Cycles through a given arc in almost regular multipartite tournaments

Lutz Volkmann Stefan Winzen

Lehrstuhl II für Mathematik
RWTH Aachen
Germany
{volkm,winzen}@math2.rwth-aachen.de

Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. In 1998, Y. Guo showed, if every arc of a regular c-partite tournament is contained in a directed cycle of length 3, then every arc belongs to a directed cycle of length n for each $n \in \{4, 5, \ldots, c\}$. Recently, L. Volkmann generalized this result for $c \geq 6$. He showed, if $V_1, V_2, \ldots, V_c$ are the partite sets of an almost regular c-partite tournament with $c \geq 6$ and $|V_1| = |V_2| = \ldots = |V_c| \geq 2$, then every arc of D is contained in a directed cycle of length n for each $n \in \{4, 5, \ldots, c\}$. In this paper we shall extend this theorem to all almost regular c-partite tournaments with $c \geq 7$ such that there are at least two vertices in each partite set. Examples will show that this result is not valid for the case that $c = 6$ or that $c = 7$ and there is only one vertex in at least one partite set.

1 Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say that x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N^-_D(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is an arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d^+_D(x) = d^+(x) = |N^+(x)|$ and $d^-_D(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigrah induced by X. If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n-cycle*. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

There are several measures of how much a digraph differs from being regular. In [14], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x\in V(D)}\{d^+(x), d^-(x)\} - \min_{y\in V(D)}\{d^+(y), d^-(y)\}.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \le 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \le |V_2| \le \ldots \le |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D, and we define $\gamma(D) = |V_1|$.

It is very easy to see that every arc of a regular tournament belongs to a 3-cycle. The next example shows that this is not valid for regular multipartite tournaments in general.

Example 1.1 *Let C, C', and C'' be three induced cycles of length 4 such that $C \to C' \to C'' \to C$. The resulting 6-partite tournament D_1 is 5-regular, but no arc of the three cycles C, C', C'' is contained in a 3-cycle.*

Let H, H_1, and H_2 be three copies of D_1 such that $H \to H_1 \to H_2 \to H$. The resulting 18-partite tournament is 17-regular, but no arc of the cycles corresponding to the cycles C, C', and C'' is contained in a 3-cycle.

If we continue this process, we arrive at regular c-partite tournaments with arbitrary large c which contain arcs that do not belong to any 3-cycle.

In 1998, Guo [3] proved the following generalization of Alspach's classical result [1] that every regular tournament is arc pancyclic.

Theorem 1.2 (Guo [3]) *Let D be a regular c-partite tournament with $c \ge 3$. If every arc of D is contained in a 3-cycle, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

Now, the aim was to carry this result forward to almost regular multipartite tournaments. To reach this, Volkmann [10], [12] started with the following theorems.

Theorem 1.3 (Volkmann [12]) *Let D be an almost regular multipartite tournament with c partite sets.*

If $c \ge 8$, then every arc of D is contained in a 4-cycle.

If $c = 7$ and there are at least two vertices in every partite set, then every arc of D is contained in a 4-cycle.

Theorem 1.4 (Volkmann [10]) *Let D be an almost regular multipartite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. If $c \geq 6$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

The main theorem of this paper is the following extension and supplement of Theorems 1.3 and 1.4.

Theorem 1.5 *Let D be an almost regular c-partite tournament with at least two vertices in every partite set. If $c \geq 7$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

This result is also a supplement to a theorem of Jacobson [5], which states that in an almost regular tournament with $c \geq 7$ vertices, every arc is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$. An example will show that the main theorem is not valid for $c = 6$ in general. A further example will demonstrate that the condition that there are at least two vertices in every partite set is necessary, at least for $c = 7$, the most difficult case.

According to Tewes, Volkmann and Yeo [7], the following lemma holds.

Lemma 1.6 *If $V_1, V_2, \ldots, V_c$ are the partite sets of an almost regular c-partite tournament D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, then $|V_c| \leq |V_1| + 2$.*

Hence, using Theorem 1.3 as the basis of induction, we will distinguish between the two cases that $|V_c| = |V_1| + 1$ and $|V_c| = |V_1| + 2$ in the main theorem. Then Theorem 1.5 follows immediately from Theorem 1.4.

For more information on multipartite tournaments, see [2, 3, 4, 6, 11, 13].

2 Preliminary results

The following results play an important role in our investigations.

Lemma 2.1 (Tewes, Volkmann, Yeo [7]) *Let D be an almost regular multipartite tournament. Then for every vertex x of D we have*

$$\frac{|V(D)| - \alpha(D) - 1}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - \gamma(D) + 1}{2}.$$

If we know the cardinality of the partite set $V(x)$, then we can improve the previous lemma.

Lemma 2.2 *If D is an almost regular multipartite tournament and x a vertex of D with $|V(x)| = p$, then*

$$\frac{|V(D)| - p - 1}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - p + 1}{2}.$$

Proof. Firstly, suppose that $d^+(x) \leq \frac{|V(D)|-p-2}{2}$. The fact that $d^+(x) + d^-(x) = |V(D)| - |V(x)| = |V(D)| - p$ implies that $d^-(x) \geq \frac{|V(D)|-p+2}{2}$, which leads to $d^-(x) - d^+(x) \geq 2$, a contradiction to $i_g(D) \leq 1$.

Now suppose that $d^+(x) \geq \frac{|V(D)|-p+2}{2}$. Since $d^+(x) + d^-(x) = |V(D)| - |V(x)| = |V(D)| - p$, we obtain $d^-(x) \leq \frac{|V(D)|-p-2}{2}$, and thus, it follows that $d^+(x) - d^-(x) \geq 2$, a contradiction to $i_g(D) \leq 1$.

Consequently, we have $\frac{|V(D)|-p-1}{2} \leq d^+(x) \leq \frac{|V(D)|-p+1}{2}$. The results for $d^-(x)$ follow analogously. □

In this article we treat the case of an almost multipartite tournament D with $\alpha(D) = r + 1$ or $\alpha(D) = r + 2$ and $\gamma(D) = r$ for any $r \geq 2$. This leads to the following remark.

Remark 2.3 *If $\alpha(D) = r + 2$, $\gamma(D) = r$ and $i_g(D) \leq 1$, then $|V(D)| - r$ is even. So the bounds in Lemma 2.2 can be improved by*

$$d^+(x), d^-(x) = \frac{|V(D)| - r - 2}{2} \quad \textit{if} \quad |V(x)| = r + 2$$

or

$$d^+(x), d^-(x) = \frac{|V(D)| - r}{2} \quad \textit{if} \quad |V(x)| = r.$$

Consequently, for the case that $\alpha(D) = r + 2$, instead of Lemma 2.1, we can use the following result:

$$\frac{|V(D)| - r - 2}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - r}{2}.$$

Now let us summarize some results of Lemma 2.2 and Remark 2.3.

Corollary 2.4 *If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r + 2$, then for every vertex x of D we have*

$$\frac{|V(D)| - r - 2}{2} \leq d^+(x), d^-(x).$$

The next result is a well-known theorem of Turán [8] (see also [9], p. 212).

Theorem 2.5 *Let D be a digraph without 2-cycles. If the underlying graph of D has no clique of order $p + 1$, then*

$$|E(D)| \leq \frac{p-1}{2p}|V(D)|^2.$$

3 Main result

Theorem 3.1 *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $2 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r + 2$ and $|V_c| \geq r + 1$. If $c \geq 7$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

Proof. We prove the theorem by induction on n. For $n = 4$ the result follows from Theorem 1.3. Now let e be an arc of D and assume that e is contained in an n-cycle $C = a_n a_1 a_2 \ldots a_{n-1} a_n$ with $e = a_n a_1$ and $4 \le n < c$. Suppose that $e = a_n a_1$ is not contained in any $(n+1)$-cycle.

Obviously, $|V(D)| = cr + k$ with $1 \le k \le c-1$, if $|V_c| = r+1$ and $2 \le k \le 2c-2$, if $|V_c| = r+2$. Firstly, we observe that $N^+(v) - V(C) \ne \emptyset$ for each $v \in V(C) = \{a_1, a_2, \ldots, a_n\}$, because otherwise Corollary 2.4, the fact that $r \ge 2$ and $k \ge 1$ yield the contradiction

$$n = |V(C)| \ge d^+(v) + 2 \ge \frac{cr + k - r - 2}{2} + 2 = \frac{(c-1)r + k + 2}{2} > c.$$

Analogously, one can show that $N^-(v) - V(C) \ne \emptyset$ for each $v \in V(C)$.

Next let S be the set of vertices that belong to partite sets not represented on C and define

$$X = \{x \in S \mid C \to x\}, \qquad Y = \{y \in S \mid y \to C\}.$$

Assume that $X \ne \emptyset$ and let $x \in X$. If there is a vertex $w \in N^-(a_n) - V(C)$ such that $x \to w$, then $a_n a_1 a_2 \ldots a_{n-2} x w a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. If $(N^-(a_n) - V(C)) \to x$, then $|N^-(x)| \ge |N^-(a_n) - V(C)| + |V(C)| \ge |N^-(a_n)| + 2$, a contradiction to the hypothesis that $i_g(D) \le 1$. If there exists a vertex $b \in (N^-(a_n) - V(C))$ such that $V(b) = V(x)$, then b is adjacent to all vertices of C. In the case that $N^-(b) \cap V(C) \ne \emptyset$, let $l = \max_{1 \le i \le n-1}\{i | a_i \to b\}$. Then $a_n a_1 \ldots a_l b a_{l+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. It remains to consider the case that $N^-(b) \cap V(C) = \emptyset$. If there is a vertex $u \in (N^-(b) - V(C)) = N^-(b)$ such that $x \to u$, then $a_n a_1 a_2 \ldots a_{n-3} x u b a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Otherwise, $N^-(b) \to x$, and we arrive at the contradiction $d^-(x) \ge d^-(b) + |V(C)|$. Altogether, we have seen that $X \ne \emptyset$ is not possible, and analogously we find that $Y \ne \emptyset$ is impossible. Consequently, from now on we shall assume that $X = Y = \emptyset$.

