On the normality of Cayley graphs of order pq*

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Abstract

A Cayley graph $\Gamma = \mathrm{Cay}(G, \mathcal{S})$ is said to be normal for a finite group G, if the right regular representation $R(G)$ is normal in the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. In this paper we investigate the normality of Cayley graphs of groups of order a product of two distinct primes, by determining all nonnormal Cayley graphs of these groups.

1 Introduction

Let G be a finite group and $\mathcal{S}$ be a subset of G not containing the identity element 1_G. The Cayley digraph $\Gamma = \mathrm{Cay}(G, \mathcal{S})$ of group G with respect to $\mathcal{S}$ is the digraph with vertex set $V(\Gamma) = G$ and arc set $Arc(\Gamma) = \{(g, sg) | g \in G, s \in \mathcal{S}\}$. If $\mathcal{S}$ is inverse-closed (i.e., $\mathcal{S}^{-1} = \mathcal{S}$), then $\mathrm{Cay}(G, \mathcal{S})$ can be viewed as an undirected graph by identifying two arcs (g, h) and (h, g) with an undirected edge $\{h, g\}$. In this case, this graph is called the Cayley graph of G with respect to the Cayley subset $\mathcal{S}$. It is easy to see that $\Gamma = \mathrm{Cay}(G, \mathcal{S})$ is connected if and only if $G = \langle \mathcal{S} \rangle$ and that the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ contains the right regular representation $R(G)$ of G. The following fact is basic for Cayley digraphs.

Proposition 1.1 *A (di)graph Γ is a Cayley (di)graph of a group G if and only if* $\mathrm{Aut}(\Gamma)$ *contains a regular subgroup isomorphic to G.*

For a Cayley digraph $\Gamma = \mathrm{Cay}(G, \mathcal{S})$, set $\mathrm{Aut}(G, \mathcal{S}) = \{\alpha \in \mathrm{Aut}(G) | \mathcal{S}^\alpha = \mathcal{S}\}$. Obviously, $\mathrm{Aut}(\Gamma) \geq R(G)\mathrm{Aut}(G, \mathcal{S})$. Let $A = \mathrm{Aut}(\Gamma)$. We have the following proposition (see [11]).

Proposition 1.2 (1) $N_A(R(G)) = R(G)\mathrm{Aut}(G, \mathcal{S})$.

(2) *The following statements are equivalent:*

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(i) $R(G) \trianglelefteq A$;

(ii) $A = R(G)\mathrm{Aut}(G, \mathcal{S})$;

(iii) $A_{1_G} \leq \mathrm{Aut}(G, \mathcal{S})$.

Xu defined so-called normal Cayley digraphs of groups in [11], which can be viewed as a generalization of the concept of *graphical regular representations*, GRRs in short, of finite groups.

Definition 1.3 *The Cayley digraph* $\Gamma = \mathrm{Cay}(G, \mathcal{S})$ *is called normal for group* G *if* $R(G) \trianglelefteq A$.

This concept is helpful for determining the full automorphism groups of Cayley digraphs, which is known to be very difficult in general. In fact, we may divide all Cayley digraphs into two classes: normal and nonnormal Cayley digraphs. As we can see, normal Cayley digraphs are just those which have the smallest possible full automorphism groups.

Recently, some results about the normality of Cayley digraphs have been obtained by several authors (see [11] for a survey). Let p and q be two distinct primes. By [11] and [9], we know that all vertex-primitive Cayley graphs of order pq and all disconnected Cayley graphs of order pq are nonnormal. The normality of Cayley digraphs for some groups of special order is known. For example, all Cayley digraphs of cyclic groups of prime order p except K_p and $p\mathrm{K}_1$ are normal by Galois and Burnside's theorems. In 1998, Du, Wang and Xu determined all imprimitive, nonnormal Cayley graphs for groups of order twice a prime ([5]). Their results partially solved the following problem posed by Xu in [11].

Problem *Determine all imprimitive nonnormal Cayley graphs of order* pq *and do the same thing for Cayley digraphs of order* pq.

In this paper, we shall study the normality of Cayley graphs of order pq for two distinct primes p and q. The results of this paper will answer the first part of the above problem. By the classification of the edge-transitive graphs of order pq (see [1, 3, 8, 9, 10]), we know all nonnormal edge transitive graphs of order pq (see [13, Theorem 2.12]). However, the results of this paper do not depend on it.

