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Minimally $(k, k-1)$-edge-connected graphs

Xiangfeng Pan

College of Sciences
Wuhan University of Technology
Wuhan 430070
People's Republic of China
xiangfengpan@yahoo.com.cn

Jingzhong Mao

Department of Mathematics
Central China Normal University
Wuhan, 430079
People's Republic of China

Huiqing Liu

Institute of Systems Science
Academy of Mathematics and Systems Sciences
Chinese Academy of Sciences
Beijing 100080
People's Republic of China

Abstract

For an interger $l > 1$, the l-edge-connectivity $\lambda_l(G)$ of G is defined to be the smallest number of edges whose removal leaves a graph with at least l components, if $|V(G)| \geq l$; and $\lambda_l(G) = |V(G)|$ if $|V(G)| \leq l$. A graph G is (k, l)-edge-connected if the l-edge-connectivity of G is at least k. A sufficient and necessary condition for G to be minimally $(k, k-1)$-edge-connected is obtained in the paper. Bounds of size of such graphs with given order are discussed.

1 Introduction

Graphs in this paper are simple and finite. See [2] for undefined terminology and notations in graph theory. Let P be a path and C a cycle; the length of P and C, denoted by $l(P)$ and $l(C)$, are defined to be the number of edges of P and C, respectively. If $S \subseteq V(G)$, we define $N(S) = \cup_{v \in S} N(v)$. If G is a connected graph, we define $B(G) = \{e : e \text{ is a cut edge of } G\}$. For an edge subset $X \subseteq E(G)$, the

contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. Let $G' = G/(E(G) - B(G))$, called the B-reduction of G. Let $e(G) = |E(G)|$, $|G| = |V(G)|$, and $\omega(G)$ denote the number of components of G.

We define the generalized edge connectivity $\lambda(G)$ of graph G to be the minimum integer k for which G has a k-edge set T such that $\omega(G-T) = \omega(G)+1$. Therefore, if graph G is connected, the generalized edge connectivity of G is just the edge connectivity of G.

For an integer $l > 2$, the l-edge-connectivity $\lambda_l(G)$, which was introduced by Boesch and Chen [1], is defined to be the smallest number of edges whose removal leaves a graph with at least l components, if $|V(G)| \geq l$; and $\lambda_l(G) = |V(G)|$ if $|V(G)| < l$.

A graph G is called (k,l)-edge-connected if $\lambda_l(G) \geq k$. A graph G is minimally (k,l)-edge-connected if $\lambda_l(G) \geq k$ but for any edge $e \in E(G)$, $\lambda_l(G-e) < k$.

Following [9], for a graph G, define a relation on $E(G)$ as follows: $\forall e, e' \in E(G), e \sim e'$ if and only if either $e = e'$, or $\{e, e'\}$ is a minimal edge cut of G. One can verify that the relation $\sim$ is an equivalence relation. Let $[e]_G$ denote the equivalence class that contains $e \in E(G)$. If there is no confusion, we simply denote $[e]_G$ by $[e]$. For any $e \in E(G)$, define $G_{[e]} = G/(E(G) - [e])$. Note that $G_{[e]}$ is obtained from G by contracting each component of $G - [e]$ into a single vertex.

If $e, e' \in E(G), e \sim e'$, then we call e is equivalent to e' in G. If for any equivalence class $[e']$ that contains $e' \in E(G)$, $|[e]| \geq |[e']|$, then $[e]$ is called a maximum equivalence class of the graph G,

Let $\mu(G) = \max\{|[e]| : e \in E(G)\}$.

A sequence of edge sets $S_1, S_2, \ldots, S_t$ is called an ordered edge-cut-set decomposition of $E(G)$, if it satisfies each of the following.

(i) $S_1 \subseteq E(G), |S_1| = \lambda(G)$ and $\omega(G - S_1) = \omega(G) + 1$.

(ii) $S_{m+1} \subseteq E(G - \cup_{i=1}^{m} S_i)$, $|S_{m+1}| = \lambda(G - \cup_{i=1}^{m} S_i)$ and $\omega(G - \cup_{i=1}^{m+1} S_i) = \omega(G - \cup_{i=1}^{m} S_i) + 1$, where $m = 1, 2, \ldots, t-1$.

(iii) $E(G) = \cup_{i=1}^{t} S_i$.

Note: If $|G| = n$, and if $S_1, S_2, \ldots, S_t$ is an ordered edge-cut-set decomposition of $E(G)$, then $|S_t| = 1$ and $t = n - \omega(G)$.

Let G be a connected graph, $k \geq 3$ and $S \subseteq E(G)$. If $|S| = k$ and $\omega(G - S) = l$, then S is called a (k,l)-edge-cut set of G.

