On the differences of the independence, domination and irredundance parameters of a graph

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Abstract

In this paper we present upper bounds on the differences between the independence, domination and irredundance parameters of a graph. For example, using the Brooks theorem on the chromatic number, we show that for any graph G of order n with maximum degree $\Delta \geq 2$

$$IR(G) - \beta(G) \leq \left\lfloor \frac{\Delta - 2}{2\Delta} n \right\rfloor,$$

where $\beta(G)$ and $IR(G)$ are the independence number and the upper irredundance number of a graph G, respectively. This bound implies a conjecture posed by Rautenbach (*Discrete Math.* 203 (1999), 239–252).

1 Introduction

All graphs will be finite and undirected without multiple edges. If G is a graph, $V(G)$ denotes the set, and $|G|$ the number, of vertices in G. Let $N(x)$ denote the neighborhood of a vertex x, and let $\langle X \rangle$ denote the subgraph of G induced by $X \subseteq V(G)$. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$.

A set $I \subseteq V(G)$ is called *independent* if no two vertices of I are adjacent. A set X is called a *dominating set* if $N[X] = V(G)$. An *independent dominating set* is a vertex subset that is both independent and dominating, or equivalently, is maximal independent. The *independence number* $\beta_0(G)$ is the maximum cardinality of a (maximal) independent set of G, and the *independent domination number* $i(G)$

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is the minimum cardinality taken over all maximal independent sets of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a (minimal) dominating set of G, and the *upper domination number* $\Gamma(G)$ is the maximum cardinality taken over all minimal dominating sets of G. For $x \in X$, the set

$$PN(x, X) = N[x] - N[X - \{x\}]$$

is called the *private neighborhood* of x. If $PN(x, X) = \emptyset$, then x is said to be *redundant* in X. A set X containing no redundant vertex is called *irredundant*. The *irredundance number* $ir(G)$ is the minimum cardinality taken over all maximal irredundant sets of G, and the *upper irredundance number* $IR(G)$ is the maximum cardinality of a (maximal) irredundant set of G.

The following relationship among the parameters under consideration is well-known [4, 5]:

$$ir(G) \le \gamma(G) \le i(G) \le \beta(G) \le \Gamma(G) \le IR(G).$$

Many authors ([1],[6], [9]–[14]) investigated the differences between the above parameters for various classes of graphs. In this paper, we present further results on the differences between those parameters. For example, using the Brooks theorem on the chromatic number, we show that for any graph G of order n with maximum degree $\Delta \ge 2$,

$$IR(G) - \beta(G) \le \left\lfloor \frac{\Delta - 2}{2\Delta} n \right\rfloor.$$

This bound implies a conjecture posed by Rautenbach [12]: if G is a graph with maximum degree $\Delta \le 3$, then

$$IR(G) - \beta(G) \le \left\lfloor \frac{n}{6} \right\rfloor.$$

2 Upper Bounds for the Upper Parameters

The following bounds on the difference $IR(G) - \beta(G)$ were proved by Rautenbach [12], who also characterized extremal graphs for the first bound.

Theorem 1 ([12]) *For any graph G of order $n \ge 4$,*

$$IR(G) - \beta(G) \le \left\lfloor \frac{n-4}{2} \right\rfloor.$$

Theorem 2 ([12]) *For any graph G of order n with maximum degree $\Delta \ge 1$,*

$$IR(G) - \beta(G) \le \left\lfloor \frac{(\Delta - 1)^2}{2\Delta^2} n \right\rfloor.$$

Substituting $\Delta = 3$ into the above inequality, we obtain the upper bound for graphs with maximum degree 3:

$$IR(G) - \beta(G) \leq \left\lfloor \frac{2}{9} n \right\rfloor.$$

Rautenbach [12] conjectured that this bound can be improved.