Let G be a finite group of order pq, where p, q are distinct primes. By elementary group theory, we know that $G \cong \mathbb{Z}_{pq}$ or $\mathbb{F}_{pq}$, where $\mathbb{F}_{pq}$ is the Frobenius group of order pq, that is,

$$\mathbb{F}_{pq} = \langle a, b \mid a^p = b^q = 1, b^a = b^r \rangle$$

for $r \not\equiv 1 (\mathrm{mod}\ p)$, $r^q \equiv 1 (\mathrm{mod}\ p)$ and $q \mid p - 1$. Further, we have

Proposition 1.4 *Let* p *and* q *be two distinct primes. Then*

(1) $\mathrm{Aut}(\mathbb{Z}_{pq}) \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$;

(2) $\mathrm{Aut}(\mathbb{F}_{pq}) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$.

Then an immediate consequence of the above proposition is

Corollary 1.5 *If* Γ *is a normal Cayley graph of* G*, then* $\mathrm{Aut}(\Gamma)$ *is solvable.*

The proofs of the above proposition and corollary are omitted.

In Section 2, we shall give some lemmas and some examples of nonnormal Cayley graphs of order pq; in Section 3, we shall determine all nonnormal Cayley graphs of order pq for distinct odd primes $p > q$. By [9] and [11], we know that all vertex-primitive Cayley graphs of order pq are nonnormal. Here, we shall only consider the imprimitive case.

For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group, respectively. Let Γ be a graph and $B \le \mathrm{Aut}(\Gamma)$. Then Γ is called B-vertex-transitive (respectively, B-primitive), if B acts transitively (respectively, primitively) on $V(\Gamma)$. In the case $B = \mathrm{Aut}(\Gamma)$, we shall call Γ a vertex-transitive (respectively, vertex-primitive) graph without the prefix B.

To end this section we give some notation used in the present paper. Let X, Y be two graphs. We use $x \sim_{\mathrm{X}} x'$, sometimes $x \sim x'$ for brevity, to denote that x and x' are adjacent in X. Additionally, the *lexicographic product* $\mathrm{X}[\mathrm{Y}]$ is defined as the graph with vertex-set $V(\mathrm{Y}) \times V(\mathrm{X})$ such that $(y, x) \sim (y', x')$ if and only if either $\{x, x'\} \in E(\mathrm{X})$ or $x = x'$ and $\{y, y'\} \in E(\mathrm{Y})$; the *Cartesian product* $\mathrm{X} \times \mathrm{Y}$ is defined as the graph with vertex-set $V(\mathrm{X}) \times V(\mathrm{Y})$ such that $(x, y) \sim (x', y')$ if and only if either $y = y'$, $\{x, x'\} \in E(\mathrm{X})$ or $x = x'$, $\{y, y'\} \in E(\mathrm{Y})$. If $V(\mathrm{X}) = V(\mathrm{Y})$ and $E(\mathrm{X}) \cap E(\mathrm{Y}) = \varnothing$, then the *edge disjoint union* $\mathrm{X} \cup \mathrm{Y}$ denotes the graph having vertex-set $V(\mathrm{X})$ and edge-set $E(\mathrm{X} \cup \mathrm{Y}) = E(\mathrm{X}) \cup E(\mathrm{Y})$.

2 Lemmas and Examples

In this section and in the next section, we use Y_l to denote some Cayley graph of $\mathbb{Z}_l$ for a prime l. Hereafter, we denote x^{-1} as $-x$ for $x \in \mathbb{Z}_l$ and denote $\mathcal{S}^{-1}$ as $-\mathcal{S}$ for $\mathcal{S} \in \mathbb{Z}_l$.

Let Γ be a connected B-vertex-transitive graph of order pq, where $B \le \mathrm{Aut}(\Gamma)$. Suppose that B is imprimitive on $V(\Gamma)$. Then B has a block $\mathcal{B}$ of length $m = p$ or q. Let $\Sigma = \{\mathcal{B}_i \mid i \in \mathbb{Z}_n\}$ be the complete block system containing $\mathcal{B}$ and let K be the kernel of the action of B on Σ. Set $\overline{B} = B/K$. Then $\overline{B}$ acts faithfully and transitively on Σ. Let $\overline{\Gamma}$ be the corresponding block graph. Then $\overline{\Gamma}$ is $\overline{B}$-vertex-transitive graph.