By the definition of S, every vertex of $V(C)$ is adjacent to every vertex of S, and from our assumption $n < c$, we deduce that $S \ne \emptyset$. Now we distinguish different cases.

Case 1. There exists a vertex $v \in S$ with $v \to a_n$. Since $Y = \emptyset$, there is a vertex $a_i \in V(C)$ such that $a_i \to v$. If $l = \max_{1 \le i \le n-1}\{i | a_i \to v\}$, then $a_n a_1 \ldots a_l v a_{l+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $a_n \to S$.

Case 2. There exists a vertex $v \in S$ with $a_1 \to v$. Since $X = \emptyset$, there is a vertex $a_i \in V(C)$ such that $v \to a_i$. If $l = \min_{2 \le i \le n-1}\{i | v \to a_i\}$, then $a_n a_1 \ldots a_{l-1} v a_l \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $S \to a_1$.

If $C = a_n a_1 a_2 \ldots a_n$ and $v \in S$, then the following three sets play an important role in our investigations

$$H = N^+(a_1) - V(C), \qquad F = N^-(a_n) - V(C), \qquad Q = N^-(v) - V(C).$$

Case 3. There exists a vertex $v \in S$ such that $v \to a_{n-1}$. If there is a vertex $a_i \in V(C)$ with $2 \le i \le n-2$ such that $a_i \to v$, then we obtain as above an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that

$v \to \{a_1, a_2, \ldots, a_{n-1}\}$. Because of $S \to a_1$, we note that every vertex of $N^+(a_1)$ is adjacent to v. If there is a vertex $x \in H$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore we assume now that $v \to (N^+(a_1) - V(C))$. This leads to $d^+(v) \geq d^+(a_1) + 1$, and thus, because of $i_g(D) \leq 1$, it follows that $N^+(v) = N^+(a_1) \cup \{a_1\}$ and $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$.

It is a simple matter to verify that $H \cap Q = \emptyset$, $S \cap H = \emptyset$ and $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$.

If there is an arc xa_2 with $x \in H$, then $a_n a_1 x a_2 a_3 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction.

Subcase 3.1. Firstly, let H consist of vertices of only one partite set V_z. At least one vertex of V_z belongs to $V(C)$, that means $|H| \leq r+1$, if $|V_z| = r+2$, $|H| \leq r$, if $|V_z| = r+1$ and $|H| \leq r-1$, if $|V_z| = r$.

Because of Corollary 2.4 and $n \leq c-1$, we have

$$\frac{cr + k - r - 2}{2} - (c-3) \leq d^+(a_1) - (n-2) = |H|. \tag{1}$$

If $|V_z| = r$, then because of $|H| \leq r-1$, (1) yields $(c-3)r + k + 6 \leq 2c$. Since $r \geq 2$ and $k \geq 1$, this leads to the contradiction $2c + 1 \leq 2c$.

If $n = 4$, then we observe that $n \leq c-3$, and this implies

$$\frac{cr + k - r - 2}{2} - (c-5) \leq d^+(a_1) - (n-2) = |H| \leq r+1.$$

This leads again to $(c-3)r + k + 6 \leq 2c$, a contradiction. Consequently, it remains to treat the cases with $|V_z| \geq r+1$ and $n \geq 5$.

Subcase 3.1.1. Assume that $|V_c| = r+1$ and $|V_z| = r+1$. If $|V(a_1)| = r+1$ (and therefore $k \geq 2$), then (1) leads to $r = 2$, $|H| = r = 2$ and $k = 2$.

If $|V(a_1)| = r$, then together with Lemma 2.2 and $n \leq c-1$, we arrive at

$$\frac{cr + k - r - 1}{2} - (c-3) \leq d^+(a_1) - (n-2) = |H| \leq r,$$

and hence $(c-3)r + k + 5 \leq 2c$. This leads to no contradiction, only if $r = 2$, $|H| = r = 2$ and $k = 1$.

Consequently, it remains to consider the case that $|H| = r = 2$ and $k = 1$ or $k = 2$ and $|V(a_1)| = r+1$. Therefore, we observe that $|V(v)| = r$.

Since $n \geq 5$, we have $Q \leadsto H$, because otherwise, if there are vertices $q \in Q$ and $h \in H$ such that $h \to q$, then $a_n a_1 h q v a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Thus, for every vertex $h \in H$, we conclude that $d^+(h) \leq r - 1 + n - 2 = n - 1$. Since $d^+(v) = d^+(a_1) + 1 = r + n - 1 = n + 1$, this is a contradiction to $i_g(D) \leq 1$.

Subcase 3.1.2. Now let $|V_c| = r+2$. If $|V_z| = r+1$, then, because of $|H| \leq r$, (1) leads to $(c-3)r + k + 4 \leq 2c$. Since in this case $k \geq 3$ and $r \geq 2$, this yields the contradiction $2c + 1 \leq 2c$.

Finally, let $|V_z| = r+2$. Then (1) leads to the contradiction $c \leq 5$, if $r \geq 3$, and to the contradiction $1 \leq 0$, if $r = 2$ and $k \geq 5$. Therefore, let $r = 2$ and $k \in \{2, 3, 4\}$. Since $cr + k - r$ is even, the case $k = 3$ is not possible.

Furthermore, we have a contradiction in (1), if $|H| \leq r$. Therefore, let $|H| = r+1$. Since $d^+(v) = d^+(a_1)+1$, we conclude that $|V(v)| \leq r+1$. Because of $n \geq 5$, analogously as in Subcase 3.1.1, we see that $(Q \cup \{a_1, a_2, v\}) \rightsquigarrow H$, and thus $d^+(h) \leq r+n-2 = n$, if $h \in H$. On the other hand, we have seen that $d^+(v) = d^+(a_1)+1 = r+1+n-1 = n+2$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2. Let $n \geq 5$ and let H consist of more than one partite set. Then there is at least one arc $pq \in E(D[H])$. Let L be the set of all vertices in H with an inner neighbor in H, and $M = H - L$. Then we note that $L \neq \emptyset$. M consists of vertices of at most one partite set and $M \rightsquigarrow L$. If we take a vertex $q \in L$ with an inner neighbor $p \in H$, then it cannot be that $qa_3 \in E(D)$, because otherwise $a_n a_1 p q a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Therefore let $a_3 \rightsquigarrow L$. If there is an arc xy with $x \in H$ and $y \in Q$, then $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Altogether, we have seen that $(Q \cup M \cup \{a_1, a_2, a_3\}) \rightsquigarrow L$.

First, let $|V(v)| = r+2$. Then, because of $d^+(v) \geq d^+(a_1)+1$, Remark 2.3 yields the contradiction

$$\frac{cr+k-r-2}{2} + 1 \leq d^+(a_1)+1 \leq d^+(v) = \frac{cr+k-r-2}{2}.$$

Now let $|V(v)| \leq r+1$. Since $|R| = 0$, for every vertex $q \in L$, we conclude that $d(q, V(D)-L) \leq n+r-3$, and thus, it follows with Corollary 2.4 that $d^+_{D[L]}(q) = d^+(q) - d(q, V(D)-L) \geq \frac{cr+k-r-2}{2} - r - n + 3$. This implies

$$\begin{aligned} \frac{|L|(|L|-1)}{2} &\geq |E(D[L])| \\ &= \sum_{q \in L} d^+_{D[L]}(q) \;\geq\; |L| \left\{ \frac{cr+k-r-2}{2} - r - n + 3 \right\}. \end{aligned} \tag{2}$$

The conditions $d^+(v) \geq d^+(a_1)+1$, $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$, and Lemma 2.1 (respectively, Remark 2.3, if $|V_c| = r+2$) yield $|L| = |H| - |M| = d^+(a_1) - n + 2 - |M| \leq d^+(v) - 1 - n + 2 - |M| \leq \frac{cr+k-r+1}{2} - n + 1 - |M|$ (respectively, $|L| \leq \frac{cr+k-r}{2} - n + 1 - |M|$, if $|V_c| = r+2$). Combining this with inequality (2), we obtain

$$\frac{cr+k-r+1}{2} - n - |M| \geq |L| - 1 \geq 2\left\{ \frac{cr+k-r-2}{2} - r - n + 3 \right\},$$

if $|V_c| = r+1$ and

$$\frac{cr+k-r}{2} - n - |M| \geq |L| - 1 \geq 2\left\{ \frac{cr+k-r-2}{2} - r - n + 3 \right\},$$

if $|V_c| = r+2$. This leads to $2n \geq (c-5)r + k + 7 + 2|M|$ (respectively, $2n \geq (c-5)r + k + 8 + 2|M|$, if $|V_c| = r+2$). Because of $k \geq 1$, $r \geq 2$ and $n \leq c-1$, this is a contradiction, if $|M| \geq 1$ (a contradiction, if $|V_c| = r+2$).