Theorem 1 [11] *Let G be a minimally k-edge-connected graph, $|G| = n$, $k \geq 2$ and $n \geq 3k$; then $e(G) \leq k(n-k)$. Furthermore, equality holds if and only if $G \cong K_{k,n-k}$.*

Theorem 2 [15] *Let G be a minimally k-edge-connected graph, $|G| = n$, $k \geq 2$ and $k + 2 \leq n < 3k$; then $e(G) \leq \lfloor (n+k)^2/8 \rfloor$.*

By Theorem 1 and Theorem 2, one can obtain the following proposition immediately:

Proposition 1 *Let G be a minimally 2-edge-connected graph and $|G| = n \geq 4$. Then $e(G) \leq 2n - 4$.*

Proposition 2 [9] *Each of the following holds:*
(i) If G has a cut edge, then G' is a tree with edge set $B(G)$.
(ii) If G has no cut edges, then $G_{[e]}$ is a cycle with edge set $[e]$.
(iii) If G has no cut edges, then for any $e' \in [e]$, $B(G-e')$ is a path in $(G-e')'$ with edge set $[e]-\{e'\}$.

Furthermore, we get the following proposition:

Proposition 2′ *Let G be a connected graph, $e \in E(G)$ and $|[e]| \geq 2$. Then $G_{[e]}$ is a cycle with edge set $[e]$.*

2 A sufficient and necessary condition

Proposition 3 *Let G be a connected graph. The following are equivalent.*
(i) For any edge $e \in E(G)$, $\lambda_l(G-e) = k-1$.
(ii) G is mimimally (k, l)-edge-connected.

Proof. To show that Proposition 3 (i) implies Proposition 3 (ii), it suffices to show $\lambda_l(G) \geq k$. Since $\lambda_l(G-e) = k-1$, $\lambda_l(G) \leq k$. Assume $\lambda_l(G) \leq k-1$. Then there exists a $T \subseteq E(G)$ such that $|T| \leq k-1$ and $\omega(G-T) = l$. Thus for each $e \in T$, $\omega(G-e-(T-e)) = \omega(G-T) = l$. However $|T-e| \leq k-2$, contrary to the assumption that $\lambda_l(G-e) = k-1$.

Conversely, assume Proposition 3 (ii). By definition, $\lambda_l(G) \geq k$ and for any $e \in E(G)$, $\lambda_l(G-e) < k$. Assume there exists an edge $e \in E(G)$ such that $\lambda_l(G-e) \leq k-2$; then there exists a $T \subseteq E(G-e)$ with $|T| \leq k-2$ and $\omega(G-e-T) = l$. However, $|T \cup \{e\}| \leq k-1$, contrary to $\lambda_l(G) \geq k$. Thus for any edge $e \in E(G)$, $\lambda_l(G-e) = k-1$. □

Theorem 3 *Let G be a connected graph, $|G| \geq k-1$. Then G is minimally $(k, k-1)$-edge-connected if and only if each of the following holds.*
(i) $|B(G)| \leq k-3$;
(ii) $\mu(G) \leq k-|B(G)|-2$;
(iii) for any $e \in E(G)-B(G)$, $\mu(G-e) \geq k-|B(G)|-|[e]_G|$.

Proof. Let G be a minimally $(k, k-1)$-edge-connected graph; then $\lambda_{k-1}(G) \geq k$.

Assume $|B(G)| \geq k-2$. Then one can choose some $T \subseteq B(G)$ with $|T| = k-2$ and $\omega(G-T) = k-1$, contrary to $\lambda_{k-1}(G) \geq k$. So $|B(G)| \leq k-3$.

Assume $\mu(G) \geq k-|B(G)|-1$. Then we can choose $S \subseteq E(G)$ which consists of all edges in $B(G)$ and $k-|B(G)|-1$ edges in some maximum equivalence class of the graph G. Then $|S| = k-1$ and $\omega(G-S) = 1+|B(G)|+(k-|B(G)|-2) = k-1$, contrary to $\lambda_{k-1}(G) \geq k$. Therefore $\mu(G) \leq k-|B(G)|-2$.

Assume for some edge $e \in E(G)-B(G)$, $\mu(G-e) \leq k-|B(G)|-|[e]_G|-1$. By Proposition 3, $\lambda_{k-1}(G-e) = k-1$, so there exists an edge set $T \subseteq E(G-e)$ with $|T| = k-1$ and $\omega((G-e)-T) = k-1$. Let $T = \{e_1, e_2, \ldots, e_{k-1}\}$. Let $J = \{e_i | \omega(G-e-\{e_1, e_2, \ldots, e_i\}) = \omega(G-e-\{e_1, e_2, \ldots, e_{i-1}\}), 1 \leq i \leq k-1\}$, then $T-J = \{e_i | \omega(G-e-\{e_1, e_2, \ldots, e_i\}) = \omega(G-e-\{e_1, e_2, \ldots, e_{i-1}\})+1, 1 \leq i \leq k-1\}$. So $|J| = |T|-|T-J| = |T|-[\omega((G-e)-T)-1] = 1$. Without loss of generality, assume $J = \{e_1\}$. By Proposition $2'$, T must consist of edges in $B(G-e) \cup [e_1]_{G-e}$. Then, by $\mu(G-e) \leq k - |B(G)| - |[e]_G| - 1$, $|T| \leq |B(G-e) \cup [e_1]_{G-e}| \leq (|B(G)| + |[e]_G| - 1) + (k - |B(G)| - |[e]_G| - 1) = k-2$, contrary to $|T| = k-1$. Thus for any $e \in E(G) - B(G)$, $\mu(G-e) \geq k - |B(G)| - |[e]_G|$.

