradiction. Hence we have a cyclic subsystem. Carnes, Dye, and Reed have shown in Lemmas 1 and 2 of [2] that if M is even then $M \equiv 4 \pmod{12}$. □

Lemma 3.3 *If a DTS(v) admits a bicyclic antiautomorphism with cycles of length M and N, where $1 < M < N$, then $M|N$.*

Proof: For $N \neq 2M$ a type 3 triple must occur. Without loss of generality, assume the triple (x_0, y_1, z_1) occurs.

If N is even, we have $\alpha^N((x_0, y_1, z_1)) = ((x + N)_0, y_1, z_1)$. But then $x + N = x$, so that $M|N$.

If N is odd, then $\alpha^{2N}((x_0, y_1, z_1)) = ((x+2N)_0, y_1, z_1)$. But then $x + 2N = x$, so that $M|2N$. In the case where M is odd, we have $M|N$. In the case where M is even, we must have $M \equiv 4 \pmod{12}$. Since $M|2N$ we have that N is even, a contradiction. □

In the remainder of this paper we consider the case where $N = 2M$.

4 M even

Lemma 4.1 *If $v = M + N$, $N = 2M$, and $M \equiv 4 \pmod{12}$, there exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles.*

Proof: Let $M = 12k + 4$, $N = 24k + 8$.

For $k = 0$ the base triples are $(0_1, 4_1, 2_1)$, and the following with their reverses: $(0_0, 0_1, 3_1), (0_0, 1_1, 2_1)$, and the triples for a cycle of length 4 in [2].

For $k \geq 1$ the base triples are $(0_1, (12k+4)_1, (6k+2)_1)$, along with the following and their reverses:
$(0_0, (6k-t+1)_1, (6k+t+2)_1)$ for $t = 0, 1, \ldots, 6k+1$,
$(0_1, (6k-2t)_1, (6k+2t+4)_1)$ for $t = 0, 1, \ldots, k-1$,
$(0_1, (10k-2t+2)_1, (10k+2t+4)_1)$ for $t = 0, 1, \ldots, k-1$,
and the triples for a cycle of length M in [2]. □

5 M odd

Lemma 5.1 *If $v = M + N$ and $M \equiv 1 \pmod{12}$, there exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles.*

Proof: Let $M = 12k + 1$, $N = 24k + 2$.

For $k = 0$ the base triples are $(0_1, 0_0, 1_1)$ and $(1_1, 0_0, 0_1)$.

For $k \geq 2$, k even, the base triples are the following, along with their reverses:
$(0_1, 0_0, (12k+1)_1)$,
$(0_0, (3k-t)_1, (9k+t+1)_1)$ for $t = 0, 1, \ldots, 3k-1$,
$(0_0, (9k-t)_1, (15k+t+2)_1)$ for $t = 0, 1, \ldots, 3k-1$,
and the triples in the cycle of length M from the corresponding DTS(M) of a cyclic STS(M). The remaining triples in the cycle of length N are formed using an $(A, 2k)$-system.

For $k = 1$ the base triples are the following, along with their reverses:
$(0_1, 0_0, 13_1), (0_0, 1_1, 12_1), (0_0, 2_1, 11_1), (0_0, 7_1, 19_1), (0_0, 8_1, 18_1), (0_0, 9_1, 17_1)$,
$(0_0, 10_1, 16_1)$, and the triples in the cycle of length 13 from the corresponding DTS(13) of a cyclic STS(13). The remaining triples in the cycle of length N are formed using a $(B, 2)$-system.

For $k \geq 3$, k odd, the base triples are the following, along with their reverses:
$(0_1, 0_0, (12k+1)_1)$,

$(0_0, (3k-t-1)_1, (9k+t+2)_1)$ for $t = 0, 1, \ldots, 3k-2$,
$(0_0, (9k-t+1)_1, (15k+t+1)_1)$ for $t = 0, 1, \ldots, 3k$,
and the triples in the cycle of length M from the corresponding DTS(M) of a cyclic STS(M). The remaining triples in the cycle of length N are formed using a $(B, 2k)$-system.
□

Lemma 5.2 *If $v = M + N$ and $M \equiv 3 \pmod{12}$, there exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles.*

Proof: Let $M = 12k + 3$, $N = 24k + 6$.

For $k = 0$, the base triples are the following:
$(0_1, 2_0, 3_1), (3_1, 2_0, 0_1), (0_1, 1_1, 2_1), (0_0, 1_0, 3_1), (0_0, 2_1, 0_1)$.

For $k \geq 2$, k even, the base triples include the following:
$(0_0, (6k+1)_0, (12k+2)_1), (0_0, (6k+1)_1, (18k+3)_1), (0_0, (24k+5)_1, (6k)_1)$,
$(0_0, (18k+6)_1, (6k+2)_1), (1_0, (12k+2)_1, (6k+3)_1), (0_0, (6k+3)_1, (12k+1)_1)$,
$((6k-1)_1, 0_1, (6k+1)_1), (0_1, (6k-2)_1, (6k)_1))$.
Also included are the following triples and their reverses:
$(0_1, (12k+2)_0, (12k+3)_1), (0_0, 2_1, (12k+3)_1), (0_1, (12k-1)_1, (24k-1)_1)$,
$(0_0, (3k-t)_1, (9k+t+3)_1)$ for $t = 0, 1, \ldots, 3k-3$,
$(0_0, (9k-t+2)_1, (15k+t+4)_1)$ for $t = 0, 1, \ldots, 3k-2$.
The remaining triples in the cycle of length M are formed using an $(A, 2k)$-system.