Lemma 2.1 *Let* $\mathcal{B}$, Σ *and* K *be as above. For* $i, j \in \mathbb{Z}_n$, *let* Γ_{ij} *denote the subgraph of* Γ *induced by* $\mathcal{B}_i \cup \mathcal{B}_j$.

(1) *If* K *acts unfaithfully on some block, then* $\Gamma \cong Y_n[Y_m]$ *and* $Y_n \ne nK_1$.

(2) *Assume that* K *is faithful and* 2*-transitive on each block of length* m. *We have*

  (i) $K^{\mathcal{B}_i}$ *and* $K^{\mathcal{B}_j}$ *are equivalent for any* i, $j \in \mathbb{Z}_n$.

  (ii) *If* $\mathcal{B}_i$ *and* $\mathcal{B}_j$ *are adjacent in* $\overline{\Gamma}$, *then*

$$\Gamma_{ij} \cong K_2[Y_m],\ K_2[Y_m] - mK_2\ , or\ K_2 \times Y_m,$$

  where $Y_m = K_m$ *or* mK_1.

(iii) *If the complement* Γ^c *of* Γ *is connected, then*

$$\Gamma, \textit{or } \Gamma^c \cong Y_n^{(1)}[mK_1] \cup (Y_n^{(2)}[mK_1] - m\,Y_n^{(2)}) \cup m\,Y_n^{(3)},$$

where $Y_n^{(i)} = \mathrm{Cay}(\mathbb{Z}_n, \mathcal{S}_n^{(i)})$, $\mathcal{S}_n^{(i)} = -\mathcal{S}_n^{(i)}$ *for* $i = 1, 2, 3$, $\mathcal{S}_n^{(i)} \cap \mathcal{S}_n^{(j)} = \varnothing$ *for* $i \neq j$, *and* $\mathcal{S}_n^{(1)} \neq \varnothing$ *or* $\mathcal{S}_n^{(2)} \neq \varnothing$.

(iv) *If* Γ^c*is not connected, then* $\Gamma^c \cong m\,Y_n$.

Proof (1) Suppose that K acts unfaithfully on some block $\mathcal{B}_{i_0}$. Then $K_{(\mathcal{B}_{i_0})} \neq 1$, where $K_{(\mathcal{B}_{i_0})}$ is the kernel of the action of K on $\mathcal{B}_{i_0}$. Since $\overline{\Gamma}$ is connected and since the length of blocks is a prime, there exist two blocks which are adjacent such that $K_{(\mathcal{B}_{i_0})}$ acts trivially on one block and transitively on the other one. We set

$$\begin{aligned}
&E_1 = \{E([\mathcal{B}_i]) \mid i \in \mathbb{Z}_n\}, \quad \text{where } [\mathcal{B}_i] \text{ is the induced subgraph by } \mathcal{B}_i\\
&E_2 = \{E(\Gamma_{ij}) \mid \Gamma_{ij} \ncong \mathrm{K}_2[\mathrm{Y}_m],\ i \neq j,\ i, j \in \mathbb{Z}_n,\ \mathcal{B}_i \text{ and } \mathcal{B}_j \text{ are adjacent}\} \backslash E_1\\
&E_3 = \{E(\Gamma_{ij}) \mid \Gamma_{ij} \cong \mathrm{K}_2[\mathrm{Y}_m],\ i \neq j,\ i, j \in \mathbb{Z}_n,\ \mathcal{B}_i \text{ and } \mathcal{B}_j \text{ are adjacent}\} \backslash E_1
\end{aligned}$$

Clearly, $E_3 \neq \varnothing$. It is easy to see that $E_i^\alpha \cap E_j^\beta = \varnothing$, for $i \neq j$ and for all $\alpha, \beta \in B$.

Let Γ_1 be the graph of order pq with vertex set $V(\Gamma_1) = V(\Gamma)$ and edge set $E(\Gamma_1) = E(\Gamma) \backslash E_3$. If $E_2 \neq \varnothing$, then the block graph $\overline{\Gamma_1}$ of Γ_1 is connected. Obviously, $B \leq \mathrm{Aut}(\Gamma_1)$. It follows from the argument as in the first paragraph that Γ_1 has edges of type E_3, a contradiction. Hence $E_2 = \varnothing$, and $\Gamma \cong \mathrm{Y}_n[\mathrm{Y}_m]$. Finally, the connectedness of Γ leads to $\mathrm{Y}_n \neq n\mathrm{K}_1$.