Conjecture 1 ([12]) *If G is a graph with maximum degree $\Delta \leq 3$, then*

$$IR(G) - \beta(G) \leq \left\lfloor \frac{n}{6} \right\rfloor.$$

Note here that $IR(G) = \beta(G)$ for any graph with maximum degree $\Delta \leq 2$. Thus, it is sufficient to prove the conjecture for graphs with maximum degree 3.

We first prove the upper bound for $IR(G) - \beta(G)$, improving Theorem 2. This bound immediately implies Conjecture 1.

Theorem 3 *For any graph G of order n with maximum degree $\Delta \geq 2$,*

$$IR(G) - \beta(G) \leq \left\lfloor \frac{\Delta - 2}{2\Delta} n \right\rfloor.$$

Proof: Consider a maximum irredundant set X of G. Let us denote all isolated vertices of the graph $\langle X \rangle$ by X^0 and let $X^* = X - X^0$. Since X has no redundant vertex, it follows that $PN(x, X) \neq \emptyset$ for any $x \in X$. Note that $PN(x, X) \cap X = \emptyset$ for any vertex $x \in X^*$. For each vertex $x \in X^*$, take one vertex from the set $PN(x, X)$ and generate the set Y^*. Obviously, $X^* \cap Y^* = \emptyset$. Moreover, all edges between the sets X^* and Y^* form a perfect matching of the graph $F = \langle X^* \cup Y^* \rangle$. Denote $|X^*| = t$; thus $|F| = 2t$.

We denote by I a maximum independent set of the graph F. Let us prove that

$$|I| = \beta(F) \geq \frac{2t}{\Delta}.$$

We will need Brooks theorem [3]: *If H is a connected graph which is not complete and $\Delta(H) \geq 3$, then*

$$\chi(H) \leq \Delta(H).$$

Denote by $F_1, F_2, ..., F_m$ connected components of the graph F. Let us consider an arbitrary component F_i. Since any vertex $x \in X^*$ is adjacent to $x' \in X^*$, we see that

$$\Delta(F_i) \geq 2.$$

Suppose that $\Delta(F_i) = 2$. Since F has a perfect matching, it follows that F_i is isomorphic to either an even cycle or a chain. Hence we obtain

$$\beta(F_i) \geq \frac{|F_i|}{2} = \frac{|F_i|}{\Delta(F_i)}.$$

Assume now that $\Delta(F_i) \geq 3$. Since F_i cannot be isomorphic to a complete graph, we obtain, by Brooks theorem,

$$\chi(F_i) \leq \Delta(F_i).$$

Therefore,

$$\beta(F_i) \geq \frac{|F_i|}{\chi(F_i)} \geq \frac{|F_i|}{\Delta(F_i)}.$$

We obtain

$$\beta(F) = \sum_{i=1}^{m} \beta(F_i) \geq \sum_{i=1}^{m} \frac{|F_i|}{\Delta(F_i)} \geq \frac{1}{\Delta(G)} \sum_{i=1}^{m} |F_i| = \frac{|F|}{\Delta(G)} = \frac{2t}{\Delta}.$$

It is not difficult to see that $I \cup X^0$ is an independent set of G. Hence

$$\beta(G) \geq |I| + |X^0|.$$

Thus,

$$\begin{aligned} IR(G) - \beta(G) &\leq |X^*| + |X^0| - |I| - |X^0| \\ &= |X^*| - |I| \leq t - \frac{2t}{\Delta} = \frac{\Delta - 2}{2\Delta} 2t \leq \frac{\Delta - 2}{2\Delta} n, \end{aligned}$$

as required. ∎

We now prove the upper bound on the difference $IR(G) - \beta(G)$ as a function of the chromatic number and order of G. It may be pointed out that Theorem 4 as well as Theorem 3 imply the same upper bounds for the differences $IR(G) - \Gamma(G)$ and $\Gamma(G) - \beta(G)$.