For $k = 2$, the remaining triples are the following:
$(1_1, 5_1, 13_1), (0_1, 8_1, 9_1), (1_1, 6_1, 10_1), (0_1, 5_1, 6_1), (1_1, 4_1, 7_1)$.

For $k \geq 4$, k even, the base triples also include the following:
$((6k)_1, 6_1, 0_1), (0_1, 3_1, 6_1), (0_1, (3k-3)_1, (6k-6)_1)$.

For $k = 4$ the remaining triples are the following, along with their reverses:
$(0_1, 15_1, 16_1), (0_1, 13_1, 17_1), (0_1, 14_1, 19_1), (0_1, 12_1, 20_1), (0_1, 11_1, 21_1)$.

For $k = 6$, the remaining triples are the following, along with their reverses:
$(0_1, 25_1, 26_1), (0_1, 17_1, 21_1), (0_1, 24_1, 29_1), (0_1, 18_1, 27_1), (0_1, 20_1, 31_1)$,
$(0_1, 16_1, 28_1), (0_1, 19_1, 32_1), (0_1, 14_1, 22_1), (0_1, 23_1, 33_1)$.

For $k \geq 8$, k even, the base triples also include the following, along with their reverses:
$(0_1, (4k-1)_1, (6k-3)_1)$,
$(0_1, (3k-t-4)_1, (3k+t+4)_1)$ for $t = 0, 1, \ldots, k-6$.

For $k = 8$ the remaining triples are the following, along with their reverses:
$(0_1, 33_1, 44_1), (0_1, 34_1, 43_1), (0_1, 35_1, 36_1), (0_1, 23_1, 27_1), (0_1, 32_1, 37_1)$,
$(0_1, 26_1, 39_1), (0_1, 25_1, 40_1), (0_1, 24_1, 41_1), (0_1, 22_1, 38_1)$.

For $k \geq 10$, $k \equiv 0 \pmod 6$ the remaining triples are the following, along with their reverses:
$(0_1, (5k-5)_1, (5k-4)_1), (0_1, (3k-1)_1, (3k+3)_1), (0_1, (5k-6)_1, (5k-1)_1)$,
$(0_1, (3k)_1, (5k-3)_1), (0_1, (3k+2)_1, (5k+1)_1), (0_1, (3k-2)_1, (5k-2)_1)$,
$(0_1, (3k+1)_1, (5k+2)_1)$,
$(0_1, (5k-3t-8)_1, (5k+3t+5)_1)$ for $t = 0, 1, \ldots, \frac{k-9}{3}$,
$(0_1, (5k-3t-7)_1, (5k+3t+4)_1)$ for $t = 0, 1, \ldots, \frac{k-9}{3}$,
$(0_1, (5k-3t-9)_1, (5k+3t)_1)$ for $t = 0, 1, \ldots, \frac{k-9}{3}$.

For $k \geq 10$, $k \equiv 2 \pmod 6$ the remaining triples are the following, along with their reverses:
$(0_1, (5k-5)_1, (5k-4)_1), (0_1, (3k-1)_1, (3k+3)_1), (0_1, (5k-8)_1, (5k-3)_1),$
$(0_1, (3k+2)_1, (5k-1)_1), (0_1, (3k+1)_1, (5k)_1), (0_1, (3k)_1, (5k+1)_1),$
$(0_1, (3k-2)_1, (5k-2)_1),$
$(0_1, (5k-3t-7)_1, (5k+3t+4)_1)$ for $t = 0, 1, \ldots, \frac{k-8}{3}$,
$(0_1, (5k-3t-6)_1, (5k+3t+3)_1)$ for $t = 0, 1, \ldots, \frac{k-8}{3}$,
$(0_1, (5k-3t-11)_1, (5k+3t+2)_1)$ for $t = 0, 1, \ldots, \frac{k-11}{3}$.

For $k \geq 10$, $k \equiv 4 \pmod 6$ the remaining triples are the following, along with their reverses:
$(0_1, (5k-5)_1, (5k-4)_1), (0_1, (5k-7)_1, (5k-3)_1), (0_1, (3k-2)_1, (3k+3)_1),$
$(0_1, (3k+1)_1, (5k-2)_1), (0_1, (3k)_1, (5k-1)_1), (0_1, (3k-1)_1, (5k)_1),$
$(0_1, (3k+2)_1, (5k+2)_1),$
$(0_1, (5k-3t-6)_1, (5k+3t+3)_1)$ for $t = 0, 1, \ldots, \frac{k-7}{3}$,
$(0_1, (5k-3t-8)_1, (5k+3t+5)_1)$ for $t = 0, 1, \ldots, \frac{k-10}{3}$,
$(0_1, (5k-t-10)_1, (5k+3t+1)_1)$ for $t = 0, 1, \ldots, \frac{k-10}{3}$.

For $k \geq 1$, k odd, the base triples include the following:
$(0_0, (6k)_0, (12k+2)_1), ((12k+2)_1, (6k+2)_1, 0_0)$.
Also included are the following triples and their reverses:
$(0_1, (12k+2)_0, (12k+3)_1), (0_0, 0_1, (12k+1)_1), (0_0, (3k)_1, (3k+1)_1),$
$(0_0, (3k-t-1)_1, (9k+t+3)_1)$ for $t = 0, 1, \ldots, 3k-3$,
$(0_0, (9k-t+2)_1, (15k+t+5)_1)$ for $t = 0, 1, \ldots, 3k-2$.
The remaining triples in the cycle of length M are formed using a $(B, 2k)$-system.