(2) Suppose that K is 2-transitive and faithful on each block. It follows from the classification of 2-transitive groups of prime degree, (see, for example, [2]), that K has at most two inequivalent permutation representations of degree m. We assume, without loss of generality, that the actions of K on $\mathcal{B}_0$ and on $\mathcal{B}_1$ are equivalent. Since n is an odd prime and since the set of all such $\mathcal{B}_j$ for which the actions of K on $\mathcal{B}_0$ and on $\mathcal{B}_j$ are equivalent is a block for the action of B on Σ, it follows that the actions of K on all $\mathcal{B}_i \in \Sigma$ are equivalent. Then we have (i) and (ii). Obviously, $[\mathcal{B}_i] = \mathrm{Y}_m = \mathrm{K}_m$ or $m\mathrm{K}_1$ as required.

We first assume that $[\mathcal{B}_i] \cong m\mathrm{K}_1$. Set

$$\begin{aligned}
E_4 &= \{E(\Gamma_{ij}) \mid \Gamma_{ij} \cong \mathrm{K}_2[m\mathrm{K}_1],\ i \neq j,\ i, j \in \mathbb{Z}_n\},\\
E_5 &= \{E(\Gamma_{ij}) \mid \Gamma_{ij} \cong \mathrm{K}_2[m\mathrm{K}_1] \backslash m\mathrm{K}_2,\ i \neq j,\ i, j \in \mathbb{Z}_n\},\\
E_6 &= \{E(\Gamma_{ij}) \mid \Gamma_{ij} \cong m\mathrm{K}_2,\ i \neq j,\ i, j \in \mathbb{Z}_n\}.
\end{aligned}$$

Then $E_4 \cup E_5 \cup E_6 = E(\Gamma)$, and $E_i^\alpha \cap E_j^\beta = \varnothing$, for $i \neq j$ and for all $\alpha, \beta \in B$.

Let Γ_i be the graph of order pq with vertex set $V(\Gamma_i) = V(\Gamma)$ and edge set $E(\Gamma_i) = E_i$, for $i = 4, 5, 6$. Then $B \leq \mathrm{Aut}(\Gamma_i)$ and $\Gamma = \Gamma_4 \cup \Gamma_5 \cup \Gamma_6$. We have

$$\overline{\Gamma} = \overline{\Gamma_4} \cup \overline{\Gamma_5} \cup \overline{\Gamma_6} = \mathrm{Y}_n^{(1)} \cup \mathrm{Y}_n^{(2)} \cup \mathrm{Y}_n^{(3)},$$

where $\mathrm{Y}_n^{(i)} = \mathrm{Cay}(\mathbb{Z}_n, \mathcal{S}^{(i)})$, $\mathcal{S}^{(i)} = -\mathcal{S}^{(i)}$ for $i = 1, 2, 3$, $\mathcal{S}^{(i)} \cap \mathcal{S}^{(j)} = \varnothing$ for $i \neq j$, and at least one of $\mathcal{S}^{(i)}$'s is not empty. Since K is primitive on each block, we have

$$|\mathrm{fix}_{V(\Gamma)}(K_{x_i}) \cap \mathcal{B}_j| = 1, \forall\, i, j \in \mathbb{Z}_n,\ \forall\, x_i \in \mathcal{B}_i.$$

It follows that

$$\Gamma_4 \cong \mathrm{Y}_n^{(1)}[m\mathrm{K}_1],\ \Gamma_5 \cong \mathrm{Y}_n^{(2)}[m\mathrm{K}_1] - m\mathrm{Y}_n^{(2)},\ \Gamma_6 \cong m\mathrm{Y}_n^{(3)},$$

and hence

$$\Gamma \cong \mathrm{Y}_n^{(1)}[m\mathrm{K}_1] \cup (\mathrm{Y}_n^{(2)}[m\mathrm{K}_1] - m\mathrm{Y}_n^{(2)}) \cup m\mathrm{Y}_n^{(3)}.$$

Since Γ is connected, we have $\mathcal{S}^{(1)} \neq \varnothing$ or $\mathcal{S}^{(2)} \neq \varnothing$.