Consequently, it remains to consider the case that $|M| = 0$. This means that every vertex in $H = L$ has an inner neighbor in H. Therefore, $|L| = |H| \geq 3$, and every vertex in H is the last point of a path of length 2. If $a_4 \rightsquigarrow H$, then, because of $d(q, V(D) - L) \leq r + n - 4$, we obtain a contradiction as above. Thus, let $q_3 a_4 \in E(D)$ with $q_3 \in H$, and let q_3 be the last point of the path $q_1 q_2 q_3$ in H, then $a_n a_1 q_1 q_2 q_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction.

Subcase 3.3. Finally, let $n = 4$ and let H consist of more than one partite set. Let us define the set G by $G = N^+(a_3) - V(C)$. If there is a vertex $w \in F \cap G$, then $a_4 a_1 a_2 a_3 w a_4$ is a 5-cycle through $a_4 a_1$, a contradiction. If there is an arc xy with $x \in G$ and $y \in F$, then $a_4 a_1 a_3 x y a_4$ is a 5-cycle, a contradiction. Consequently, it remains to consider the case that $F \cap G = \emptyset$ and $F \rightsquigarrow (G \cup \{a_3, a_4\})$.

According to Corollary 2.4, we have

$$|G| = |N^+(a_3)| - 1 \geq \frac{cr + k - r - 2}{2} - 1 = \frac{cr + k - r - 4}{2},$$

and thus, it follows for every vertex $x \in F$ that

$$\begin{aligned} d(V(D) - F, x) &\leq cr + k - |F| - |G| - 2 \\ &\leq \frac{cr + k + r + 4}{2} - |F| - 2 = \frac{cr + k + r}{2} - |F|. \end{aligned}$$

This leads to

$$d^-_{D[F]}(x) \geq \frac{cr + k - r - 2}{2} - \frac{cr + k + r}{2} + |F| = |F| - r - 1$$

for every $x \in F$. Hence, we conclude on the one hand that

$$|E(D[F])| = \sum_{x \in F} d^-_{D[F]}(x) \geq |F|(|F| - r - 1).$$

On the other hand, since $S \cap F = \emptyset$, the subdigraph $D[F]$ is 3-partite, and thus, Theorem 2.5 yields

$$|E(D[F])| \leq \frac{1}{3}|F|^2.$$

The last two inequalities imply $r \geq \frac{2}{3}|F| - 1$. Since $|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 2$, we deduce from Corollary 2.4 that

$$\begin{aligned} r &\geq \frac{2|F|}{3} - 1 \geq \frac{cr + k - r - 6}{3} - 1 = \frac{cr + k - r - 9}{3} \\ \Leftrightarrow 3r &\geq (c-1)r + k - 9. \end{aligned} \tag{3}$$

Subcase 3.3.1. Let $|V_c| = r + 1$. Then, (3) leads to no contradiction, only if $c = 8$, $r = 2$ and $k = 1$ or if $c = 7$, $r = 2$ and $k \leq 3$.

Firstly, let $c = 8$, $r = 2$ and $k = 1$. Then we note that $|H| \leq 4$, and thus, it follows that

$$9 \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 = |H| + 3 \leq 7,$$

a contradiction.

Therefore, it remains to consider the case that $c = 7$, $r = 2$ and $k \leq 3$. If $D[V(C)]$ is no tournament (that means that $V(a_2) = V(a_4)$), then we have $|S| \geq 4r = 8$ and $|H| \leq 3$, and therefore we arrive at the contradiction

$$9 \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 = |H| + 3 \leq 6.$$

Consequently, we investigate the case that $D[V(C)]$ is a tournament. Then we see that

$$7 \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 = |H| + 3,$$

and this yields $|H| \geq 4$. If $|H| = 4$, then we have equality in the last inequality chain, which implies $H \rightsquigarrow a_4$ and $a_2 \to a_4$. Let $x \in N^+(h) - V(C)$ with $h \in H$ such that $x \to a_2$, then $a_4a_1hxa_2a_4$ is a 5-cycle, a contradiction. Consequently, $a_2 \rightsquigarrow N^+(h) - V(C)$ for every vertex $h \in H$. If every element of H has an outer neighbor in H, then there exists a 3-cycle or a 4-cycle in H. Now, we take a vertex $h_3 \in H - V(a_4)$ such that h_3 is contained in a cycle $h_3h_1h_2h_3$ or $h_4h_1h_2h_3h_4$ in H. This leads to the 5-cycle $a_4a_1h_1h_2h_3a_4$, a contradiction. Hence, there exists a vertex $h_0 \in H$ such that $N^+_{D[H]}(h_0) = \emptyset$. Since $a_2 \rightsquigarrow H$, $a_2 \to \{a_3, a_4\}$ and $N^+(h_0) \cap V(C) \subseteq \{a_3, a_4\}$, it follows that

$$\begin{aligned} d^+(a_2) &\geq |H| + 2 + |N^+(h_0) - V(C)| - |V(a_2) - \{a_2\}| \\ &\geq 4 + |N^+(h_0) - V(C)| \geq d^+(h_0) + 2, \end{aligned}$$

a contradiction to $i_g(D) \leq 1$.

Therefore, let $5 \leq |H| \leq 6$. Then H contains vertices of exactly three partite sets and $k \geq 2$. In the case that $|H| = 5$ (respectively, $|H| = 6$), the vertex a_4 has at most one (respectively, two, if $|H| = 6$) further outer neighbors except S and a_1. If $a_2 \to a_4$, then $H_1 = H - N^+(a_4)$ consists of at least four elements and $H_1 \rightsquigarrow a_4$. Then, analogously to the case $|H| = 4$, we arrive at a contradiction.

Consequently, let $a_4 \to a_2$. Then, because $|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 1$, we get instead of (3) the better bound $r \geq \frac{cr+k-r-7}{3}$. Since $c = 7$, this yields $7 \geq 3r + k$, a contradiction to $k \geq 2$.

Subcase 3.3.2. Now let $|V_c| = r + 2$. Then (3) leads to no contradiction, only if $c = 7$, $r = 2$ and $2 \leq k \leq 3$. Since, with respect to Remark 2.3, $k = 3$ is impossible, it remains to treat the cases when $|V(a_3)| = r$ or $|V(a_4)| = r$.

If $|V(a_3)| = r$, then we obtain with Remark 2.3 that

$$|G| = |N^+(a_3)| - 1 = \frac{cr + k - r}{2} - 1 = \frac{cr + k - r - 2}{2}.$$

Following the same lines as above, we arrive at the inequality $(c-4)r + k \leq 6$ which leads to the contradiction $c \leq 6$.

If $|V(a_4)| = r$, then, according to Remark 2.3, we obtain the estimation

$$|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 2 \geq \frac{cr + k - r}{2} - 2 = \frac{cr + k - r - 4}{2}.$$

In this case, following the same way as above, we get the inequality $(c-4)r + k \leq 7$, which leads to the contradiction $c \leq 13/2$.

Summarizing the investigations of Case 3, we see that it remains to consider the case that $a_{n-1} \to S$.

Case 4. There exists a vertex $v \in S$ such that $a_2 \to v$. If we consider the converse of D, then, analogously to Case 3, it remains to treat the case that $S \to a_2$.

Summarizing the investigations in the Cases $1-4$, we can assume in the following, usually without saying so, that

$$\{a_{n-1}, a_n\} \to S \to \{a_1, a_2\} \rightsquigarrow H. \tag{4}$$

Case 5. Let $n = 4$. Because of (4), we have $a_4 \to S$ and thus $S \cup \{a_1\} \subseteq N^+(a_4)$. If $D[V(C)]$ is 3-partite or 2-partite, then, in the case that $|V_c| = r+1$, we see that

$$1 + (c-3)r \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 \leq |H| + 3 \leq 2r + 3,$$

and in the case that $|V_c| = r+2$, we obtain

$$\begin{array}{rcl} 1 + (c-3)r & \leq & |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 \leq |H| + 2 \leq 2r + 4 \\ \text{if} & & V(a_1) = V(a_3) \quad \text{and} \\ 1 + (c-3)r & \leq & |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 \leq |H| + 3 \leq 2r + 4 \\ \text{if} & & V(a_2) = V(a_4). \end{array}$$

All these cases yield a contradiction to $c \geq 7$. Consequently, it remains to consider the case that $D[V(C)]$ is a tournament.

Firstly, let $a_2 \to a_4$. If $a_1 \to a_3$ and $v \in S$, then $a_4 a_1 a_3 v a_2 a_4$ is a 5-cycle, a contradiction. Now let $a_3 \to a_1$. If there are vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_4 a_1 x v a_2 a_4$ is a 5-cycle, a contradiction. Otherwise, we have $S \to H$. If we choose $v, w \in S$ such that $v \to w$, then $N^+(a_1) = H \cup \{a_2\}$ and $N^+(v) \supseteq H \cup \{a_1, a_2, w\}$, a contradiction to $i_g(D) \leq 1$.