For $k = 1$ the remaining triples are the following:
$(0_1, 8_1, 14_1), (1_1, 13_1, 27_1), (0_1, 5_1, 12_1), (0_1, 3_1, 7_1), (1_1, 9_1, 4_1)$.

For $k \geq 3$, k odd, the base triples also include $(0_1, (3k)_1, (6k)_1)$, along with the following triples and their reverses:
$(0_0, (6k+1)_1, (6k+3)_1), (0_1, (12k)_1, (24k+2)_1)$.

For $k = 3$ the remaining triples are the following:
$(0_1, 16_1, 36_1), (0_1, 14_1, 29_1), (0_1, 8_1, 13_1), (0_1, 11_1, 17_1), (0_1, 12_1, 19_1)$.

For $k \geq 5$, k odd, the base triples also include the following triples and their reverses:
$(0_1, (2k+3)_1, (4k+5)_1), (0_1, (4k+2)_1, (6k+2)_1), (0_1, (4k+3)_1, (6k+1)_1),$
$(0_1, (3k+1)_1, (3k+4)_1), (0_1, (3k+2)_1, (5k+3)_1), (0_1, (3k+3)_1, (5k+2)_1),$
$(0_1, (3k+5)_1, (5k+1)_1), (0_1, (5k-1)_1, (5k+4)_1),$
$(0_0, (3k-t-1)_1, (3k+t+6)_1)$ for $t = 0, 1, \ldots, k-5$.

For $k \geq 7$, k odd, the base triples also include the following and their reverses:
$(0_0, (5k-2t-3)_1, (5k+2t+5)_1)$ for $t = 0, 1, \ldots, \frac{k-7}{2}$,
$(0_0, (5k-2t)_1, (5k+2t+6)_1)$ for $t = 0, 1, \ldots, \frac{k-7}{2}$. □

Lemma 5.3 *If $v = M + N$ and $M \equiv 5 \pmod{12}$, there exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles.*

Proof: For $k = 0$, the base triples are
$(0_0, 0_1, 1_0), (0_0, 2_1, 2_0), (1_0, 4_1, 0_1), (1_0, 3_1, 9_1)$

and the following with their reverses: $(0_1, 4_0, 5_1), (0_1, 1_1, 3_1)$.

For $k \geq 2$, k even, the base triples include the following:
$(0_0, (6k+2)_0, (12k+4)_1), ((12k+4)_1, (6k+1)_0, 0_0), ((6k+3)_1, 0_0, 0_1)$,
$(0_1, 0_0, (18k+7)_1)$.
Also included are the following triples and their reverses:
$(0_1, (12k+4)_0, (12k+5)_1), (0_0, (12k+3)_1, (12k+4)_1)$,
$(0_0, (3k-t)_1, (9k+t+5)_1)$ for $t = 0, 1, \ldots, 3k-2$,
$(0_0, (9k-t+4)_1, (15k+t+6)_1)$ for $t = 0, 1, \ldots, 3k$.
The remaining triples in the cycle of length M are formed using an $(A, 2k)$-system.

For $k = 2$, the remaining triples are the following, along with their reverses:
$(0_1, 6_1, 11_1), (0_1, 7_1, 9_1), (0_1, 8_1, 12_1), (0_1, 10_1, 13_1)$.

For $k \geq 4$, k even, the base triples also include the following, along with their reverses:
$(0_1, (2k+2)_1, (4k+2)_1), (0_1, (5k+1)_1, (5k+4)_1), (0_1, (3k+2)_1, (5k+3)_1)$,
$(0_1, (3k+3)_1, (5k+2)_1)$,
$(0_1, (5k-t-1)_1, (5k+t+5)_1)$ for $t = 0, 1, \ldots, k-4$.

For $k = 4$, the remaining triples are the following, along with their reverses:
$(0_1, 11_1, 13_1), (0_1, 12_1, 17_1), (0_1, 16_1, 20_1)$.

For $k \geq 6$, k even, the base triples also include the following, along with their reverses:
$(0_1, (4k+1)_1, (5k)_1), (0_1, (\frac{5}{2}k+2)_1, (\frac{5}{2}k+4)_1)$,
$(0_1, (\frac{5}{2}k-t+1)_1, (\frac{7}{2}k+t+2)_1)$ for $t = 0, 1, \ldots, \frac{k-4}{2}$.

For $k = 6$, the remaining triples are $(0_1, 18_1, 22_1)$ and its reverse.

For $k = 8$, the remaining triples are the following, along with their reverses:
$(0_1, 22_1, 24_1), (0_1, 23_1, 28_1), (0_1, 25_1, 29_1)$.

For $k \geq 10$, $k \equiv 0 \pmod 4$ the remaining triples are the following, along with their reverses:
$(0_1, (3k+1)_1, (3k+5)_1)$,
$(0_1, (3k-2t-1)_1, (3k+2t+4)_1)$ for $t = 0, 1, \ldots, \frac{k-8}{4}$,
$(0_1, (3k-2t)_1, (3k+2t+7)_1)$ for $t = 0, 1, \ldots, \frac{k-12}{4}$.