Theorem 4 *For any graph G of order n with maximum degree $\Delta \geq 1$,*

$$IR(G) - \beta(G) \leq \left\lfloor \frac{\chi(G) - 2}{2\chi(G)} n \right\rfloor.$$

Proof: Using the same notation as in the proof of Theorem 3, we obtain

$$|I| = \beta(F) \geq \frac{|F|}{\chi(F)} \geq \frac{2t}{\chi(G)}.$$

Therefore,

$$\begin{aligned} IR(G) - \beta(G) &\leq |X^*| + |X^0| - |I| - |X^0| \\ &= |X^*| - |I| \leq t - \frac{2t}{\chi(G)} = \frac{\chi(G)-2}{2\chi(G)}2t \leq \frac{\chi(G)-2}{2\chi(G)}n, \end{aligned}$$

as required. ∎

Theorem 4 immediately implies the following well-known result on bipartite graphs.

Corollary 1 ([4]) *If G is a bipartite graph, then*

$$\beta(G) = \Gamma(G) = IR(G).$$

We now take use of Fajtlowicz' and Staton's lower bounds for the independence number to prove our next theorem. It can be shown that the first bound of the theorem is sharp; an example of such a graph is the generalized Peterson graph of order 14.

Theorem 5 *(i) If G is a K_3-free graph and $\Delta(G) = 3$, then*

$$IR(G) - \beta(G) \leq \left\lfloor \frac{n}{7} \right\rfloor.$$

(ii) If G contains no K_q $(q \geq 3)$ and $\Delta \geq 1$, then

$$IR(G) - \beta(G) \leq \left\lfloor \frac{\Delta + q - 4}{2\Delta + 2q} n \right\rfloor.$$

Proof: For any graph G containing no K_q, Fajtlowicz [7, 8] showed that

$$\beta(G) \geq \frac{2n}{\Delta + q}.$$

For the particular case $\Delta = q = 3$, Staton [13] proved a stronger result:

$$\beta(G) \geq \frac{n}{\Delta - 1/5}.$$

Suppose now that G contains no K_q. Using the same notation as in the proof of Theorem 3, we obtain

$$|I| = \beta(F) \geq \frac{2|F|}{\Delta(F) + q} \geq \frac{4t}{\Delta(G) + q} = \frac{4t}{\Delta + q}.$$

Therefore,

$$\begin{aligned} IR(G) - \beta(G) &\leq |X^*| + |X^0| - |I| - |X^0| \\ &= |X^*| - |I| \leq t - \frac{4t}{\Delta + q} = \frac{\Delta + q - 4}{2\Delta + 2q} 2t \leq \frac{\Delta + q - 4}{2\Delta + 2q} n. \end{aligned}$$

For the case $\Delta = q = 3$, we apply Staton's bound:

$$|I| = \beta(F) \geq \frac{|F|}{\Delta(F) - 1/5} \geq \frac{|F|}{\Delta(G) - 1/5} = \frac{2t}{3 - 1/5} = \frac{5}{7} t.$$

Thus,

$$IR(G) - \beta(G) \leq |X^*| - |I| \leq t - \frac{5}{7} t \leq \frac{n}{7},$$

as required. ∎

3 Upper Bounds for the Lower Parameters

Let us consider the differences between the lower parameters. Notice here that trivial bounds

$$i(G) - \gamma(G) \leq n - \Delta - 1$$

and

$$i(G) - ir(G) \leq n - \Delta - 1$$

are good for graphs whose maximum degree is close to the order, while for graphs with relatively small maximum degree it is better to use the bounds below.

Theorem 6 *For any graph G of order n with maximum degree $\Delta \geq 3$,*

$$i(G) - \gamma(G) \leq \left\lfloor \frac{\Delta - 2}{\Delta} n \right\rfloor - 1.$$