We now assume that $[\mathcal{B}_i] \cong \mathrm{K}_m$. If the complement Γ^c of Γ is connected, then

$$\Gamma^c \cong \mathrm{Y}_n^{(1)}[m\mathrm{K}_1] \cup (\mathrm{Y}_n^{(2)}[m\mathrm{K}_1] - m\mathrm{Y}_n^{(2)}) \cup m\mathrm{Y}_n^{(3)}$$

for three Cayley subsets of $\mathbb{Z}_n$ satisfying the conditions as required. If Γ^c is disconnected, then Γ^c is necessarily isomorphic to $m\mathrm{Y}_n$. $\square$

The following lemmas are helpful to determine the normality of Cayley graphs of order pq. By Theorem 3.5A of [4], we have Lemma 2.2.

Lemma 2.2 *Let Γ be a Cayley graph of $\mathbb{Z}_{pq}$. Then $A = \mathrm{Aut}(\Gamma)$ acts primitively on $V(\Gamma)$ if and only if $\Gamma \cong K_{pq}$, or pqK_1.*

Lemma 2.3 *Suppose that $\{m, n\} = \{p, q\}$. Then*

(1) *$Y_n[Y_m]$ is a Cayley graph of $\mathbb{Z}_{pq}$.*

(2) *If one of Y_n, Y_m is connected and one of them is not a complete graph, then $\mathrm{Aut}(Y_n[Y_m]) = \mathrm{Aut}(Y_m) \wr \mathrm{Aut}(Y_n)$, where $\mathrm{Aut}(Y_m) \wr \mathrm{Aut}(Y_n)$ denotes the wreath product of $\mathrm{Aut}(Y_m)$ and $\mathrm{Aut}(Y_n)$.*

(3) *If Y_n, Y_m are connected, then $\mathrm{Aut}(Y_p \times Y_q) = \mathrm{Aut}(Y_p) \times \mathrm{Aut}(Y_q)$.*

Proof (1) It is easy to see that $\mathrm{Aut}(\mathrm{Y}_n[\mathrm{Y}_m]) \geq \mathrm{Aut}(\mathrm{Y}_m) \wr \mathrm{Aut}(\mathrm{Y}_n)$. Let π be an element in $\mathrm{Aut}(\mathrm{Y}_m)$ of order m, and let σ be an element in $\mathrm{Aut}(\mathrm{Y}_n)$ of order n. We set $\alpha = (\pi, \cdots, \pi; 1)$ and $\beta = (1, \cdots, 1; \sigma)$. Then $G = \langle \alpha, \beta \rangle \cong \mathbb{Z}_{pq}$, and G acts regularly on $V(\mathrm{Y}_n[\mathrm{Y}_m])$. So $\mathrm{Y}_n[\mathrm{Y}_m]$ is a Cayley graph of $\mathbb{Z}_{pq}$.

(2) By the conditions in this lemma, $\Gamma = \mathrm{Y}_n[\mathrm{Y}_m] \not\cong pq\mathrm{K}_1, \mathrm{K}_{pq}$. It follows that $A = \mathrm{Aut}(\mathrm{Y}_n[\mathrm{Y}_m])$ acts imprimitively on the vertex set $\Omega = \{(i, j) \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$. Since any block of A is a block of $\mathrm{Aut}(\mathrm{Y}_m) \wr \mathrm{Aut}(\mathrm{Y}_n)$, A has only imprimitive blocks of length m. It follows that $\mathrm{Aut}(\mathrm{Y}_m) \wr \mathrm{Aut}(\mathrm{Y}_n) \leq A \leq \mathrm{S}_m \wr \mathrm{S}_n$. Let $\rho = (\pi_1, \cdots, \pi_n; \sigma) \in A$. Then

$$(i, j_1) \sim (i, j_2) \Leftrightarrow (i, j_1)^\rho \sim (i, j_2)^\rho \Leftrightarrow (i^{\pi_{j_1}}, j_1^\sigma) \sim (i^{\pi_{j_2}}, j_2^\sigma) \Leftrightarrow j_1^\sigma \sim_{\mathrm{Y}_n} j_2^\sigma;$$

$$\begin{aligned} i_1 \sim_{\mathrm{Y}_m} i_2 &\Leftrightarrow (i_1, j) \sim (i_2, j) \Leftrightarrow (i_1, j)^\rho \sim (i_2, j)^\rho \Leftrightarrow (i_1^{\pi_j}, j^\sigma) \sim (i_2^{\pi_j}, j^\sigma) \\ &\Leftrightarrow i_1^{\pi_j} \sim_{\mathrm{Y}_m} i_2^{\pi_j}. \end{aligned}$$

It follows that $\pi_1,\ \pi_2, \cdots, \pi_n \in \mathrm{Aut}(\mathrm{Y}_m)$ and $\sigma \in \mathrm{Aut}(\mathrm{Y}_n)$. Then $A = \mathrm{Aut}(\mathrm{Y}_m) \wr \mathrm{Aut}(\mathrm{Y}_n)$.