Conversely, assume Theorem 3 (i), (ii) and (iii). We first show that $\lambda_{k-1}(G) \geq k$. Let $T = \{f_1, f_2, \ldots, f_{k-1}\} \subseteq E(G)$ be a $k-1$ edge set. since $|B(G)|+\mu(G) \leq k-2$, T includes at least two edges(without loss of generality, assume they are f_1, f_2) which belong to different equivalence classes and are not cut edge. By Proposition $2'$, $\omega(G) = \omega(G-f_1) = \omega(G-f_1-f_2)$. Then, for any $\{f_1, f_2, \ldots, f_{k-1}\} \subseteq E(G)$, $\omega(G - \{f_1, f_2, \ldots, f_{k-1}\}) \leq 1 + (k-3) = k-2$. So $\lambda_{k-1}(G) \geq k$. To show that G is minimally $(k, k-1)$-edge-connected, it suffices to show that, for any $e \in E(G)$, $\lambda_{k-1}(G-e) < k$. Assume there exists an edge $f \in E(G)$, $\lambda_{k-1}(G-f) \geq k$. Choose some $S \subseteq E(G)$ which consists of all edges in $B(G) \cup [f]_G$ and $k - |B(G)| - |[f]_G|$ edges in some maximum equivalence class of $G-f$. Then $|S| = |B(G)| + |[f]_G| + k - |B(G)| - |[f]_G| = k$ and $\omega(G-S) = 1 + |B(G)| + |[f]_G| - 1 + (k - |B(G)| - |[f]_G| - 1) = k-1$. Thus $\omega(G - f - (S-f)) = k-1$ and $|S-f| = k-1$, contrary to the assumption that $\lambda_{k-1}(G-f) \geq k$. So G is minimally $(k, k-1)$-edge-connected. □

Corollary 1 *Let G be a 2-edge-connected graph, $|G| \geq k-1$. The following are equivalent.*

(i) G is minimally $(k, k-1)$-edge connected.

(ii) $\mu(G) \leq k-2$, and for any $e \in E(G)$, $\mu(G-e) \geq k - |[e]_G|$.

Corollary 2 *Let G be a connected graph, $|G| \geq k-1$, $|B(G)| = k-3$, and $k \geq 4$. Then G is minimally $(k, k-1)$-edge connected if and only if every nontrivial component of $G - B(G)$ is minimally 3-edge-connected.*

Proof. By Theorem 3 and $|B(G)| = k-3$, G is minimally $(k, k-1)$-edge-connected $\Leftrightarrow \mu(G) \leq k - |B(G)| - 2 = 1$ and for any $e \in E(G) - B(G)$, $\mu(G-e) \geq k - |B(G)| - |[e]_G| = 2 \Leftrightarrow$ every nontrivial component of $G - B(G)$ is minimally 3-edge-connected. □

It is easy to obtain the following.

Corollary 3 *If G is 2-edge-connected and $\mu(G) \leq k-2$, then G is $(k, k-1)$-edge-connected.*

3 Bounds of size of minimally $(k, k-1)$-edge-connected graphs with given order

Lemma 1 *If $H \subseteq G$ is 2-edge-connected, $e \in E(H)$, and $\lambda(H-e) \geq 2$, then $\mu(G-e) \leq max\{\mu(G), \mu(H-e)\}$.*

Proof. We claim that for any $f \in E(H-e)$, f is not equivalent to any edge in $E(G-e) - E(H-e)$ in $G-e$. Assume there exists an edge $f = uv \in E(H-e)$ which is equivalent to some $g \in E(G) - E(H)$ in $G - e$; then $G - \{e, g\}$ is connected. Since $\lambda(H - e) \geq 2$, $H - \{e, f\}$ is connected. Then $G - \{e, g, f\}$ is connected. (Otherwise, f is a cut edge of $G - \{e, g\}$. Thus there is no $(u,\ v)$-path in $G - \{e, g, f\}$. Then, by $H - \{e, f\} \subseteq G - \{e, g, f\}$, there is no $(u,\ v)$-path in $H - \{e, f\}$. So $H - \{e, f\}$ is not connected, a contradiction.) However, by the assumption that f is equivalent to g in $G - e$, $G - \{e, g, f\}$ is not connected. So the claim must hold. Then, for any $h \in E(G - e) - E(H - e)$, $[h]_{G-e} = [h]_G$ and $|[h]_{G-e}| \leq \mu(G)$. And for any $i \in E(H - e)$, $[i]_{G-e} \subseteq [i]_{H-e}$ and $|[i]_{G-e}| \leq \mu(H - e)$. Thus $\mu(G - e) = \max\{|[f]_{G-e}| : f \in E(G - e)\} = \max\{\max\{|[f]_{G-e}| : f \in E(G - e) - E(H - e)\}, \max\{|[i]_{G-e}| : i \in E(H - e)\}\} \leq \max\{\mu(G), \mu(H - e)\}$. □

Lemma 2 *Let G be a 2-edge-connected graph, $e \in E(G)$, and $\mu(G) = k$. Then $\mu(G - e) \leq 2k$.*

Proof. Assume $\mu(G - e) \geq 2k + 1$ and $[f]_{G-e}$ is a maximum equivalence class of $G - e$. Then $|[f]_{G-e}| = \mu(G - e)$. Let $C_1, C_2, \ldots, C_{\mu(G-e)}$ denote the components of $G - e - [f]_{G-e}$. Let u and v denote the ends of e. There are two cases.