For $k \geq 10$, $k \equiv 2 \pmod 4$ the remaining triples are the following, along with their reverses:
$(0_1, (3k)_1, (3k+4)_1)$,
$(0_1, (3k-2t-2)_1, (3k+2t+5)_1)$ for $t = 0, 1, \ldots, \frac{k-10}{4}$,
$(0_1, (3k-2t+1)_1, (3k+2t+6)_1)$ for $t = 0, 1, \ldots, \frac{k-10}{4}$.

For $k \geq 1$, k odd, the base triples include the following:
$(0_0, (6k+2)_0, (12k+4)_1), ((12k+4)_1, (6k)_0, 0_0), (0_0, (6k+4)_1, (9k+4)_1)$,
$((9k+4)_1, (6k+2)_1, 0_0)$.
Also included are the following triples and their reverses:
$(0_0, 0_1, (6k+3)_1), (0_1, (12k+4)_0, (12k+5)_1)$,
$(0_0, (3k-t+1)_1, (15k+t+7)_1)$ for $t = 0, 1, \ldots, 3k-1$,
$(0_0, (9k-t+3)_1, (21k+t+10)_1)$ for $t = 0, 1, \ldots, 3k-2$.
The remaining triples in the cycle of length M are formed using a $(B, 2k)$-system.

For $k = 1$, the remaining triples are the following:
$(0_1, 2_1, 5_1), (0_1, 7_1, 11_1), (0_1, 6_1, 10_1), (1_1, 7_1, 9_1), (7_1, 0_1, 8_1), (1_1, 12_1, 11_1)$.

For $k \geq 3$, k odd, the base triples also include the following:
$((3k)_1, 0_1, (3k+2)_1), (0_1, 1_1, 2_1)$.
Also included are the following triples and their reverses:
$(0_1, (4k+3)_1, (6k+5)_1), (0_1, (4k+4)_1, (6k+4)_1)$.

For $k = 3$, the remaining triples are the following, along with their reverses:
$(0_1, 17_1, 20_1), (0_1, 10_1, 14_1), (0_1, 13_1, 18_1), (0_1, 12_1, 19_1)$.

For $k = 5$, the remaining triples are the following, along with their reverses:
$(0_1, 28_1, 31_1), (0_1, 27_1, 32_1), (0_1, 16_1, 20_1), (0_1, 13_1, 19_1), (0_1, 22_1, 29_1)$,
$(0_1, 18_1, 26_1), (0_1, 21_1, 30_1), (0_1, 14_1, 25_1)$.

For $k \geq 7$, k odd, the base triples also include the following, along with their reverses:
$(0_1, (3k+1)_1, (5k+2)_1), (0_1, (3k+3)_1, (4k+2)_1), (0_1, (\frac{7}{2}k+\frac{3}{2})_1, (\frac{11}{2}k+\frac{1}{2})_1)$,
$(0_1, (\frac{9}{2}k - t + \frac{7}{2})_1, (\frac{11}{2}k + t + \frac{7}{2})_1)$ for $t = 0, 1, \ldots, \frac{k-3}{2}$,
$(0_1, (\frac{5}{2}k - t + \frac{3}{2})_1, (\frac{7}{2}k + t + \frac{5}{2})_1)$ for $t = 0, 1, \ldots, \frac{k-3}{2}$,
$(0_1, (3k - t - 1)_1, (3k + t + 4)_1)$ for $t = 0, 1, \ldots, \frac{k-7}{2}$.

For $k = 7$, the remaining triples are the following, along with their reverses:
$(3_1, 38_1, 41_1), (4_1, 36_1, 40_1)$.

For $k = 9$, the remaining triples are the following, along with their reverses:
$(0_1, 46_1, 52_1), (0_1, 48_1, 51_1), (0_1, 45_1, 49_1)$.

For $k = 11$, the remaining triples are the following, along with their reverses:
$(0_1, 56_1, 59_1), (0_1, 58_1, 62_1), (0_1, 54_1, 60_1), (0_1, 55_1, 63_1)$.

For $k \geq 13$, $k \equiv 1 \pmod 6$ the remaining triples are the following, along with their reverses:
$(0_1, (5k+1)_1, (5k+5)_1), (0_1, (5k+3)_1, (5k+6)_1)$,
$(0_1, (5k - 3t - 1)_1, (5k + 3t + 9)_1)$ for $t = 0, 1, \ldots, \frac{k-13}{6}$,
$(0_1, (5k - 3t)_1, (5k + 3t + 8)_1)$ for $t = 0, 1, \ldots, \frac{k-13}{6}$,
$(0_1, (5k - 3t - 2)_1, (5k + 3t + 4)_1)$ for $t = 0, 1, \ldots, \frac{k-13}{6}$.

For $k \geq 13$, $k \equiv 3 \pmod 6$ the remaining triples are the following, along with their reverses:
$(0_1, (5k+1)_1, (5k+7)_1), (0_1, (5k+3)_1, (5k+6)_1), (0_1, (5k)_1, (5k+4)_1)$,
$(0_1, (5k - 3t - 2)_1, (5k + 3t + 10)_1)$ for $t = 0, 1, \ldots, \frac{k-15}{6}$,
$(0_1, (5k - 3t - 1)_1, (5k + 3t + 9)_1)$ for $t = 0, 1, \ldots, \frac{k-15}{6}$,
$(0_1, (5k - 3t - 3)_1, (5k + 3t + 5)_1)$ for $t = 0, 1, \ldots, \frac{k-15}{6}$.