Now assume that $a_4 \to a_2$. Firstly, let $a_1 \to a_3$. If there are vertices $v \in S$ and $x \in F = N^-(a_4) - V(C)$ such that $v \to x$, then $a_4 a_1 a_3 v x a_4$ is a 5-cycle, a contradiction. Otherwise, we have $F \to S$. If we choose $v, w \in S$ such that $v \to w$, then we see that $N^-(a_4) = F \cup \{a_3\}$ and $N^-(w) \supseteq F \cup \{a_3, a_4, v\}$, a contradiction to $i_g(D) \leq 1$. In the remaining case that $a_3 \to a_1$, it follows from Corollary 2.4 that

$$\begin{array}{rcl} cr + k & = & |V(D)| \geq |H| + |F| + |S| + |V(C)| - |H \cap F| \\ & \geq & \dfrac{cr + k - r - 2}{2} - 1 + \dfrac{cr + k - r - 2}{2} - 1 \\ & & +(c-4)r + 4 - |H \cap F| \\ & = & 2cr + k - 5r - |H \cap F|. \end{array}$$

Consequently, $|H \cap F| \geq (c-5)r \geq 2r$ and thus, $H \cap F$ consists of at least two partite sets. If we choose $u_2, u_3 \in H \cap F$ such that $u_2 \to u_3$, then $C' = a_4 a_1 u_2 u_3 a_4$ is also a 4-cycle through $a_4 a_1$. Since $u_2 \to a_4$, we arrive, analogously to above, at a contradiction.

Altogether, we have shown in the meantime that every arc of D belongs to a 5-cycle.

Case 6. Let $n \geq 5$ and assume that there exists a vertex $v \in S$ such that $v \to a_{n-2}$. If there is a vertex $a_i \in V(C)$ with $3 \leq i \leq n-3$ such that $a_i \to v$, then we obtain, as in Case 1, an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-2}\}$. If there is a vertex $h \in H$ such that $h \to v$, then $a_n a_1 h v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore, we assume now that $v \to H$. This leads to $d^+(v) \geq d^+(a_1)$, and thus, because of $i_g(D) \leq 1$, it follows that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$ or $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\}$ for some $j \in \{3, 4, \ldots, n-1\}$ and $a_j \to a_1$ or $V(a_1) = V(a_j)$.

Subcase 6.1. Assume that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$. If there is a vertex $h \in H$ such that $h \to a_n$, then $a_n a_1 a_3 a_4 \ldots a_{n-1} v h a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we may assume now that $a_n \to (H - V(a_n))$. If $a_{i-1} \to a_n$ for $3 \leq i \leq n-1$, then $a_n a_1 a_i a_{i+1} \ldots a_{n-1} v a_2 a_3 \ldots a_{i-1} a_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains to treat the case that $a_n \to a_{i-1}$ or $a_{i-1} \in V(a_n)$ for $2 \leq i \leq n-1$. Let $\{a_1, a_2, \ldots, a_{n-2}\} = A \cup B$ such that $a_n \to A$ and $B \subseteq V(a_n)$. Then $N^+(a_1) = H \cup \{a_2, a_3, \ldots, a_{n-1}\}$ and $N^+(a_n) \supseteq A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$. This leads to

$$d^+(a_n) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) = d^+(a_1) + |S| - r,$$

if $|V_c| = r+1$ (and $d^+(a_n) \geq d^+(a_1) + |S| - (r+1)$, if $|V_c| = r+2$). To get no contradiction, S has to consist of only one partite set, which means $n = c-1$, $D[V(C)]$ is a tournament, $B = \emptyset$ and $a_n \to \{a_1, a_2, \ldots, a_{n-2}\}$ (respectively, $n = c-1$, $D[V(C)]$ is a tournament or $n = c-2$, $r = 2$, $|S| = 2r = 4$, $|V(a_n)| = r+2 = 4$, $d^+(a_n) = d^+(a_1) + 1$). Now define $R = V(D) - (H \cup F \cup S \cup V(C))$. Since $H \cap F = \emptyset$, we obtain by Corollary 2.4

$$|R| \leq cr + k - \left\{ \frac{cr + k - r - 2}{2} - (n-2) + \frac{cr + k - r - 2}{2} - 1 + |S| + n \right\}.$$

This yields $|R| \leq 1$, if $|S| = r$, $|R| = 0$, if $|S| = r+1$, and $|R| \leq -1$, if $|S| = 2r$ or $|S| = r+2$. Thus, it follows that $n = c-1$ and $|S| \leq r+1$ in all cases. Furthermore, we see that $|S| + |R| \leq r+1$.

If there is an arc $h \to y$ with $h \in H$ and $y \in F$, then we observe that $a_n a_1 a_4 \ldots a_{n-1} v h y a_n$ is an $(n+1)$-cycle, a contradiction. Hence let $(F \cup \{a_1, a_2, a_n, v\}) \leadsto H$. Now let L be the set of vertices in H having an inner neighbor in H, and let $M = H - L$. In the case that $L \neq \emptyset$ and $b \in L$, it cannot be that $b a_3 \in E(D)$, because otherwise $a_n a_1 a b a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle, if $a \in H$ is an inner neighbor of b, a contradiction. Furthermore, we note that $M \leadsto L$ and that M consists of vertices of at most one partite set.

Hence, for every vertex $b \in L$, we conclude that $d(b, V(D) - L) \leq n - 4 + |S| - 1 + |R| \leq r + n - 4 = r + c - 5$. Now it follows from Corollary 2.4 that

$$d^+_{D[L]}(b) = d^+(b) - d(b, V(D) - L) \geq \frac{cr + k - r - 2}{2} - r - c + 5.$$

This implies

$$\frac{|L|(|L|-1)}{2} \geq |E(D[L])| = \sum_{b \in L} d^+_{D[L]}(b)$$

$$\geq \; |L|\left\{\frac{cr+k-r-2}{2}-r-c+5\right\}.$$

Furthermore, because of Lemma 2.1, we observe that $|L| = |H| - |M| = d^+(a_1) - (n-2) - |M| \leq \frac{cr+k-r+1}{2} - |M| - c + 3$. Combining these results, we arrive at

$$\frac{cr+k-r+1}{2} - |M| - c + 2 \geq |L| - 1 \geq 2\left\{\frac{cr+k-r-2}{2} - r - c + 5\right\}.$$

The last inequality is equivalent to $(c-5)r \leq -k-2|M|+2c-11 \leq -2|M|+2c-12$. Since $r \geq 2$, this leads to the contradiction $|M| \leq -1$.

Consequently, it remains to consider the case that $L = \emptyset$, which means that H consists of vertices of only one partite set. This partite set has to be $V(a_n)$, because otherwise, we observe that $N^+(a_n) \supseteq \{a_1, \ldots, a_{n-2}\} \cup H \cup S$ and $N^+(a_1) = H \cup \{a_2, \ldots, a_{n-1}\}$, a contradiction to $i_g(D) \leq 1$. This implies that $a_2 \to H$ and even $\{a_3, \ldots, a_{n-1}\} \to H$, because otherwise, let $i = \min_{3 \leq l \leq n-1}\{l | h \to a_l\}$ with $h \in H$, then $a_n a_1 \ldots a_{i-1} h a_i \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we have $(\{a_1, a_2, \ldots, a_{n-1}, a_n, v\} \cup F) \rightsquigarrow H$. Then we conclude for every vertex $h \in H$ that $\frac{cr+k-r-2}{2} \leq d^+(h) = d(h, V(D) - H) \leq |S| - 1 + |R| \leq r$, a contradiction to $c \geq 7$.

Subcase 6.2. Assume that there exists exactly one $j \in \{3, 4, \ldots, n-1\}$ such that $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$ and that $n \geq 6$. This condition implies $d^+(v) \geq d^+(a_1) + 1$ and thus, because of $i_g(D) \leq 1$, $d^+(v) = d^+(a_1) + 1$. Furthermore, we note that $H \cap Q = \emptyset$ and $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$.

If there are vertices $x \in H$ and $y \in Q$ such that $x \to y$, then, because of $n \geq 6$, $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Hence, we assume that $(Q \cup \{a_1, a_2, v\}) \rightsquigarrow H$. Let L be the set of vertices q in H which have an inner neighbor p in H. Furthermore, let $M = H - L$ and $|L| \neq 0$. Then we have $(Q \cup M \cup \{a_1, a_2, a_3, v\}) \rightsquigarrow L$.

Firstly, let $|V(v)| = r + 2$. Then Remark 2.3 yields the contradiction

$$\frac{cr+k-r-2}{2} + 1 \leq d^+(a_1) + 1 = d^+(v) = \frac{cr+k-r-2}{2}.$$

Secondly, let $|V(v)| = r + 1$. Then, for every vertex $q \in L$, we conclude that $d(q, V(D) - L) \leq |V(v)| + |V(C)| - 4 = r + n - 3$, and thus, it follows from Lemma 2.2 and Corollary 2.4 that

$$\begin{aligned} d^+_{D[L]}(q) &= d^+(q) - d(q, V(D) - L) \\ &\geq \frac{cr+k-r-2}{2} - r - n + 3, \quad \text{if} \quad k \geq 2 \\ \text{and} \quad d^+_{D[L]}(q) &\geq \frac{cr+k-r-1}{2} - r - n + 3, \quad \text{if} \quad k = 1. \end{aligned}$$

This implies

$$\frac{|L|(|L|-1)}{2} \geq |E(D[L])| \; = \; \sum_{q \in L} d^+_{D[L]}(q)$$

$$\geq \quad |L|\left\{\frac{cr+k-r-2}{2}-r-n+3\right\}$$

$$\text{and} \quad \frac{|L|(|L|-1)}{2} \quad \geq \quad |L|\left\{\frac{cr+k-r-1}{2}-r-n+3\right\},$$

respectively. The conditions $d^+(v) = d^+(a_1) + 1$, $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and Lemma 2.2 yield $|L| = |H| - |M| = d^+(a_1) - n + 3 - |M| = d^+(v) - n + 2 - |M| \leq \frac{cr+k-r}{2} - |M| - n + 2$. Combining these results, we arrive at the inequalities

$$\frac{cr+k-r}{2} - |M| - n + 1 \geq |L| - 1 \quad \geq \quad 2\left\{\frac{cr+k-r-2}{2}-r-n+3\right\}$$

$$\text{and} \quad \frac{cr+k-r}{2} - |M| - n + 1 \quad \geq \quad 2\left\{\frac{cr+k-r-1}{2}-r-n+3\right\},$$

respectively. A transformation leads to $2n \geq (c-5)r + k + 2|M| + 6$ and $2n \geq (c-5)r + k + 2|M| + 8$, respectively. Since $n \leq c - 1$, $k \geq 2$ (respectively, $k = 1$) and $r \geq 2$, this yields a contradiction, if $|M| \geq 1$.