Case 1 For some $C_i, i \in \{1, \ldots, \mu(G - e)\}$, $u \in C_i$ and $v \in C_i$. Then $\mu(G) = \mu(G - e) \geq 2k + 1$.

Case 2 For some C_i and $C_j, i, j \in \{1, \ldots, \mu(G - e)\} (i \neq j)$, $u \in C_i$ and $v \in C_j$. Then, by Propsition $2'$, $\mu(G) \geq \frac{\mu(G-e)}{2} > k$.
So $\mu(G) > k$, contrary to the assumption $\mu(G) = k$. Thus $\mu(G - e) \leq 2k$. □

Lemma 3 *Let G be a minimally $(k, k-1)$-edge-connected graph, $|G| \geq k - 1$, $|B(G)| \leq k - 4$, $k \geq 5$. Then for any $H \subseteq G$, $\lambda(H) \leq 2$.*

Proof. Assume there exists a $H \subseteq G$, $\lambda(H) \geq 3$. Without loss of generality, assume H is connected. Then $\mu(H) = 1$, and for some $e \in E(H)$, $[e]_G = 1$ and $\lambda(H - e) \geq 2$. Obviously H is 2-edge-connected. Thus, by Lemma 2, $\mu(H - e) \leq 2$. By Theorem 3, $\mu(G) \leq k - |B(G)| - 2$. And, by $|B(G)| \leq k - 4$, $k - |B(G)| - 2 \geq 2$. Then, by Lemma 1, $\mu(G - e) \leq \max\{\mu(G), \mu(H - e)\} \leq k - |B(G)| - 2$. However, by Theorem 3, $\mu(G - e) \geq k - |B(G)| - |[e]_G| = k - |B(G)| - 1$, a contradiction. □

Proposition 4 *If $H \subseteq G$ is connected and $e \in E(H) - B(H)$, then $[e]_G \subseteq [e]_H$.*

Proof. The proof is similar to that of Lemma 1. □

Lemma 4 *If G is 2-edge-connected and minimally $(k, k-1)$-edge-connected, $k \geq 6$, then G does not contain such a subgraph H that satisfies each of the following.*

(i) $\mu(H) \leq 2$.

(ii) H is 2-edge-connected but not minimally 2-edge-connected.

Proof. Assume there exists some $H \subseteq G$ which satisfies both (i) and (ii). Then for some $e \in E(H)$, $\lambda(H - e) \geq 2$. Obviously, $|[e]_H| = 1$ and $e \notin B(H)$. Then, by Proposition 4, $[e]_G \subseteq [e]_H$. By Proposition 2 (iii), $B(G - e) \subseteq [e]_H - \{e\} = \emptyset$. Therefore $G - e$ is 2-edge-connected. Now we show $G - e$ is $(k, k-1)$-edge-connected.

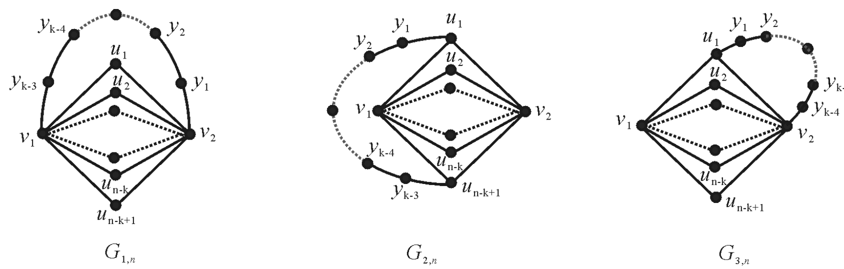

Figure 1

Since $\mu(H) \leq 2$, by Lemma 2, $\mu(H-e) \leq 4$. By Lemma 1 and Corollary 1, $\mu(G-e) \leq \max\{\mu(G), \mu(H-e)\} \leq \{\mu(G), 4\} \leq k-2$. Then, by Corollary 3, $G-e$ is $(k, k-1)$-edge-connected, contrary to the fact that G is minimally $(k, k-1)$-edge-connected. □

Let E^t denote an edgeless graph with order t. Let $E^t \vee H$ denote the join of E^t and H.