For $k \geq 13$, $k \equiv 5 \pmod 6$ the remaining triples are the following, along with their reverses:
$(0_1, (5k+1)_1, (5k+4)_1), (0_1, (5k+3)_1, (5k+7)_1), (0_1, (5k-1)_1, (5k+5)_1)$,
$(0_1, (5k)_1, (5k+8)_1)$,
$(0_1, (5k - 3t - 3)_1, (5k + 3t + 11)_1)$ for $t = 0, 1, \ldots, \frac{k-17}{6}$,
$(0_1, (5k - 3t - 2)_1, (5k + 3t + 10)_1)$ for $t = 0, 1, \ldots, \frac{k-17}{6}$,
$(0_1, (5k - 3t - 4)_1, (5k + 3t + 6)_1)$ for $t = 0, 1, \ldots, \frac{k-17}{6}$. □

Lemma 5.4 *If* $v = M + N$ *and* $M \equiv 7 \pmod{12}$, *there exists a DTS(v) which admits a bicyclic antiautomorphism where* $v = M + N$, $N = 2M$, M *and* N *being the lengths of the cycles.*

Proof: Let $M = 12k + 7$, $N = 24k + 14$.

For $k \geq 0$, k even, the base triples are the following, along with their reverses:
$(0_1, 0_0, (12k+7)_1), (0_0, (9k+5)_1, (15k+9)_1)$,
$(0_0, (3k-t+1)_1, (9k+t+6)_1)$ for $t = 0, 1, \ldots, 3k$,
$(0_0, (9k-t+4)_1, (15k+t+10)_1)$ for $t = 0, 1, \ldots, 3k$,
and the triples in the cycle of length M from the corresponding DTS(M) of a cyclic STS(M). The remaining triples in the cycle of length N are formed using an $(A, 2k+1)$-system.

For $k \geq 1$, k odd, the base triples are the following, along with their reverses:
$(0_1, 0_0, (12k+7)_1), (0_0, (3k+2)_1, (9k+5)_1)$,
$(0_0, (3k-t+1)_1, (9k+t+6)_1)$ for $t = 0, 1, \ldots, 3k$,
$(0_0, (9k-t+4)_1, (15k+t+10)_1)$ for $t = 0, 1, \ldots, 3k$,
and the triples in the cycle of length M from the corresponding DTS(M) of a cyclic STS(M). The remaining triples in the cycle of length N are formed using a $(B, 2k+1)$-system. □

Lemma 5.5 *If $v = M + N$ and $M \equiv 9 \pmod{12}$, there exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles.*

Proof: Let $M = 12k + 9$, $N = 24k + 18$.

For $k \geq 0$, k even, the base triples include the following:
$(0_0, (6k+4)_0, (12k+8)_1), (1_0, (6k+5)_1, (18k+13)_1), (1_0, (6k+4)_1, (12k+9)_1)$,
$(0_0, (6k+5)_1, (18k+15)_1), (1_0, (12k+8)_1, (6k+6)_1), (0_0, (6k+6)_1, (12k+7)_1)$.
Also included are the following, along with their reverses:
$(0_0, 2_1, (12k+9)_1), (0_1, (12k+8)_0, (12k+9)_1)$.
The remaining triples in the cycle of length M are formed using an $(A, 2k+1)$-system.

For $k = 0$, the remaining triples are the following:
$(0_1, 3_1, 6_1), (6_1, 0_1, 2_1), (5_1, 0_1, 4_1)$.

For $k \geq 2$, k even, the following are also included, along with their reverses:
$(0_0, (3k-t+1)_1, (9k+t+8)_1)$ for $t = 0, 1, \ldots, 3k-2$,
$(0_0, (9k-t+7)_1, (15k+t+11)_1)$ for $t = 0, 1, \ldots, 3k$.

For $k = 2$, the remaining triples are the following:
$(31_1, 1_1, 18_1), (0_1, 29_1, 59_1), (29_1, 0_1, 14_1), (7_1, 0_1, 15_1), (0_1, 8_1, 6_1), (6_1, 0_1, 10_1)$,
$(0_1, 5_1, 9_1), (9_1, 0_1, 12_1), (10_1, 0_1, 11_1), (11_1, 12_1, 0_1), (5_1, 0_1, 3_1)$.

For $k \geq 4$, k even, the base triples also include the following:
$((6k+5)_1, 0_1, (6k+2)_1), ((6k+1)_1, 0_1, 3_1), ((6k-2)_1, (3k-1)_1, 0_1)$.

For $k = 4$, the remaining triples are the following, along with their reverses:
$(0_1, 10_1, 12_1), (0_1, 13_1, 18_1), (0_1, 14_1, 20_1), (0_1, 15_1, 24_1), (0_1, 16_1, 23_1)$,
$(0_1, 17_1, 21_1), (0_1, 19_1, 27_1), (0_1, 53_1, 54_1)$.

For $k = 6$, the remaining triples are the following, along with their reverses:
$(0_1, 14_1, 27_1), (0_1, 15_1, 26_1), (0_1, 16_1, 25_1), (0_1, 18_1, 24_1), (0_1, 19_1, 23_1), (0_1, 20_1, 22_1)$,
$(0_1, 21_1, 33_1), (0_1, 28_1, 36_1), (0_1, 29_1, 39_1), (0_1, 30_1, 35_1), (0_1, 31_1, 32_1), (0_1, 77_1, 155_1)$.