Thirdly, let $|V(v)| = r$. Then, for every vertex $q \in L$, we conclude ($|R| = 0$) that $d(q, V(D) - L) \leq r + n - 4$, and analogously to above, we get the contradiction $|M| \leq -1$.

The case that $|M| = 0$ yields a contradiction, analogously as in Subcase 3.2.

Consequently it remains to consider the possibility that $|L| = 0$, which means that H consists of vertices of only one partite set V_z. Firstly, let $|V_z| = r + 2$ and $|V(a_1)| \geq r + 1$ (this means $k \geq 3$). Since $|N^+(a_1) \cap V(C)| = n - 3, n \leq c - 1$ and Corollary 2.4, this leads to

$$\frac{cr+k-r-2}{2} - (c-4) \leq d^+(a_1) - (n-3) = |H| \leq r + 1,$$

which is equivalent to $2c \geq (c-3)r + k + 4$, a contradiction, because of $r \geq 2$ and $k \geq 3$. Now let $|V_z| = r + 2$ and $|V(a_1)| = r$. Then Remark 2.3 yields

$$\frac{cr+k-r}{2} - (c-4) \leq d^+(a_1) - (n-3) = |H| \leq r + 1,$$

hence $2c \geq (c-3)r + k + 6$, a contradiction. Finally, let $|V_z| \leq r + 1$; then we arrive at

$$\frac{cr+k-r-2}{2} - (c-4) \leq d^+(a_1) - (n-3) = |H| \leq r,$$

hence $2c \geq (c-3)r + k + 6$, a contradiction.

Subcase 6.3. Assume that $n = 5$ and there is exactly one $j \in \{3, 4\}$ such that $a_1 \to (\{a_2, a_3, a_4\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$.

Subcase 6.3.1. Let $a_1 \to \{a_2, a_3\}$ and $a_4 \to a_1$ or $V(a_4) = V(a_1)$. If there is a vertex $h \in H$ such that $h \to a_5$, then $a_5a_1a_3a_4vha_5$ is a 6-cycle, a contradiction. Therefore, we may assume that $a_5 \to (H - V(a_5))$. If $a_2 \to a_5$, then $a_5a_1a_3a_4va_2a_5$ is a 6-cycle, a contradiction. Hence, it remains to treat the case that $a_5 \to a_2$ or

$V(a_5) = V(a_2)$. Let $\{a_1, a_2\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_3\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - (V(a_5) - (B \cup \{a_5\})))$. This leads to

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) = d^+(a_1) + |S| - r,$$

if $|V(a_5)| = r + 1$ and

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 2 - (|B| + 1)) = d^+(a_1) + |S| - (r + 1), \qquad (5)$$

if $|V(a_5)| = r + 2$. Since $i_g(D) \leq 1$, the set S consists of one ($n = c - 1$, if $|V(a_5)| = r+1$) or of at most two ($n = c-2$, if $|V(a_5)| = r+2$) partite sets. Firstly, let $n = c - 1$. Then, since $n = 5$, this leads to a contradiction to $c \geq 7$. In the remaining case that $n = c - 2$ and $|V(a_5)| = r + 2$, we have $|V_c| = r + 2$, $r = 2$ and $|S| = 2r = 4$. In this case, because of (5) and Remark 2.3, we arrive at the contradiction

$$\frac{cr + k - r - 2}{2} + 1 \leq d^+(a_1) + 1 = d^+(a_5) = \frac{cr + k - r - 2}{2}.$$

Subcase 6.3.2. Let $n = 5$ and assume that $a_1 \to \{a_2, a_4\}$ and $a_3 \to a_1$ or $V(a_3) = V(a_1)$. Analogously to Subcase 6.2, H consists of at least two partite sets. Hence, there exist vertices $x, y \in H$ such that $x \to y$. If $y \to a_5$, then $a_5a_1a_4vxya_5$ is a 6-cycle, a contradiction. Now let $W = H - V(a_5)$ and $U = \{x \in W | d^-_{D[H]}(x) = 0\}$. It follows that U is a subset of one partite set, which means $|U| \leq r$ (respectively, $|U| \leq r + 1$, if $|V_c| = r + 2$), and $a_5 \to (W - U)$. If $a_3 \to a_5$, then $a_5a_1a_4va_2a_3a_5$ is a 6-cycle, a contradiction. Hence, it remains to consider the case that $a_5 \to a_3$ or $V(a_5) = V(a_3)$. Let $\{a_1, a_3\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_4\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - ((V(a_5) - (B \cup \{a_5\})) \cup U))$ and therefore

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) - |U| \geq d^+(a_1) + |S| - 2r,$$

if $|V_c| \leq r + 1$ and

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 2 - (|B| + 1)) - |U| \geq d^+(a_1) + |S| - 2(r + 1),$$

if $|V_c| = r + 2$. Since $i_g(D) \leq 1$, this yields a contradiction, if S consists of more than two (respectively, three, if $|V_c| = r + 2$) partite sets. Let $|V_c| = r + 2$ and let S consist of three partite sets; then we get a contradiction, if $r \geq 4$. If $r = 3$ and $|V(a_5)| = r + 2$, then, because of Remark 2.3, we arrive at the contradiction

$$\frac{cr + k - r - 2}{2} + 1 \leq d^+(a_1) + 1 = d^+(a_5) = \frac{cr + k - r - 2}{2}.$$

If $r = 3$ and $|V(a_5)| \leq r + 1$, then we have the contradiction

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) - |U| \geq d^+(a_1) + r - 1 = d^+(a_1) + 2.$$

Consequently, it remains to treat the cases $n = c - 2$, $|B| = 0, D[V(C)]$ is a tournament or $|V_c| = r+2$, $n = c-3$ and $r = 2$. If we define $U' = (N^+(a_1) \cap N^-(a_5)) - V(C)$, then $U' \subseteq U$ and U' consists of vertices of only one partite set V_y. Now let $J = N^-(a_5) - (U' \cup V(C))$ and $G = N^+(a_1) - (V_y \cup \{a_2, a_4\})$. In this case, we note that $G \neq \emptyset$, because otherwise $H = N^+(a_1) - \{a_2, a_4\} \subseteq V_y$, hence, it follows from Corollary 2.4 that

$$\frac{cr + k - r - 2}{2} - 2 \leq d^+(a_1) - 2 = |H| \leq r + 1,$$

a contradiction to $c \geq 7$. Therefore, assume that $G \neq \emptyset$. If there are vertices $x \in G$ and $y \in J \cup U'$ such that $x \to y$, then $a_5 a_1 a_4 v x y a_5$ is a 6-cycle, a contradiction.

Suppose next that there exist vertices $b \in G$ and $w \in S$ such that $b \to w$. If $w \to a_3$, then $a_5 a_1 b w a_3 a_4 a_5$ is a 6-cycle, a contradiction. So, we can assume that $a_3 \to w$. If there is a vertex $x \in (N^-(a_5) - V(C))$ such that $w \to x$, then $a_5 a_1 a_2 a_3 w x a_5$ is a 6-cycle, a contradiction. Thus, we can assume that $(N^-(a_5) - V(C)) \to w$. Altogether, we see that $N^-(a_5) \subseteq (N^-(a_5) - V(C)) \cup \{a_2, a_4\}$ and $N^-(w) \supseteq (N^-(a_5) - V(C)) \cup \{a_3, a_4, a_5, b\}$ and this yields the contradiction $d^-(w) \geq d^-(a_5) + 2$. Consequently, it remains to treat the case that $S \to G$. If we define $R = V(D) - (H \cup J \cup S \cup V(C))$, then, because of

$$\begin{aligned} |J| &\geq |N^-(a_5)| - |U'| - 2 \geq \frac{cr + k - r - 2}{2} - |U'| - 2 \\ &= \begin{cases} \frac{6r+k-2}{2} - |U'| - 2, & \text{if} \quad n = c - 2 = 5 \\ \frac{7r+k-2}{2} - |U'| - 2, & \text{if} \quad n = c - 3 = 5 \end{cases}, \end{aligned}$$

we obtain $|R| \leq$

$$\begin{cases} 7r + k - \left\{\frac{6r+k-2}{2} - |U'| - 2 + \frac{6r+k-2}{2} - 2 + 2r + 5\right\}, & \text{if} \quad n = c - 2 \\ 16 + k - \left\{\frac{12+k}{2} - |U'| - 2 + \frac{12+k}{2} - 2 + 6 + 5\right\}, & \text{if} \quad n = c - 3 \end{cases}$$

$$= \begin{cases} |U'| - r + 1, & \text{if} \quad n = c - 2 \\ |U'| - 3, & \text{if} \quad n = c - 3 \end{cases}.$$

Thus, we also see that $U' \neq \emptyset$. Let there be a vertex $y \in G$ such that $y \to a_3$. Because of $U' \subseteq U$ and $V_y \subseteq V(D) - G$, there exists a vertex $x \in U'$ such that $x \to y$. This leads to the 6-cycle $a_5 a_1 x y a_3 a_4 a_5$, a contradiction. Hence, it remains that $(S \cup J \cup U' \cup \{a_1, a_2, a_3, a_5\}) \leadsto G$.