Corollary 4 *Let G be a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph; then $E^2 \vee K_2 \nsubseteq G$.*

Proof. Assume $H \cong E^2 \vee K_2 \subseteq G$, then H satisfies Lemma 4 (i) and (ii), a contradiction. □

Proposition 5 *Let G be a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph with $k \geq 4$; then $\mu(G) \geq 2$.*

Proof. Assume $\mu(G) = 1$, by Lemma 2, then for any $e \in E(G)$, $\mu(G-e) \leq 2$. However, by Corollary 1, $\mu(G-e) \geq k - |[e]| \geq 4 - 1 = 3$, a contradiction. □

Theorem 4 *Let G be a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph, $|G| = n$, $n \geq k+2$, and $k \geq 6$. Then $e(G) \leq 2n-k$.*

Proof. Since G is minimally $(k, k-1)$-edge-connected, by Proposition 3, there exists a $(k, k-1)$-edge-cut set S of G. Then $|S| = k$ and $\omega(G-S) = k-1$. Choose an ordered edge-cut-set decomposition $S_1, S_2, \ldots, S_t$ of $E(G-S)$, then $t = n - \omega(G-S) = n-k+1$ and $|S_{n-k+1}| = 1$. By Lemma 3, for all $i = 1, 2, \ldots, n-k$, $|S_i| \leq 2$. So $e(G) = |S| + \sum_{i=1}^{n-k+1} |S_i| \leq k + (n-k) \times 2 + 1 = 2n-k+1$. Assume $e(G) = 2n-k+1$, then for all $i = 1, 2, \ldots, n-k$, $|S_i| = 2$ and $|S_{n-k+1}| = 1$. Hence all graphs $G-S, G-S-\cup_{i=1}^{m} S_i$, where $m = 1, 2, \ldots, n-k$, have only a nontrivial component. Let H_0 and H_m denote the nontrivial component of $G-S$ and $G-S-\cup_{i=1}^{m} S_i$ respectively. Since $n \geq k+2$, $|S_{n-k-1}| = |S_{n-k}| = 2$ and $|S_{n-k+1}| = 1$, $H_{n-k-2} \cong E^2 \vee K_2$. Then, by $H_{n-k-2} \subseteq G$ and Corollary 4, a contradiction. □

Let n and k be two positive integers with $n \geq k+4$ and $k \geq 6$. Let $G_{1,n}$, $G_{2,n}$ or $G_{3,n}$ denote the union of a complete bipartite graph $K_{n-k+1,2}$ with bipartition

$(\{u_1, u_2, \ldots, u_{n-k+1}\}, \{v_1, v_2\})$ and a path $v_2y_1y_2\ldots y_{k-3}v_1, u_1y_1y_2\ldots y_{k-3}u_{n-k+1}$ or $u_1y_1y_2\ldots y_{k-3}v_2$ of length $k-2$ respectively (see Figure 1).

Theorem 5 *Let G be a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph, $|G| = n$, $n \geq k+4$ and $k \geq 6$. Then $e(G) = 2n-k$ if and only if $G \cong G_{1,n}, G_{2,n}$, or $G_{3,n}$.*

Proof. Let G be a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph with $e(G) = 2n-k$. (The existence of G can be seen in Figure 1.)

Let S be a $(k, k-1)$-edge-cut set of G. Let $S_1, S_2, \ldots, S_t$ be an ordered edge-cut-set decomposition of $E(G-S)$. Then $t = n-k+1$, $|S_{n-k+1}| = 1$ and, by Lemma 3, for any $i \in \{1, 2, \ldots, n-k\}$, $|S_i| \leq 2$.

Firstly, we show that $G-S$ has only one nontrivial component. Assume that $G-S$ has more than one nontrivial components. There are two cases.

Case 1 Assume $G-S$ has at least three nontrivial components. Then there exist at least two edge sets S_i, S_j such that $|S_i| = |S_j| = 1$ and $i, j \neq n-k+1$. So $e(G) = |S| + \sum_{i=1}^{n-k+1} |S_i| \leq k + 3 \times 1 + (n-k-2) \times 2 = 2n-k-1$, contrary to $e(G) = 2n-k$.

Case 2 Assume $G-S$ has exactly two nontrivial components. Then one of these two nontrivial components, denoted by G_1, satisfies $|V(G_1)| \geq [n-(k-3)]/2 \geq [k+4-(k-3)]/2 > 3$. Then there exists some $i_0 \neq n-k+1$ with $|S_{i_0}| = 1$ and for all $j \neq i_0, n-k+1$, $|S_j| = 2$. There must exist $1 \leq i_1 < i_2 < \ldots < i_l \leq n-k+1$ such that $\cup_{j=1}^{l} S_{i_j} = E(G_1)$. Then $S_{i_1}, S_{i_2}, \ldots, S_{i_l}$ is an ordered edge-cut-set decomposition of $E(G_1)$, $l = |V(G_1)| - \omega(G_1) = |V(G_1)| - 1 \geq 4-1 = 3$ and for each $m \in \{1, 2, \ldots, l-1\}$, $G_1 - \cup_{j=1}^{m} S_{i_j}$ has only one nontrivial component. Since $|S_{i_{l-2}}| = |S_{i_{l-1}}| = 2$ and $|S_{i_l}| = 1$, the nontrvial component of $G_1 - \cup_{j=1}^{l-3} S_{i_j}$ is isomorphic to $E^2 \vee K_2$, by Corollary 4, a contradiction.