Proof: Let X be a minimum dominating set of G such that the number of edges in $\langle X \rangle$ is minimum. Denote all isolated vertices of the graph $\langle X \rangle$ by X^0 and let $X^* = X - X^0$. We put $w = |N[X^0] - N(X^*)|$ and $t = |X^*|$. Also, let

$$r = \max_{x \in X^*} |N(x) - X^*| = |N(x^*) - X^*|.$$

Note that $r \geq 2$, for otherwise the private neighborhood of x^* consists of exactly one vertex y^* and the set $(X - x^*) \cup y^*$ is a minimum dominating set containing fewer edges than X, contrary to hypothesis. Also, it is easy to see that $\Delta \geq r + 1$. Since X is a dominating set, we obtain

$$n \leq t + \sum_{x \in X^*} |N(x) - X^*| + w \leq t(1 + r) + w$$

or

$$t \geq \frac{n-w}{1+r}.$$

Consider the graph

$$G' = G - (N[x^*] \cup N[X^0]).$$

Each vertex $x \in X^*$ has a private neighbour. Thus, the graph G' possesses a matching M consisting of $t - z - 1$ edges, where $z = |N(x^*) \cap X^*|$. We have

$$i(G') \leq |G'| - |M| \leq (n - r - z - 1 - w) - (t - z - 1) = n - r - w - t.$$

Taking into account the above inequalities, we deduce

$$\begin{aligned}
i(G) - \gamma(G) &\leq 1 + i(G') + |X^0| - |X^*| - |X^0| \\
&= 1 + i(G') - t \\
&\leq 1 + n - r - w - 2t \\
&\leq 1 + n - r - w - 2\frac{n-w}{1+r} \\
&= 1 + n - r - w - \frac{2n}{1+r} + \frac{2w}{1+r} \\
&= (1 - \frac{2}{1+r})n + 1 - r + \frac{1-r}{1+r}w \\
&\leq (1 - \frac{2}{\Delta})n - 1 \\
&= \frac{\Delta - 2}{\Delta}n - 1.
\end{aligned}$$

∎

Theorem 7 *For any graph G of order n with maximum degree $\Delta \geq 3$,*

$$i(G) - ir(G) \leq \min\left\{ \left\lfloor \frac{2\Delta - 3}{2\Delta - 1} n \right\rfloor, \left\lfloor \frac{\Delta - 1}{\Delta} n - \frac{\Delta}{2} \right\rfloor \right\} - 1.$$

Proof: We first prove that

$$i(G) - ir(G) \leq \left\lfloor \frac{2\Delta - 3}{2\Delta - 1} n \right\rfloor - 1.$$

Let X be a minimum maximal irredundant set of G. We denote all isolated vertices of the graph $\langle X \rangle$ by X^0 and put $X^* = X - X^0$. Since X has no redundant vertex, it follows that $PN(x, X) \neq \emptyset$ for any $x \in X$. Note that $PN(x, X) \cap X = \emptyset$ for any vertex $x \in X^*$. For each vertex $x \in X^*$, take one vertex from the set $PN(x, X)$ and generate the set Y^*. Obviously, $X^* \cap Y^* = \emptyset$. Moreover, all edges between the sets X^* and Y^* form a perfect matching of the graph $\langle X^* \cup Y^* \rangle$. Denote $|X^*| = |Y^*| = t$ and $U = V(G) - N[X]$. We put $w = |N[X^0] - N(X^*)|$. Also, let

$$r = \max_{x \in X^*} |N(x) - X^*|$$

and

$$s = \max_{y \in Y^*} |N(y) \cap U| = |N(y^*) \cap U|.$$

Note that

$$r \leq \Delta - 1$$

and

$$s \leq \Delta - 1.$$

If $s = 0$, then $U = \emptyset$. Hence $ir(G) = \gamma(G)$ and the result follows by Theorem 6. Therefore we may assume that $s \geq 1$.