Firstly, let us observe the case that $n = c - 2$. Then, for every vertex $x \in G$, we get $d(x, V(D) - G) \leq |R| + 1 + |V_y \cap H| - |U'| \leq 2 - r + |V_y| - |V_y \cap V(C)| \leq 1 - r + |V_y| \leq 3$ and thus, it follows that

$$d^+_{D[G]}(x) = d^+(x) - d(x, V(D) - G) \geq \frac{6r + k - 2}{2} - 3 = \frac{6r + k - 8}{2}.$$

This implies

$$\frac{|G|(|G| - 1)}{2} \geq |E(D[G])| = \sum_{x \in G} d^+_{D[G]}(x) \geq |G| \frac{6r + k - 8}{2}.$$

In view of Lemma 2.1, we have $|G| = d^+(a_1) - |V_y \cap H| - 2 \le d^+(a_1) - 2 \le \frac{6r+k-3}{2}$. Altogether, this leads to $\frac{6r+k-5}{2} \ge |G| - 1 \ge 6r + k - 8$, and thus, we obtain the inequality $6r + k \le 11$, a contradiction.

Now let $n = c - 3$. Then, for every vertex $x \in G$, we conclude that $d(x, V(D) - G) \le |R| + 1 + |V_y \cap H| - |U'| \le -2 + |V_y| - |V_y \cap V(C)| \le -3 + |V_y| \le 1$ and thus, it follows that $d^+(x) \le |G| = d^+(a_1) - |V_y \cap H| - 2 \le d^+(a_1) - 2$, a contradiction to $i_g(D) \le 1$.

Summarizing the investigations of Case 6, we see that it remains to treat the case when $a_{n-2} \to S$.

Case 7. Let $n = 5$. If we consider the cycle $C^{-1} = a_1a_5a_4a_3a_2a_1 = b_5b_1b_2b_3b_4b_5$ in the converse D^{-1} of D, then $\{b_4, b_5\} \to S \to \{b_1, b_2, b_3\}$. Since this is exactly the situation of Case 6, there exists in D^{-1} a 6-cycle, containing the arc $b_5b_1 = a_1a_5$, and hence there exists in D a 6-cycle through a_5a_1.

Case 8. Let $n \ge 6$. Assume that there exists a vertex $v \in S$ such that $a_3 \to v$. If we consider the converse of D, then in view of Case 6, it remains to consider the case that $S \to a_3$.

Case 9. Let $c > n \ge 6$. If there exist vertices $y \in S$ and $x \in H$ such that $x \to y$, then $a_na_1xya_3a_4\dots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, we assume now that $S \to H$. Let $y \in S$. If there exists a vertex $x \in H$ such that $x \to a_n$, then $a_na_1a_2\dots a_{n-2}yxa_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains to treat the case that $(S \cup \{a_1, a_2, a_n\}) \leadsto H$.

If $a_1 \to a_i$ and $a_{i-1} \to a_n$ for $i \in \{3, 4, \dots, n-1\}$, then the $(n+1)$-cycle $a_na_1a_i\dots a_{n-1}ya_2\dots a_{i-1}a_n$ yields a contradiction. Thus, if $a_1 \to a_i$ for some $i \in \{3, 4, \dots, n-1\}$, then we may assume that $a_n \to a_{i-1}$ or $V(a_{i-1}) = V(a_n)$. Let $N = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$ be exactly the subset of $V(C) - \{a_2\}$ with the property that $a_1 \to N$. Then we define $A \cup B = \{a_{i_1-1}, a_{i_2-1}, \dots, a_{i_k-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. This definition and the fact that $a_n \to (H - V(a_n))$ lead to $N^+(a_1) = \{a_2\} \cup N \cup H$ and $N^+(a_n) \supseteq \{a_1\} \cup A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$. This implies

$$\begin{aligned} d^+(a_n) &\ge |A| + |S| + 1 + |H| - (r + 1 - (|B| + 1)) \\ &= |A| + |B| + |H| + |S| - r + 1 \\ &= d^+(a_1) + |S| - r, \end{aligned} \tag{6}$$

if $|V(a_n)| \le r + 1$ and

$$d^+(a_n) \ge d^+(a_1) + |S| - (r + 1), \tag{7}$$

if $|V(a_n)| = r + 2$. If $|V(a_n)| = r + 2$ and S consists of two partite sets, then by (7), we conclude that $r = 2$ and $|S| = 2r = 4$, and thus, Remark 2.3 leads to the contradiction

$$\frac{cr + k - r - 2}{2} + 1 \le d^+(a_1) + 1 \le d^+(a_n) = \frac{cr + k - r - 2}{2}.$$

Hence, because of the bounds (6) and (7), we conclude that the case $n = c-1$, $|B| = 0$ and $D[V(C)]$ is a tournament, remains to be considered.

Subcase 9.1. There exists a vertex $v \in S$ such that $v \to a_{n-3}$. If there is a vertex $a_i \in V(C)$ with $4 \leq i \leq n-4$ such that $a_i \to v$, then we obtain, as in Case 1, an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-3}\}$. If $R_1 = V(D) - (H \cup Q \cup V(v) \cup V(C))$, then because of $|H| = |N^+(a_1) - V(C)| \geq d^+(a_1) - (n-2)$ and $|Q| = |N^-(v) - V(C)| \geq d^-(v) - 3$, we see with respect to Lemma 2.2 and Corollary 2.4 that

$$\begin{aligned} |R_1| &\leq cr + k \\ &\quad - \left\{ \frac{cr+k-r-2}{2} - (n-2) + \frac{cr+k-r-1}{2} - 3 + r + n \right\} = \frac{5}{2}, \end{aligned}$$

if $|V(v)| = r$,

$$\begin{aligned} |R_1| &\leq cr + k \\ &\quad - \left\{ \frac{cr+k-r-2}{2} - (n-2) + \frac{cr+k-r-2}{2} - 3 + r + 1 + n \right\} = 2, \end{aligned}$$

if $|V(v)| = r+1$, and

$$\begin{aligned} |R_1| &\leq cr + k \\ &\quad - \left\{ \frac{cr+k-r-2}{2} - (n-2) + \frac{cr+k-r-2}{2} - 3 + r + 2 + n \right\} = 1, \end{aligned}$$

if $|V(v)| = r+2$. Altogether, we see that $|R_1| \leq 2$, if $|V(v)| \leq r+1$ and $|R_1| \leq 1$, if $|V(v)| = r+2$.

Subcase 9.1.1. Firstly, let H consist of vertices of only one partite set. Because of $|B| = 0$, according to (6) (respectively, (7)), this partite set has to be $V(a_n)$. If there are vertices $h \in H$ and $y \in F$ such that $h \to y$, then $a_n a_1 a_4 \ldots a_{n-1} v h y a_n$ is an $(n+1)$-cycle, a contradiction. Hence, $F \to H$. Since $H \subseteq V(a_n) - \{a_n\}$, we even have $a_2 \to H$ and thus $\{a_3, a_4, \ldots, a_{n-1}\} \to H$. Consequently, $(N^-(a_n) \cup S) \to H$. Therefore, for $x \in H$, it follows that $d^-(x) \geq d^-(a_n) + |S| \geq d^-(a_n) + 2$, a contradiction to $i_g(D) \leq 1$.

Subcase 9.1.2. Now we assume that H consists of vertices of more than one partite set. Let L be the set of vertices in H which have an inner neighbor in H and $M = H - L$. If there are vertices $q \in L$ and $p \in H$ such that $p \to q \to a_3$, then $a_n a_1 p q a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, $a_3 \rightsquigarrow L$.

Firstly, let $n \geq 7$. Then, we have $Q \rightsquigarrow L$, because otherwise, if there are vertices $x \in Q$ and $q \in L$ such that $q \to x$, then $a_n a_1 q x v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Altogether, we observe that $(Q \cup V(v) \cup M \cup \{a_1, a_2, a_3, a_n\}) \rightsquigarrow L$. Since $|R_1| \leq 2$, for every vertex $q \in L$, it follows that $d(q, V(D) - L) \leq n - 2 = c - 3$, and thus Corollary 2.4 leads to

$$d^+_{D[L]}(q) = d^+(q) - d(q, V(D) - L) \geq \frac{cr+k-r-2}{2} - c + 3.$$

This implies

$$\frac{|L|(|L|-1)}{2} \geq |E(D[L])| = \sum_{q \in L} d^+_{D[L]}(q) \geq |L| \left\{ \frac{cr+k-r-2}{2} - c + 3 \right\}.$$

Since $d^+(v) \geq |H| + (n-3) = |H| + (c-4)$, we conclude together with Lemma 2.1 that $|L| \leq d^+(v) - (n-3) - |M| = d^+(v) - c + 4 - |M| \leq \frac{cr+k-r+1}{2} - c + 4 - |M|$. Combining these results, we arrive at

$$\frac{cr+k-r+1}{2} - c + 3 - |M| \geq |L| - 1 \geq 2\left\{\frac{cr+k-r-2}{2} - c + 3\right\}.$$

This results in $(c-1)r + k + 2|M| + 1 \leq 2c$, a contradiction, if $|M| \geq 1$.

The case $|M| = 0$ leads to a contradiction, analogously to Subcase 3.2.