Secondly, we show that $|S_1| = 2$. Assume $|S_1| = 1$, then for each $i \in \{2, 3, \ldots, n-k\}$, $|S_i| = 2$. Hence, for each $m \in \{1, 2, \ldots, n-k\}$, $G-S-\cup_{i=1}^{m} S_i$ has only one nontrivial component. Thus the nontrvial component of $G-S-\cup_{i=1}^{n-k-2} S_i$ is isomorphic to $E^2 \vee K_2$, by Corollary 4, a contradiction.

Thirdly, let H denote the nontrivial component of $G-S$, we show that $H \cong K_{n-k,2}$.

Claim: For any $e \in E(H)$, $|[e]_H| = 2$.

Assume there exists some edge $e \in E(H)$, $|[e]_H| \geq 3$. By $\lambda(G-S) = |S_1| = 2$ and Proposition 2$'$, we can choose an ordered edge-cut-set decomposition $T_1, T_2, \ldots, T_{n-k+1}$ of $E(H)$ with $T_1 \subseteq [e]_H$, $|T_1| = 2$ and $T_2 \subseteq [e]_H - T_1$, $|T_2| = \lambda(G-S-T_1) = 1$. By $e(G) = |S| + \sum_{i=1}^{n-k+1} |T_i| = 2n-k$, for all $i \in \{3, 4, \ldots, n-k\}$, $|T_i| = 2$. So the nontrivial component of $G-S-\cup_{i=1}^{n-k-2} T_i$ is isomorphic to $E^2 \vee K_2$, by Corollary 4, a contradiction.

Assume there exists an edge $f \in E(H)$, $|[f]_H| = 1$. Since $\lambda(G-S) = 2$, H is 2-edge-connected and f is not cut edge of H. Then, by $|[f]_H| = 1$, $H-f$ is still 2-edge-connected. And $\mu(H) = \max\{|[e]_H| : e \in E(H)\} \leq 2$. So $H \subseteq G$ satisfies both Lemma 4 (i) and Lemma 4 (ii), a contradiction. Thus the claim must hold.

Then H is minimally 2-edge-connected. By Proposition 1, $e(H) \leq 2|V(H)|-4 = 2(n-(k-2))-4 = 2n-2k$. So $e(G) = |S| + e(H) \leq k + 2n - 2k = 2n - k$. By $e(G) = 2n-k$, $e(H) = 2n-2k$. Since $|V(H)| = n-(k-2) \geq 6$, by Theorem 1, $H \cong K_{n-k,2}$.

Fourthly, we show $\mu(G) = k-2$. Let H still denote the nontrivial component of $G-S$. Then $\cong K_{n-k,2}$. Choose some $e \in E(H)$, by $|[e]_H| = 2$ and Proposition 4, $|[e]_G| \leq |[e]_H| = 2$. There are two cases.

Case 1 $|[e]_G| = 2$, then $[e]_G = [e]_H$. For any $f \in S = E(G-e) - E(H-e)$, since f is not equivalent to e in G, $[f]_{G-e} \supseteq [f]_G$. Similar to the proof of Proposition 4, one can show $[f]_{G-e} \cap E(H-e) = \emptyset$. Then $[f]_{G-e} \subseteq [f]_G$. Thus $[f]_{G-e} = [f]_G$

Claim: for any $h \in E(H-e)$, $|[h]_{G-e}| \leq 2$.

If $h \in B(H-e)$, by $|[e]_G| = 2$, then $\{h\} = B(G-e)$. Thus $|[h]_{G-e}| = 1 < 2$. If $h \notin B(H-e)$, by Proposition 4, $|[h]_{G-e}| \leq |[h]_{H-e}| \leq 2$. So the claim must hold.

By Proposition 5 and Corollary 1, $\mu(G-e) = \max\{|[f]_{G-e}| : f \in E(G-e)\} = \max\{\max\{|[f]_{G-e}| : f \in S\}, \max\{|[h]_{G-e}| : h \in E(H-e)\}\} \leq \max\{\max\{|[f]_G| : f \in S\}, 2\} \leq \mu(G) \leq k-2$. By Corollary 1, $\mu(G-e) \geq k - |[e]_G| = k-2$, so $\mu(G) = \mu(G-e) = k-2$.

Case 2 $|[e]_G| = 1$. By $H \cong K_{n-k,2}$, let $\{g\} = B(H-e)$. For any $f \in S$, by Proposition 4, $[f]_{G-e} \subseteq [f]_G \cup \{g\}$. Then $|[f]_{G-e}| \leq |[f]_G| + 1$. By Corollary 1, $\mu(G-e) = \max\{|[f]_{G-e}| : f \in E(G-e)\} = \max\{\max\{|[f]_{G-e}| : f \in S\}, \max\{|[f]_{G-e}| : f \in E(H-e)\}\} \leq \max\{\max\{|[f]_G| + 1 : f \in S\}, 2\} \leq \mu(G) + 1 \leq k-1$. And, by Corollary 1, $\mu(G-e) \geq k - |[e]_G| = k-1$. So $\mu(G) = k-2$.