For $k \geq 8$, k even, the remaining triples are the following, along with their reverses:

$(0_1, (12k+5)_1, (12k+6)_1), (0_1, (4k+2)_1, (6k+3)_1), (0_1, (6k-5)_1, (6k-3)_1),$
$(0_1, (6k-4)_1, (6k)_1), (0_1, (6k-6)_1, (6k-1)_1),$
$(0_1, (3k-t-2)_1, (3k+t+5)_1)$ for $t = 0, 1, \dots, k-4,$
$(0_1, (3k-t+4)_1, (5k+t-4)_1)$ for $t = 0, 1, 2, 3, 4,$
$(0_1, (5k-t-5)_1, (5k+t+1)_1)$ for $t = 0, 1, \dots, k-8.$

For $k \geq 1$, k odd, the base triples include the following:
$(0_0, (6k+3)_0, (12k+8)_1), (1_0, (6k+4)_1, (18k+14)_1), (1_0, (6k+5)_1, (12k+9)_1),$
$(0_0, (6k+3)_1, (6k+5)_1), (0_1, 2_1, (12k+8)_1), ((12k+6)_1, 0_1, (6k+4)_1),$
$((3k+5)_1, 1_1, (3k-1)_1), (0_1, (3k-2)_1, (6k+2)_1), (0_1, 3_1, 6_1).$
Also included are the following, along with their reverses:
$(0_1, (12k+8)_0, (12k+9)_1), (0_0, 0_1, (12k+7)_1),$
$(0_0, (3k-t+1)_1, (9k+t+7)_1)$ for $t = 0, 1, \dots, 3k-1,$
$(0_0, (9k-t+6)_1, (15k+t+11)_1)$ for $t = 0, 1, \dots, 3k.$
The remaining triples in the cycle of length M are formed using a $(B, 2k+1)$-system.

For $k = 1$ the remaining triples are $(0_1, 5_1, 9_1)$ and its reverse.

For $k = 3$ the remaining triples are the following, along with their reverses:
$(0_1, 16_1, 17_1), (0_1, 14_1, 18_1), (0_1, 10_1, 15_1), (0_1, 11_1, 19_1), (0_1, 12_1, 21_1).$

For $k = 5$ the remaining triples are the following, along with their reverses:
$(0_1, 14_1, 26_1), (0_1, 15_1, 25_1), (0_1, 16_1, 23_1), (0_1, 17_1, 21_1), (0_1, 18_1, 27_1),$
$(0_1, 20_1, 28_1), (0_1, 22_1, 33_1), (0_1, 24_1, 29_1), (0_1, 30_1, 31_1).$

For $k = 7$ the remaining triples are the following, along with their reverses:
$(0_1, 17_1, 31_1), (0_1, 18_1, 30_1), (0_1, 20_1, 29_1), (0_1, 21_1, 28_1), (0_1, 22_1, 27_1),$
$(0_1, 23_1, 39_1), (0_1, 24_1, 35_1), (0_1, 26_1, 41_1), (0_1, 32_1, 45_1), (0_1, 33_1, 43_1),$
$(0_1, 34_1, 42_1), (0_1, 36_1, 40_1), (0_1, 37_1, 38_1).$

For $k = 9$ the remaining triples are the following, along with their reverses:
$(0_1, 21_1, 37_1), (0_1, 22_1, 36_1), (0_1, 23_1, 35_1), (0_1, 24_1, 34_1), (0_1, 26_1, 33_1),$
$(0_1, 27_1, 32_1), (0_1, 28_1, 47_1), (0_1, 29_1, 49_1), (0_1, 30_1, 48_1), (0_1, 38_1, 53_1),$
$(0_1, 39_1, 52_1), (0_1, 41_1, 42_1), (0_1, 43_1, 51_1), (0_1, 44_1, 55_1), (0_1, 45_1, 54_1),$
$(0_1, 46_1, 50_1), (0_1, 40_1, 57_1).$

For $k = 11$ the remaining triples are the following, along with their reverses:
$(0_1, 25_1, 47_1), (0_1, 26_1, 46_1), (0_1, 27_1, 45_1), (0_1, 28_1, 44_1), (0_1, 29_1, 43_1),$
$(0_1, 30_1, 42_1), (0_1, 32_1, 40_1), (0_1, 33_1, 56_1), (0_1, 34_1, 58_1), (0_1, 35_1, 39_1),$
$(0_1, 36_1, 41_1), (0_1, 38_1, 57_1), (0_1, 49_1, 59_1), (0_1, 48_1, 69_1), (0_1, 50_1, 67_1),$
$(0_1, 51_1, 66_1), (0_1, 52_1, 65_1), (0_1, 53_1, 64_1), (0_1, 54_1, 63_1), (0_1, 55_1, 62_1),$
$(0_1, 60_1, 61_1).$