It remains to treat the case that $n = 6$ and $c = n + 1 = 7$. We remember that $\{a_4, a_5, a_6\} \to S \to \{a_1, a_2, a_3\}$. We note that $H \cap F = \emptyset$, since $F \to a_6 \leadsto H$. If there are vertices $f \in F$ and $w \in S$ such that $w \to f$ then $a_6a_1a_2a_3a_4wfa_6$ is a 7-cycle, a contradiction. Therefore, we have $F \to S$. Since $H \cap F = \emptyset$, we see that $F \leadsto a_1$. Let $R_2 = V(D) - (H \cup F \cup S \cup V(C))$. Since $|B| = 0$ and $a_6 \to a_{i-1}$, if $a_1 \to a_i$ for $2 \leq i \leq n-1$, we observe that $|N^+(a_1) \cap V(C)| + |N^-(a_6) \cap V(C)| \leq l + 5 - l = 5$, if $|N^+(a_1) \cap V(C)| = l$. Hence, Corollary 2.4 yields

$$|R_2| \leq cr + k - \left\{\frac{cr+k-r-2}{2} + \frac{cr+k-r-2}{2} - 5 + |S| + n\right\} \leq 1.$$

From the fact that $v \to H$ and $N^+(v) \cap V(C) = \{a_1, a_2, a_3\}$, we deduce that $|N^+(a_1) \cap V(C)| \geq 2$. If $\{a_3\} \subseteq N^+(a_1)$ or $\{a_4\} \subseteq N^+(a_1)$, then $F \leadsto H$, because otherwise, if there are vertices $h \in H$ and $f \in F$ such that $h \to f$, then either $a_6a_1a_3a_4vhfa_6$ or $a_6a_1a_4a_5vhfa_6$ is a 7-cycle, a contradiction. Let L be the set of vertices in H which have an inner neighbor in H and let $M = H - L$. Then it follows that $(M \cup F \cup S \cup \{a_1, a_2, a_3, a_6\}) \leadsto L$, and thus, since $|R_2| \leq 1$, for every vertex $q \in L$, we observe that $d(q, V(D) - L) \leq 3 = n - 3 = c - 4$ and, analogously as above, we get a contradiction. Consequently, let $N^+(a_1) \cap V(C) = \{a_2, a_5\}$, and thus $a_6 \to a_4$ and $d^+(a_1) = d^+(v) - 1$.

Assume that F consists of vertices of only one partite set V_b. In this case, we observe that $N^-(a_6) \subseteq F \cup (N^-(a_6) \cap V(C))$. Since $|N^+(a_6) \cap V(C)| \geq |N^+(a_1) \cap V(C)| = 2$, it follows that $|N^-(a_6) \cap V(C)| \leq 3$ and thus $\frac{6r+k-2}{2} \leq d^-(a_6) \leq r + 3$, if $|V_c| = r + 1$. This yields the contradiction $4r + k \leq 8$. Hence, let us investigate the case that $|V_c| = r+2$. If $|V_b| = r+2$ and $|V(a_6)| \geq r+1$ (that means $k \geq 3$), then we arrive at the contradiction $\frac{6r+k-2}{2} \leq d^-(a_6) \leq r+4$. On the other hand, if $|V_b| \leq r+1$ or $|V(a_6)| = r$, we see that $\frac{6r+k-2}{2} \leq d^-(a_6) \leq r + 3$ or $\frac{6r+k}{2} \leq d^-(a_6) \leq r + 4$, in both cases a contradiction.

Consequently, it remains to consider the case that F consists of more than one partite set. Hence, there exists an arc $f_1f_2 \in E(D[F])$, and the set F_1 of vertices in F having an outer neighbor in F is non-empty. Let $F_2 = F - F_1$. If there are vertices $f_1 \in F_1$, $h \in H$ and $f_2 \in F$ such that $h \to f_1 \to f_2$, then $a_6a_1a_5vhf_1f_2a_6$ is a 7-cycle, a contradiction. Therefore, we may assume that $F_1 \leadsto H$. Furthermore, we see that $F_1 \leadsto a_4$, because otherwise $a_6a_1a_2a_3a_4f_1f_2a_6$ is a 7-cycle, a contradiction. Because of $H \cap F = \emptyset$, we conclude that $F \leadsto a_1$. It is also easy to see that $F \leadsto a_5$ and $F \to S$, since otherwise we are able to construct a 7-cycle, a contradiction. Summarizing, we see that $F_1 \leadsto (H \cup S \cup F_2 \cup \{a_1, a_4, a_5, a_6\})$. Hence, since $|R_2| \leq 1$, for every vertex $f_1 \in F_1$, we conclude that $d(V(D)-F_1, f_1) \le 3$, and thus, it follows from Corollary 2.4 that

$$d^-_{D[F_1]}(f_1) = d^-(f_1) - d(V(D)-F_1, f_1) \ge \frac{6r+k-2}{2} - 3.$$

This implies

$$\frac{|F_1|(|F_1|-1)}{2} \ge |E(D[F_1])| = \sum_{f_1\in F_1} d^-_{D[F_1]}(f_1) \ge |F_1| \left\{\frac{6r+k-2}{2} - 3\right\}.$$

We see that $d^-(a_6) \ge |F| + 2$, because otherwise, we arrive at the contradiction $d^+(a_6) \ge 4 + |H| - |V(a_6) - \{a_6\}| + |S| \ge d^+(a_1) + 2 + |S| - r \ge d^+(a_1) + 2$, if $|V(a_6)| \le r+1$. If $|V(a_6)| = r+2$, then we obtain $d^+(a_6) \ge d^+(a_1)+1$, a contradiction to Remark 2.3. Thus, it follows that $|F_1| \le d^-(a_6) - 2 - |F_2| \le \frac{6r+k+1}{2} - 2 - |F_2|$. Combining these results, we obtain

$$\frac{6r+k+1}{2} - 3 - |F_2| \ge |F_1| - 1 \ge 2\left\{\frac{6r+k-2}{2} - 3\right\},$$

which can be transformed to $6r + k + 2|F_2| \le 11$, a contradiction.

Subcase 9.2. Finally, we assume that $a_{n-3} \to S$. Then we see that $n = c-1 \ge 7$. Let $R = V(D) - (H \cup F \cup S \cup V(C))$. If there is a vertex $w \in H \cap F$, then $a_n a_1 a_2 \dots a_{n-2} v w a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $H \cap F = \emptyset$. We have seen above that $|H| = d^+(a_1) - |N| - 1$ and $|N^+(a_n) \cap V(C)| \ge |N| + 1$. Hence $|N^-(a_n) \cap V(C)| \le n - |N| - 2$, and thus $|F| = |N^-(a_n) - V(C)| \ge d^-(a_n) - (n - 2 - |N|)$. It follows from Corollary 2.4 that

$$\begin{aligned}|R| \le{}& cr + k \\ &- \left\{\frac{cr+k-r-2}{2} - |N| - 1 + \frac{cr+k-r-2}{2} - n + 2 + |N| + |S| + n\right\},\end{aligned}$$

and thus $|R| \le 1$, if $|S| = r$; $|R| = 0$, if $|S| = r+1$; and $|R| \le -1$, if $|S| = r+2$. If there is an arc xy with $x \in H$ and $y \in F$, then $a_n a_1 a_2 \dots a_{n-3} v x y a_n$ is an $(n+1)$-cycle, a contradiction. If there is an arc uy with $u \in S$ and $y \in F$, then $a_n a_1 a_2 \dots a_{n-2} u y a_n$ is an $(n+1)$-cycle, a contradiction. Furthermore, if there is an arc $x a_{n-1}$ with $x \in H$, then $a_n a_1 a_2 \dots a_{n-3} v x a_{n-1} a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, it remains to treat the case that $(F \cup S \cup \{a_1, a_2, a_{n-1}, a_n\}) \rightsquigarrow H$ and $F \rightsquigarrow (\{a_1, a_{n-1}, a_n\} \cup S \cup H)$.

Subcase 9.2.1. Firstly, we investigate the case that $r = 2$. As seen above, for every vertex $h \in H$, we conclude that $d(h, V(D) - H) \le n - 3 = c - 4$ and thus $d^+_{D[H]}(h) \ge \frac{cr+k-r-2}{2} - c + 4 = \frac{k+4}{2} \ge \frac{5}{2}$ and therefore $d^+_{D[H]}(h) \ge 3$. Hence, H contains at least 7 vertices. Furthermore, there is at least one vertex h_1 in H such that $d^+_{D[H]}(h_1) \le \frac{|H|-1}{2}$. Since $N^+(a_1) = H \cup N \cup \{a_2\}$ and $i_g(D) \le 1$, we conclude that $d^+(h_1) \ge |H| + |N|$. In addition, $(F \cup S \cup \{a_1, a_2, a_{n-1}, a_n\}) \rightsquigarrow H$, and thus $N^+(h_1) \subseteq V(C) \cup R \cup H$, which leads to

$$|N^+(h_1) \cap V(C)| + |R| + \frac{|H|-1}{2} \ge d^+(h_1) \ge |H| + |N|.$$

This implies

$$|N^+(h_1) \cap V(C)| \geq \frac{|H|+1}{2} + |N| - |R| \geq |N| + 3.$$

Let $a_i \in N^+(h_1) \cap V(C)$ $(3 \leq i \leq n-2)$. If $a_{i-1} \to a_n$, then we observe that $a_n a_1 h_1 a_i \dots a_{n-2} v a_2 \dots a_{i-1} a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, in $V(C)$, a_n has at least $|N|+3$ further outer neighbors except a_1. According to (6) and (7), this yields

$$d^+(a_n) \geq |N| + 4 + |H| + |S| - (r+1) = d^+(a_1) + 2 + |S| - r \geq d^+(a_1) + 2,$$

a contradiction to $i_g(D) \leq 1$.