Lastly, let $[i]_G$ be a maximum equivalence class of G, then $|[i]_G| = \mu(G) = k-2$. Similar to the proof of that $G-S$ has only one nontrivial component, we can show that $G - [i]_G$ has only one nontrivial component. And similar to the proof of that $H \cong K_{n-k,2}$, one can prove that the nontrivial component of $G - [i]_G$ is isomorphic to $K_{n-k+1,2}$. So $G \cong G_{1,n}, G_{2,n}$ or $G_{3,n}$. □

Theorem 6 *Let G be a connected and minimally $(k, k-1)$-edge-connected graph, $|G| \geq k-1$, $1 \leq |B(G)| \leq k-4$, $k \geq 5$. Then $e(G) \leq 2n-k+1$.*

Proof. The proof is similar to that of Theorem 4. □

Theorem 7 *Let G be a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph, $|G| = n$, $k \geq 4$. Then each of the following holds.*

(i) If $k-1 \leq n \leq 3k-7$, then $e(G) \geq n+1$.

(ii) If $(m-1)(3k-7) < n \leq m(3k-7)$ for some integer $m \geq 2$, then $e(G) \geq n+m$.

Proof. Assume $k-1 \leq n \leq 3k-7$. Since G is 2-edge-connected, $e(G) = (\sum_{v \in V(G)} d(v))/2 \geq 2n/2 = n$. Assume $e(G) = n$, then G must be a cycle. Thus $\mu(G) = n \geq k-1$. However, by Corollary 1, $\mu(G) \leq k-2$, a contradiction. So $e(G) \geq n+1$.

Assume $(m-1)(3k-7) < n \leq m(3k-7)$ for some integer $m \geq 2$.

Let G_i denote a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph which satisfies that $e(G_i) - |V(G_i)| = i$ and $|V(G_i)|$ reachs maximum, where $i = 1, 2, \ldots$. (For their existence, see Q_i following Theorem 7.)

Let us first study G_i.

Claim 1 G_i has no cut vertex.

Assume there exists a cut vertex v of G_i. Then $G_i - v$ has at least two components C_1, C_2. Since G is 2-edge-connected, there are some $u_1, u_2 \in V(C_1)$ and $v_1, v_2 \in V(C_2)$ with $\{u_1, u_2, v_1, v_2\} \subseteq N(v)$. Let G'_i denote the graph obtained from G_i by splitting v into two vertices v', v'', and connecting u_1, v_1 with v' and the others in $N(v)$ with v'' and joining v', v'' by a path $v'y_1y_2 \ldots y_{k-3}v''$ of length $k-2$. By Proposition 5 and Corollary 1, it is not difficult to show that G'_i is also a 2-edge-connected and minimally $(k, k-1)$-edge-connected graph. However, $e(G'_i) - |V(G'_i)| = (e(G_i) + k - 2) - (|V(G_i)| + k - 2) = i$, and $|V(G'_i)| = |V(G_i)| + k - 2 > |V(G_i)|$, contrary to the choice of G_i.

Claim 2 For any $v \in V(G_i)$, $d(v) \leq 3$.

Assume there exists a vertex $v \in V(G_i)$ with $d(v) \geq 4$. Let $\{u_1, u_2, u_3, u_4\} \subseteq N(v)$.

Case 1 $G_i - v$ is 2-edge-connected. Then let G'_i denote the graph obtained from G_i by splitting v into two vertices v', v'', and connecting u_1, u_2 with v' and the others in $N(v)$ with v'' and joining v', v'' by a path $v'y_1y_2 \ldots y_{k-3}v''$ of length $k-2$.

Case 2 $B(G_i - v) \neq \emptyset$. By Proposition 2, $(G_i - v)'$ is a tree with edge set $B(G_i - v)$. Let $C_1, C_2, \ldots, C_t (t \geq 2)$ denote all components of $G_i - v - B(G_i - v)$ and $v_j \in (G_i - v)'$ denote the vertex obtained from C_j in the course of transforming $G_i - v$ into $(G_i - v)'$, where $j = 1, 2, \ldots, t$. Let $F = \{u : u \in V((G_i - v)') \text{ and } d(u) = 1\}$.

Case 2A $|F| = 2$. Without loss of generality, assume $F = \{v_1, v_2\}$ and $u_1 \in C_1$, $u_2 \in C_2$ (because G_i is 2-edge-connected).

Case 2B$|F| \geq 4$. Without loss of generality, assume $\{v_1, v_2, v_3, v_4\} \subseteq F$, $u_j \in C_j$,where $j = 1, 2, 3, 4$, and there exists exactly one vertex with degree more than two in (v_1, v_3)-path in $(G_i - v)'$.

Case 2C $|F| = 3$. Then there exists just one vertex with degree three in $(G_i - v)'$. Without loss of generality, assume $F = \{v_1, v_2, v_3\}$ and $v_4 \in (G_i - v)'$, $d(v_4) = 3$.

Case 2C1 For some $j \in \{1, 2, 3\}$, $|V(C_j) \cap N(v)| \geq 2$. Without loss of generality, assume $|V(C_1) \cap N(v)| \geq 2$ and $u_1, u_4 \in C_1$, $u_2 \in C_2$, $u_3 \in C_3$.