(3) We know, by Lemma 2.2, that $A = \mathrm{Aut}(\mathrm{Y}_p \times \mathrm{Y}_q)$ acts imprimitively on $V(\mathrm{Y}_p \times \mathrm{Y}_q)$. Suppose that A has blocks of length m. Then $\mathrm{Aut}(\mathrm{Y}_m) \times \mathrm{Aut}(\mathrm{Y}_n) \leq A \leq \mathrm{S}_m \wr \mathrm{S}_n$. Let $\rho = (\pi_1, \cdots, \pi_n; \sigma) \in A$. Then

$$\begin{aligned} i_1 \sim_{\mathrm{Y}_m} i_2 &\Leftrightarrow (i_1, j) \sim_{\mathrm{Y}_m \times \mathrm{Y}_n} (i_2, j) \Leftrightarrow (i_1, j)^\rho \sim_{\mathrm{Y}_m \times \mathrm{Y}_n} (i_2, j)^\rho \\ &\Leftrightarrow i_1^{\pi_j} \sim_{\mathrm{Y}_m} i_2^{\pi_j}; \\ j_1 \sim_{\mathrm{Y}_n} j_2 &\Leftrightarrow (i, j_1) \sim_{\mathrm{Y}_m \times \mathrm{Y}_n} (i, j_2) \Leftrightarrow (i, j_1)^\rho \sim_{\mathrm{Y}_m \times \mathrm{Y}_n} (i, j_2)^\rho \\ &\Leftrightarrow (i^{\pi_{j_1}}, j_1^\sigma) \sim (i^{\pi_{j_2}}, j_2^\sigma) \Leftrightarrow i^{\pi_{j_1}} = i^{\pi_{j_2}},\ j_1^\sigma \sim_{\mathrm{Y}_n} j_2^\sigma. \end{aligned}$$

It follows that $\pi_1, \cdots, \pi_n \in \mathrm{Aut}(\mathrm{Y}_m)$ and $\sigma \in \mathrm{Aut}(\mathrm{Y}_n)$, and it follows from the connectedness of Y_p and Y_q that $\pi_1 = \cdots = \pi_n$. So we have

$$\mathrm{Aut}(\mathrm{Y}_p \times \mathrm{Y}_q) = \mathrm{Aut}(\mathrm{Y}_p) \times \mathrm{Aut}(\mathrm{Y}_q).$$

Corollary 2.4 *Suppose that $\{m, n\} = \{p, q\}$ and that one of Y_n, Y_m is connected and one of them is not a complete graph. Then*

(1) *$Y_n[Y_m]$ is an imprimitive, nonnormal Cayley graph of $\mathbb{Z}_{pq}$.*

(2) *$Y_p[Y_q]$ is a Cayley graph of $\mathbb{F}_{pq}$ if and only if $q \mid p-1$ and $r\mathcal{S} = \mathcal{S}$, where $r^q \equiv 1$, $r \not\equiv 1 (\mathrm{mod}\ p)$. If $Y_p[Y_q]$ is a Cayley graph of $\mathbb{F}_{pq}$, then it is imprimitive and nonnormal.*

(3) *If $q \mid p-1$, then $Y_q[Y_p]$ is an imprimitive and nonnormal Cayley graph of $\mathbb{F}_{pq}$.*

Proof (1) By Propositions 1.2, 1.4 and Lemma 2.3.

(2) We first assume that $\Gamma = \mathrm{Y}_p[\mathrm{Y}_q]$ is a Cayley graph of $\mathbb{F}_{pq}$. Then $q \mid p-1$, since $A = \mathrm{Aut}(\mathrm{Y}_q) \wr \mathrm{Aut}(\mathrm{Y}_p)$ acts imprimitively on the vertex set $\mathbb{F}_{pq}$ of $\mathrm{Y}_p[\mathrm{Y}_q]$ with blocks of length q. We may, without loss of generality, assume that $\Sigma = \{\langle b \rangle a^i \mid i \in \mathbb{Z}_p\}$ is a complete block system of A on $\mathbb{F}_{pq}$. The corresponding block graph $\overline{\Gamma} \cong \mathrm{Y}_p$. It follows, from $R(\mathbb{F}_{pq}) \leq A$, that two blocks $\langle b \rangle a^i$ and $\langle b \rangle a^j$ are adjacent if and only if $(\langle b \rangle a^i)^{R(b)} = \langle b \rangle a^{ri} \sim_{\overline{\Gamma}} (\langle b \rangle a^j)^{R(b)} = \langle b \rangle a^{rj}$. It follows that $r\mathcal{S} = \mathcal{S}$.