For $k \geq 13$, k odd, the base triples also include the following, along with their reverses:
$(0_1, (\frac{9}{2}k+\frac{9}{2})_1, (\frac{9}{2}k+\frac{11}{2})_1), (0_1, (4k+3)_1, (5k+3)_1), (0_1, (4k+2)_1, (6k+3)_1),$
$(0_1, (3k+2)_1, (5k+4)_1), (0_1, (3k+1)_1, (5k+1)_1), (0_1, (3k+6)_1, (5k+5)_1),$
$(0_1, (3k-1)_1, (3k+3)_1), (0_1, (3k)_1, (3k+5)_1),$
$(0_1, (3k-t-3)_1, (3k+t+7)_1)$ for $t = 0, 1, \dots, k-6,$
$(0_1, (\frac{9}{2}k-t+\frac{3}{2})_1, (\frac{11}{2}k+t+\frac{7}{2})_1)$ for $t = 0, 1, \dots, \frac{k-5}{2}.$

For $k = 13$ the remaining triples are the following, along with their reverses:
$(0_1, 61_1, 72_1), (0_1, 62_1, 71_1), (0_1, 65_1, 73_1), (0_1, 67_1, 74_1).$

For $k = 15$ the remaining triples are the following, along with their reverses: $(0_1, 70_1, 83_1), (0_1, 71_1, 82_1), (0_1, 74_1, 81_1), (0_1, 75_1, 84_1), (0_1, 77_1, 85_1)$.

For $k = 17$ the remaining triples are the following, along with their reverses: $(0_1, 79_1, 94_1), (0_1, 80_1, 93_1), (0_1, 83_1, 92_1), (0_1, 84_1, 91_1), (0_1, 85_1, 96_1)$, $(0_1, 87_1, 95_1)$.

For $k = 19$ the remaining triples are the following, along with their reverses: $(0_1, 88_1, 105_1), (0_1, 89_1, 104_1), (0_1, 92_1, 103_1), (0_1, 93_1, 101_1), (0_1, 94_1, 107_1)$, $(0_1, 95_1, 102_1), (0_1, 97_1, 106_1)$.

For $k = 21$ the remaining triples are the following, along with their reverses: $(0_1, 97_1, 116_1), (0_1, 101_1, 112_1), (0_1, 103_1, 118_1), (0_1, 98_1, 115_1), (0_1, 102_1, 111_1)$, $(0_1, 104_1, 117_1), (0_1, 105_1, 113_1), (0_1, 107_1, 114_1)$.

For $k \geq 23$, $k \equiv 1 \pmod 8$, the remaining triples are the following, along with their reverses:
$(0_1, (5k)_1, (5k+11)_1), (0_1, (5k-2)_1, (5k+7)_1), (0_1, (5k-1)_1, (5k+6)_1)$,
$(0_1, (5k+2)_1, (5k+10)_1)$,
$(0_1, (5k-4t-6)_1, (5k+4t+9)_1)$ for $t = 0, 1, \ldots, \frac{k-17}{8}$,
$(0_1, (5k-4t-5)_1, (5k+4t+8)_1)$ for $t = 0, 1, \ldots, \frac{k-17}{8}$,
$(0_1, (5k-4t-4)_1, (5k+4t+15)_1)$ for $t = 0, 1, \ldots, \frac{k-25}{8}$,
$(0_1, (5k-4t-3)_1, (5k+4t+14)_1)$ for $t = 0, 1, \ldots, \frac{k-25}{8}$.

For $k \geq 23$, $k \equiv 3 \pmod 8$, the remaining triples are the following, along with their reverses:
$(0_1, (5k-1)_1, (5k+12)_1), (0_1, (5k-3)_1, (5k+8)_1), (0_1, (5k+2)_1, (5k+11)_1)$,
$(0_1, (5k)_1, (5k+7)_1), (0_1, (5k-2)_1, (5k+6)_1)$,
$(0_1, (5k-4t-7)_1, (5k+4t+10)_1)$ for $t = 0, 1, \ldots, \frac{k-19}{8}$,
$(0_1, (5k-4t-6)_1, (5k+4t+9)_1)$ for $t = 0, 1, \ldots, \frac{k-19}{8}$,
$(0_1, (5k-4t-5)_1, (5k+4t+16)_1)$ for $t = 0, 1, \ldots, \frac{k-27}{8}$,
$(0_1, (5k-4t-4)_1, (5k+4t+15)_1)$ for $t = 0, 1, \ldots, \frac{k-27}{8}$.

For $k \geq 23$, $k \equiv 5 \pmod 8$, the remaining triples are the following, along with their reverses:
$(0_1, (5k)_1, (5k+8)_1), (0_1, (5k+2)_1, (5k+9)_1)$,
$(0_1, (5k-4t-4)_1, (5k+4t+7)_1)$ for $t = 0, 1, \ldots, \frac{k-13}{8}$,
$(0_1, (5k-4t-3)_1, (5k+4t+6)_1)$ for $t = 0, 1, \ldots, \frac{k-13}{8}$,
$(0_1, (5k-4t-2)_1, (5k+4t+13)_1)$ for $t = 0, 1, \ldots, \frac{k-21}{8}$,
$(0_1, (5k-4t-1)_1, (5k+4t+12)_1)$ for $t = 0, 1, \ldots, \frac{k-21}{8}$.