We have

$$n \leq t + tr + ts + w = t(1 + r + s) + w$$

or

$$t \geq \frac{n - w}{1 + r + s}.$$

Consider the graph

$$G' = G - (N[y^*] \cup N[X^0]).$$

Each vertex $x \in X^*$ has a private neighbour. Thus, the graph G' possesses a matching M consisting of $t - z - 1$ edges, where $z = |N(y^*) \cap Y^*|$. We have

$$i(G') \leq |G'| - |M| \leq (n - s - z - 2 - w) - (t - z - 1) = n - s - w - t - 1.$$

Taking into account the above inequalities, we deduce

$$\begin{aligned}
i(G) - ir(G) &\leq 1 + i(G') + |X^0| - |X^*| - |X^0| \\
&= 1 + i(G') - t \\
&\leq n - s - w - 2t \\
&\leq n - s - w - \frac{2n}{1 + r + s} + \frac{2w}{1 + r + s} \\
&\leq \Big(1 - \frac{2}{1 + r + s}\Big) n - s + \frac{1 - r - s}{1 + r + s} w \\
&\leq \Big(1 - \frac{2}{1 + r + s}\Big) n - 1 \\
&\leq \Big(1 - \frac{2}{2\Delta - 1}\Big) n - 1.
\end{aligned}$$

The second inequality easily follows from Theorem 6 and the Bollobás-Cockayne inequality [2]

$$\gamma(G) \leq 2ir(G) - 1.$$

Indeed,

$$\begin{aligned}
i(G) - ir(G) &\leq i(G) - \frac{1}{2}(\gamma(G) + 1) \\
&= \frac{1}{2}\Big(i(G) - \gamma(G) + i(G) - 1\Big) \\
&\leq \frac{1}{2}\Big((1 - \frac{2}{\Delta})n - 1 + n - \Delta - 1\Big) \\
&= (1 - \frac{1}{\Delta})n - \Delta/2 - 1.
\end{aligned}$$

∎

We complete this section with the following result.

Theorem 8 *For any graph G of order $n \geq 3$,*

$$\gamma(G) - ir(G) \leq \left\lfloor \frac{n-3}{4} \right\rfloor.$$

Proof: Let X, X^0, X^*, Y^*, U and t be the same as in the proof of Theorem 7. Consider the set $D = Y^* \cup X^0$, and put $Z = V(G) - N[D]$ and $|Z| = z$. We see that $D \cup Z$ is a dominating set and therefore

$$\gamma(G) - ir(G) \leq |Y^*| + |X^0| + |Z| - |X^*| - |X^0| = z. \tag{1}$$

Note that any vertex $u \in U$ dominates the private neighborhood of a vertex $x \in X^*$, since X is a maximal irredundant set. Hence the set U is dominated by the set Y^* and $Z \cap U = \emptyset$. We obtain

$$n \geq |U| + 2t + z.$$

We may assume that $|U| \geq 1$, for otherwise X is dominating and the result easily follows. Hence $t \geq 1$ and

$$z \leq n - 3.$$

Let us prove that

$$\gamma(G) - ir(G) \leq \frac{1}{3}(n - z - 3). \tag{2}$$

We first suppose that $|U| \leq t$. Since

$$n \geq |U| + 2t + z \geq 3|U| + z,$$

we obtain $|U| \leq \frac{1}{3}(n - z)$. The set $X \cup U$ is dominating and redundant. Therefore,

$$\gamma(G) - ir(G) \leq |X| + |U| - 1 - |X| = |U| - 1 \leq \frac{1}{3}(n - z - 3).$$

Assume now that $|U| \geq t+1$. We have

$$n \geq |U| + 2t + z \geq 3t + z + 1.$$

Hence $t \leq \frac{1}{3}(n-z-1)$. The set $X \cup Y^*$ is dominating and redundant. Consequently,

$$\gamma(G) - ir(G) \leq |X| + |Y^*| - 1 - |X| = t - 1 \leq \frac{1}{3}(n - z - 4).$$

Thus, (1) and (2) imply

$$\gamma(G) - ir(G) \leq \max_{0 \leq z \leq n-3} \min\{z, \frac{1}{3}(n - z - 3)\} \leq \frac{n-3}{4}.$$

∎

We believe that the bound of Theorem 8 can be improved. Moreover, it would be of interest to find an analogous bound depending on degree parameters and order.

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