Subcase 9.2.2. Assume that $|N| \geq \frac{c-6}{2}$ and $r \geq 3$. Since $|R| \leq 1$, for every vertex $h \in H$, we conclude that $d(h, V(D) - H) \leq n - 3 = c - 4$ and thus, it follows from Corollary 2.4 that

$$d^+_{D[H]}(h) = d^+(h) - d(h, V(D) - H) \geq \frac{cr + k - r - 2}{2} - c + 4.$$

This implies

$$\begin{aligned} \frac{|H|(|H|-1)}{2} &\geq |E(D[H])| = \sum_{h \in H} d^+_{D[H]}(h) \\ &\geq |H| \left\{ \frac{cr + k - r - 2}{2} - c + 4 \right\}. \end{aligned}$$

Since $|H| = d^+(a_1) - |N| - 1 \leq \frac{cr+k-r+1}{2} - |N| - 1 \leq \frac{cr+k-r+1}{2} - \frac{c}{2} + 2 = \frac{cr+k-r-c+5}{2}$, we obtain

$$\frac{cr + k - r - c + 3}{2} \geq |H| - 1 \geq cr + k - r - 2 - 2c + 8.$$

This inequality is equivalent to $(c-1)r + k \leq 3c - 9$, a contradiction to $r \geq 3$.

Subcase 9.2.3. Now assume that $|N| \leq \frac{c-7}{2}$ and $r \geq 3$. Since $|R| \leq 1$, for every vertex $y \in F$, we conclude that $d(V(D) - F, y) \leq n - 2 = c - 3$ and thus, it follows from Corollary 2.4 that

$$d^-_{D[F]}(y) = d^-(y) - d(V(D) - F, y) \geq \frac{(c-1)r + k + 4}{2} - c.$$

This implies

$$\frac{|F|(|F|-1)}{2} \geq |E(D[F])| = \sum_{y \in F} d^-_{D[F]}(y) \geq |F| \left\{ \frac{(c-1)r + k + 4}{2} - c \right\}.$$

Since $i_g(D) \leq 1$, we conclude from (6) and (7) that $|N^+(a_n) \cap V(C)| \leq |N| + 3$, and thus $|N^-(a_n) \cap V(C)| \geq n - |N| - 4$. Hence, it follows that $|F| = |N^-(a_n) - V(C)| \leq$

$d^-(a_n) - (n - |N| - 4) \leq \frac{cr+k-r+1}{2} - (c-1) + 4 + \frac{c-7}{2} = \frac{(c-1)r+k+4-c}{2}$. Combining these results, we observe that

$$\frac{(c-1)r+k+2-c}{2} \geq |F| - 1 \geq (c-1)r + k + 4 - 2c.$$

A transformation of this inequality leads to $3c \geq (c-1)r + k + 6 \geq (c-1)r + 7$, a contradiction to $r \geq 3$. This completes the proof of the theorem. $\Box$

From Theorem 1.4 and the theorem in this section we can immediately deduce the main theorem.

The following example, which can also be found in [12], shows that the condition $c \geq 7$ in Theorem 1.5 is best possible.

Example 3.2 *Let* $V_1 = \{u\} \cup V_1'$ *with* $|V_1'| = 2$, $V_2 = \{v\} \cup V_2'$ *with* $|V_2'| = 2$, $V_3 = V_3' \cup V_3''$ *with* $|V_3'| = |V_3''| = 2$, *and* V_4, V_5, V_6 *with* $|V_4| = |V_5| = |V_6| = 2$ *and* $V_4 = \{x, y\}$ *be the partite sets of a* 6*-partite tournament such that* $u \to v \to V_1' \to (V_4 \cup V_5 \cup V_6) \to V_2' \to u \to (V_4 \cup V_5 \cup V_6) \to v$, $V_2' \to V_3 \to u$, $v \to V_3 \to V_1'$, $V_2' \to V_1'$, $V_4 \to V_5 \to V_6 \to V_4$, *and* $V_3' \to (V_6 \cup \{x\}) \to V_3'' \to (V_5 \cup \{y\}) \to V_3'$ *(see Figure 1). The resulting* 6*-partite tournament is almost regular with at least two vertices in every partite set; however, the arc* uv *is not contained in a* 4*-cycle.*

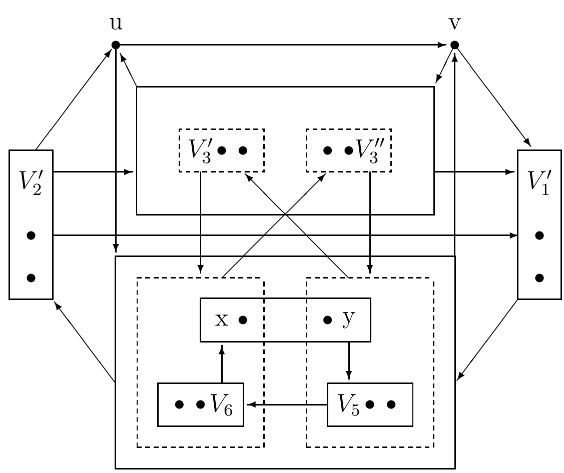

Figure 1: An almost regular 6-partite tournament with the property that the arc uv is not contained in a 4-cycle

The next example (cf. [12]) shows that the condition $r \geq 2$ is necessary for $c = 7$.

Example 3.3 *Let* $V_1 = \{u, u_2\}$*,* $V_2 = \{v, v_2\}$*,* $V_3 = \{w_1, w_2, w_3\}$*,* $V_4 = \{x\}$*,* $V_5 = \{y\}$*,* $V_6 = \{z\}$*, and* $V_7 = \{a\}$ *be the partite sets of a* 7*-partite tournament such that* $u \to v \to u_2 \to \{a, x, y, z\} \to v_2 \to u \to \{a, x, y, z\} \to v \to V_3 \to u$*,* $v_2 \to u_2$*,* $v_2 \to V_3 \to u_2$*,* $w_1 \to a \to x \to y \to z \to a \to y \to w_1 \to z \to x \to w_1$*,* $w_2 \to z \to w_3 \to a \to w_2 \to x \to w_3 \to y \to w_2$ *(see Figure 2). The resulting* 7*-partite tournament is almost regular, however, the arc* uv *is not contained in a* 4*-cycle. Consequently, the condition* $r \geq 2$ *in Theorem 3.1 is necessary, at least for* $c = 7$*.*

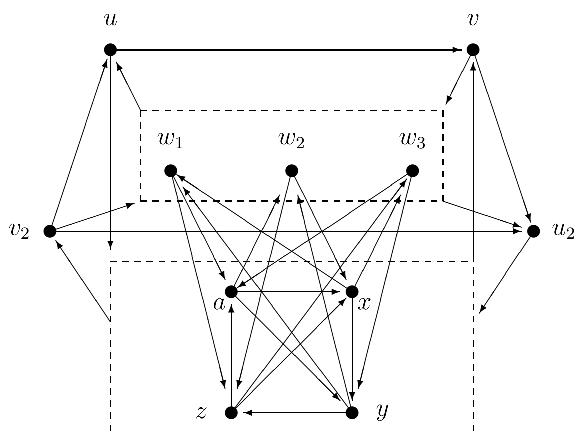

Figure 2: An almost regular 7-partite tournament with the property that the arc uv is not contained in a 4-cycle

References

[1] B. Alspach, Cycles of each length in regular tournaments, *Canad. Math. Bull.* **10** (1967), 283–286.

[2] J. Bang-Jensen, G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2000.

[3] Y. Guo, Semicomplete Multipartite Digraphs: A Generalization of Tournaments, *Habilitation thesis*, RWTH Aachen (1998).

[4] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey, *J. Graph Theory* **19** (1995), 481–505.

[5] O. S. Jacobson, Cycles and paths in tournaments, *Ph. D. Thesis*, Aarhus University (1972).

[6] M. Tewes, In-tournaments and semicomplete multipartite digraphs, *Ph. D. thesis*, RWTH Aachen, Germany, (1999).

[7] M. Tewes, L. Volkmann and A. Yeo, Almost all almost regular c-partite tournaments with $c \geq 5$ are vertex pancyclic, *Discrete Math.* **242** (2002), 201–228.

[8] P. Turán, An extremal problem in graph theory, *Mat.-Fiz. Lapok* **48** (1941) 436–452 (in Hungarian).

[9] L. Volkmann, *Fundamente der Graphentheorie*, Springer-Verlag, Vienna, New York (1996).

[10] L. Volkmann, Cycles through a given arc in certain almost regular multipartite tournaments, *Australas. J. Combin.* **26** (2002), 121–133.

[11] L. Volkmann, Cycles in multipartite tournaments: results and problems, *Discrete Math.* **245** (2002), 19–53.

[12] L. Volkmann, Cycles of length four through a given arc in almost regular multipartite tournaments, *Ars Combin.*, to appear.

[13] A. Yeo, Semicomplete Multipartite Digraphs, *Ph. D. thesis*, Odense University, (1998).

[14] A. Yeo, How close to regular must a semicomplete multipartite digraph be to secure Hamiltonicity? *Graphs Combin.* **15** (1999), 481–493.

(Received 2/1/2002)