Conversely, $r\mathcal{S} = \mathcal{S}$ implies that $\mathcal{S}$ is a union of some cosets of $\langle r \rangle$ in $\mathbb{Z}_p^*$. It follows that $q \mid |\mathrm{Aut}(\mathrm{Y}_p)|$. Let $G = \langle \sigma \rangle \langle \delta \rangle$ be a subgroup in $\mathrm{Aut}(\mathrm{Y}_p)$ of order pq such that $o(\sigma) = p$ and $o(\delta) = q$. Let π be an element in $\mathrm{Aut}(\mathrm{Y}_q)$ of order q. Set

$$\alpha = (1, \cdots, 1; \sigma),\ \beta = (\pi, \cdots, \pi; 1),\ \gamma = (1, \cdots, 1; \delta).$$

Then $\langle \alpha, \beta\gamma \rangle$ is a regular subgroup of A and $\langle \alpha, \beta\gamma \rangle \cong \mathbb{F}_{pq}$. Finally, the nonnormality of $\mathrm{Y}_p[\mathrm{Y}_q]$ is obvious.

(3) Assume that $q \mid p-1$. Let r be an element of order q in $\mathbb{Z}_p^*$, and let σ and π be as in (2). Set

$$\alpha = (\sigma^{r^q}, \sigma^{r^{q-1}}, \cdots, \sigma^{r^2}, \sigma^r; 1), \beta = (1, 1, \cdots, 1; \pi).$$

We have

$$o(\alpha) = p,\ o(\beta) = q,\ \beta^{-1}\alpha\beta = \alpha^r.$$

Then $G = \langle \alpha, \beta \rangle \cong \mathbb{F}_{pq}$ acts transitively on $V(\mathrm{Y}_q[\mathrm{Y}_p])$. It follows that $\mathrm{Y}_q[\mathrm{Y}_p]$ is a Cayley graph of $\mathbb{F}_{pq}$. The nonnormality and imprimitivity are obvious. $\square$

Lemma 2.5 *Suppose that* $\{m,n\}=\{p,q\}$. *Suppose that* $Y_n^{(t)}=\mathrm{Cay}(\mathbb{Z}_n,\mathcal{S}_n^{(t)})$ *for* $t=1,2,3$, *where* $-\mathcal{S}_n^{(t)}=\mathcal{S}_n^{(t)}\subseteq\mathbb{Z}_n\backslash\{0\}$, $\mathcal{S}_n^{(s)}\cap\mathcal{S}_n^{(t)}=\varnothing$ *for* $s\neq t$, *either* $\mathcal{S}_n^{(2)}\neq\varnothing$ *or neither of* $\mathcal{S}_n^{(1)}$ *or* $\mathcal{S}_n^{(3)}$ *is empty. Set*

$$\Gamma(n,m;\mathcal{S}_n^{(t)},3)=\,Y_n^{(1)}[mK_1]\cup(\,Y_n^{(2)}[mK_1]-m\,Y_n^{(2)})\cup m\,Y_n^{(3)}.$$

Then $\mathrm{Aut}(\Gamma(n,m;\mathcal{S}_n^{(t)},3))\geq\mathbb{Z}_n\times S_m$, *and hence* $\Gamma(n,m;\mathcal{S}_n^{(t)},3)$ *is an imprimitive Cayley graph of* $\mathbb{Z}_{pq}$. *Additionally, we have*

(1) $\Gamma(q,p;\mathcal{S}_q^{(t)},3)$ *and* $\Gamma(q,p;\mathcal{S}_q^{(t)},3)^c$ *are nonnormal for* $\mathbb{Z}_{pq}$ *and for* $\mathbb{F}_{pq}$ *when* $q\mid p-1$;