Case 2C2 For any $j \in \{1, 2, 3\}$, $|V(C_j) \cap N(v)| = 1$ and $|V(C_4) \cap N(v)| \geq 1$. Without loss of generality, assume $u_j \in C_j$, where $j = 1, 2, 3, 4$.

Case 2C3 For any $j \in \{1, 2, 3\}$, $|V(C_j) \cap N(v)| = 1$ and $|V(C_4) \cap N(v)| = 0$. Since $d(v) \geq 4$, for some $j_0 \in \{5, 6, \ldots, t\}$, $V(C_{j_0}) \cap N(v) \neq \emptyset$. Without loss of generality, assume $u_j \in C_j$, where $j = 1, 2, 3$, $u_4 \in C_{j_0}$ and there is no internal vertex with degree more than 2 in (v_1, v_{j_0})-path in $(G_i - v)'$.

For all subcases in case 2, similar to case 1, let G'_i denote the graph obtained from G_i by splitting v into two vertices v', v'', and connecting u_1, u_2 with v' and the others in $N(v)$ with v'' and joining v', v'' by a path $v'y_1y_2 \ldots y_{k-3}v''$ of length $k-2$.

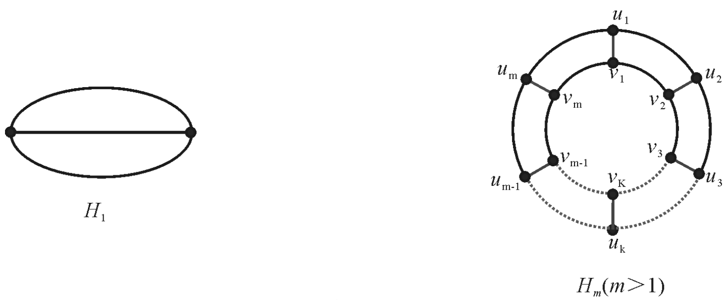

Figure 2

Obviously, G_i' is 2-edge-connected. Moreover, by Proposition 4, Propsition 5 and Corollary 1, G_i' is minimally $(k, k-1)$-edge-connected. However, $e(G_i') - |V(G_i')| = e(G_i) + k - 2 - (|V(G_i)| + k - 2) = i$ and $|V(G_i')| = |V(G_i)| + k - 2 > |V(G_i)|$, contrary to the choice of G_i.

Claim 3 $|V(G_i)| \leq i(3k-7)$.

Since G_i is 2-edge-connected, by Claim 2, $d(v) = 2$ or 3. Let $S = \{v : v \in V(G_i)$ and $d(v) = 3\}$; then $|S| = \sum_{v \in V(G_i)} d(v) - 2|V(G_i)| = 2e(G_i) - 2|V(G_i)| = 2i$. Let $T = \{(u, v) \in E(G_i) : u \in S$ or $v \in S\}$; then each of the following holds.

(i) For any $e \in E(G_i) - T$, there exists an edge $f \in T$ such that e is connected with f in G_i by some path which has no internal vertex in S. So for any $e \in E(G_i) - T$, there exists an edge $f \in T$ such that e is equivalent to f in G_i.

(ii) For any $e = (u, v) \in T$, if $\{u, v\} \not\subseteq S$, then there exists an edge $f(\neq e) \in T$ such that e is connected with f in G_i by some path which has no internal vertex in S. Thus for any $e = (u, v) \in T$, if $\{u, v\} \not\subseteq S$, then there exists an edge $f(\neq e) \in T$ such that f is equivalent to e in G_i.

Since $|S| = 2i$, $|T| \leq 3 \times 2i = 6i$. By (i) and (ii), there are no more than $6i/2 = 3i$ equivalence classes in G_i. By Corollary 1, the number of edges in each equivalence class of $E(G_i)$ is no more than $k-2$. Thus $|V(G_i)| \leq 3i \times (k-3) + 2i = i(3k-7)$.

When $(m-1)(3k-7) < n \leq m(3k-7)$, for all $i \in \{1, 2, \ldots, m-1\}$, $|V(G_i)| \leq i \times (3k-7) \leq (m-1) \times (3k-7) < n$. By the choice of G_i, $e(G) - |V(G)| > m-1$. Then $e(G) \geq n + m$. $\square$

For any integer $m(\geq 2)$, let H_m denote the graph obtained from two independent cycles $u_1u_2 \ldots u_mu_1$ and $v_1v_2 \ldots v_mv_1$ by adding n edges $u_1v_1, u_2v_2, \ldots, u_mv_m$ (see Figure 2). Let Q_m denote the graph obtained from H_m by replacing every edge in H_m with a path of length $k-2$. Obviously, Q_m is 2-edge-connected, and by Corollary 1, Q_m is minimally $(k, k-1)$-edge-connected. Since $|V(Q_m)| = m(3k-7)$ and $e(Q_m) = 3m(k-2) = |V(Q_m)| + m$, the result of Theorem 7 is the best possible.

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