For $k \geq 23$, $k \equiv 7 \pmod 8$, the remaining triples are the following, along with their reverses:
$(0_1, (5k)_1, (5k+9)_1), (0_1, (5k-1)_1, (5k+6)_1), (0_1, (5k+2)_1, (5k+10)_1)$,
$(0_1, (5k-4t-5)_1, (5k+4t+8)_1)$ for $t = 0, 1, \ldots, \frac{k-15}{8}$,
$(0_1, (5k-4t-4)_1, (5k+4t+7)_1)$ for $t = 0, 1, \ldots, \frac{k-15}{8}$,
$(0_1, (5k-4t-3)_1, (5k+4t+14)_1)$ for $t = 0, 1, \ldots, \frac{k-23}{8}$,
$(0_1, (5k-4t-2)_1, (5k+4t+13)_1)$ for $t = 0, 1, \ldots, \frac{k-23}{8}$. □

Lemma 5.6 *If $v = M + N$ and $M \equiv 11 \pmod{12}$, there exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles.*

Proof: Let $M = 12k + 11$, $N = 24k + 22$.

For $k \geq 0$, k even, the base triples include the following:
$(0_0, (6k+5)_0, (12k+10)_1), ((12k+10)_1, (6k+4)_0, 0_0), (0_0, 0_1, (6k+6)_1)$,
$(0_1, (18k+16)_1, 0_0)$.
Also included are the following, along with their reverses:
$((21k+19)_1, 0_0, (9k+8)_1)$,
$(0_0, (3k-t+2)_1, (15k+t+14)_1)$ for $t = 0, 1, \ldots, 3k+1$,
$(0_0, (9k-t+7)_1, (21k+t+20)_1)$ for $t = 0, 1, \ldots, 3k$.
The remaining triples in the cycle of length M are formed using an $(A, 2k+1)$-system.
The remaining triples in the cycle of length N are formed using a $(B, 2k+2)$-system.

For $k \geq 1$, k odd, the base triples include the following:
$(0_0, (6k+5)_0, (12k+10)_1), ((12k+10)_1, (6k+3)_0, 0_0)$,
$(0_1, (6k+6)_0, (12k+11)_1), (1_1, (6k+5)_0, (12k+12)_1)$.
Also included are the following, along with their reverses:
$(0_0, (3k+2)_1, (6k+4)_1), (0_0, 0_1, (6k+6)_1)$,
$(0_0, (3k-t+1)_1, (15k+t+14)_1)$ for $t = 0, 1, \ldots, 3k$,
$(0_0, (9k-t+8)_1, (21k+t+20)_1)$ for $t = 0, 1, \ldots, 3k$.
The remaining triples in the cycle of length M are formed using a $(B, 2k+1)$-system.

For $k = 1$, the remaining triples are the following, along with their reverses:
$(0_1, 6_1, 9_1), (0_1, 7_1, 8_1), (0_1, 10_1, 14_1), (0_1, 11_1, 13_1)$.

For $k \geq 3$, k odd, the base triples also include the following, along with their reverses:
$(0_1, (3k+3)_1, (3k+6)_1), (0_1, (3k+4)_1, (3k+5)_1)$,
$(0_1, (3k-t+1)_1, (3k+t+7)_1)$ for $t = 0, 1, \ldots, k-2$.

For $k = 3$, the remaining triples are the following, along with their reverses:
$(0_1, 19_1, 26_1), (0_1, 20_1, 25_1), (0_1, 18_1, 22_1), (0_1, 21_1, 23_1)$.

For $k \geq 5$, $k \equiv 1 \pmod 6$, the base triples also include the following, along with their reverses:
$(0_1, (5k+5)_1, (5k+9)_1), (0_1, (5k+6)_1, (5k+8)_1)$,
$(0_1, (5k-3t+3)_1, (5k+3t+12)_1)$ for $t = 0, 1, \ldots, \frac{k-4}{3}$,
$(0_1, (5k-3t+4)_1, (5k+3t+11)_1)$ for $t = 0, 1, \ldots, \frac{k-4}{3}$,
$(0_1, (5k-3t+2)_1, (5k+3t+7)_1)$ for $t = 0, 1, \ldots, \frac{k-4}{3}$.

For $k \geq 5$, $k \equiv 3 \pmod 6$, the base triples also include the following, along with their reverses:
$(0_1, (5k+3)_1, (5k+7)_1), (0_1, (5k+6)_1, (5k+8)_1)$,
$(0_1, (5k-3t+4)_1, (5k+3t+11)_1)$ for $t = 0, 1, \ldots, \frac{k-3}{3}$,
$(0_1, (5k-3t+5)_1, (5k+3t+10)_1)$ for $t = 0, 1, \ldots, \frac{k-3}{3}$,
$(0_1, (5k-3t)_1, (5k+3t+9)_1)$ for $t = 0, 1, \ldots, \frac{k-6}{3}$.

For $k \geq 5$, $k \equiv 5 \pmod 6$, the base triples also include the following, along with their reverses:
$(0_1, (5k+4)_1, (5k+9)_1), (0_1, (5k+6)_1, (5k+10)_1), (0_1, (5k+5)_1, (5k+7)_1)$,
$(0_1, (5k-3t+2)_1, (5k+3t+13)_1)$ for $t = 0, 1, \ldots, \frac{k-5}{3}$,
$(0_1, (5k-3t+3)_1, (5k+3t+12)_1)$ for $t = 0, 1, \ldots, \frac{k-5}{3}$,
$(0_1, (5k-3t+1)_1, (5k+3t+8)_1)$ for $t = 0, 1, \ldots, \frac{k-5}{3}$. □

6 Conclusion

By the lemmas in the previous sections, we have the following theorem.

Theorem 6.1 *There exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N$, $N = 2M$, M and N being the lengths of the cycles, if and only if M is odd or $M \equiv 4 \pmod{12}$.*

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