(2) $p>q>3$, *then*

- (i) $\mathrm{Aut}(\Gamma(p,q;\mathcal{S}_p^{(t)},3))=\bigcap_{t=1}^{3}\mathrm{Aut}(\,Y_p^{(t)})\times S_q$;
- (ii) $\Gamma(p,q;\mathcal{S}_p^{(t)},3)$ *and* $\Gamma(p,q;\mathcal{S}_p^{(t)},3)^c$ *are nonnormal for* $\mathbb{Z}_{pq}$;
- (iii) $\Gamma(p,q;\mathcal{S}_p^{(t)},3)$ *is a Cayley graph for* $\mathbb{F}_{pq}$ *if and only if* $q\mid p-1$ *and* $r\mathcal{S}_p^{(t)}=\mathcal{S}_p^{(t)}$ *for* $t=1,2,3$, $r\not\equiv1(\mathrm{mod}\ \ p),r^q\equiv1(\mathrm{mod}\ \ p)$;
- (iv) *if* $r\mathcal{S}_p^{(t)}=\mathcal{S}_p^{(t)}$ *for* $t=1,2,3$, $r\not\equiv1(\mathrm{mod}\ p),r^q\equiv1(\mathrm{mod}\ p)$, *then* $\Gamma(p,q;\mathcal{S}_p^{(t)},3)$ *is nonnormal for* $\mathbb{F}_{pq}$.

Proof The first part of this lemma is trivial.

(1) It is easy to check that $\mathrm{Aut}(\Gamma(q,p;\mathcal{S}_q^{(t)},3))$ has a regular subgroup isomorphic to $\mathbb{Z}_{pq}$ and has a regular subgroup isomorphic to $\mathbb{F}_{pq}$ when $q\mid p-1$. Since

$$|\mathrm{Aut}(\Gamma(q,p;\mathcal{S}_q^{(t)},3))|\geq qp!>\left\{\begin{array}{l}pq|\mathrm{Aut}(\mathbb{Z}_{pq})|\\ pq|\mathrm{Aut}(\mathbb{F}_{pq})|\end{array}\right.,$$

we have (1).

(2) (ii), (iii) and (iv) follow at once from (i). We need only to prove (i).

Since $\Gamma=\Gamma(p,q;\mathcal{S}_p^{(t)},3)$ is a Cayley graph of $\mathbb{Z}_{pq}$, it follows from Lemma 2.2 that $A=\mathrm{Aut}(\Gamma(p,q;\mathcal{S}_p^{(t)},3))$ acts imprimitively on the vertex set $V(\Gamma)$. We first assume that A has a block $\mathcal{B}$ of length p. Let Σ be the complete block system containing $\mathcal{B}$. By Lemma 2.1(1), we know that K, the kernel of the action of A on Σ, acts faithfully on each block in Σ.

Suppose that K is insolvable. By Lemma 2.1(2) and the connectedness of Γ and Γ^c, either Γ or Γ^c is isomorphic to $\Gamma(q,p;\mathcal{S}_q^{(t)},3)$ for three Cayley subsets $\mathcal{S}_q^{(t)}$ contained in $\mathbb{Z}_q\backslash\{0\}$. It follows that $\mathrm{Aut}(\Gamma)\geq\mathbb{Z}_q\times\mathrm{S}_p$, and that $\mathrm{Aut}(\Gamma)$ contains two subgroups H_1,H_2 isomorphic to S_p and S_q, respectively, such that $H_1=K$ and $H_1\cap H_2=1$, or A_q. Let α be an element of order p in $C_A(H_2)$. Then $\alpha\in K$. Since $C_K(\alpha)=\langle\alpha\rangle$ and $q>3$, therefore $H_1\cap H_2=1$. So A/K is 2-transitive on Σ. It follows that Γ is isomorphic to one of $\mathrm{K}_q[\mathrm{Y}_p]$, $\mathrm{K}_q[\mathrm{Y}_p]-p\mathrm{K}_q$ and $\mathrm{K}_q\times\mathrm{Y}_p$, where $\mathrm{Y}_p=\mathrm{K}_p$, or $p\mathrm{K}_1$. Since Γ, Γ^c are connected and since $\Gamma\ncong\mathrm{K}_q\times\mathrm{K}_p$, we have

$$\Gamma\cong\mathrm{K}_q[p\mathrm{K}_1]-p\mathrm{K}_q\Rightarrow\Gamma^c\cong\mathrm{K}_q\times\mathrm{K}_p\Rightarrow\mathrm{Aut}(\Gamma)=\mathrm{Aut}(\mathrm{K}_q)\times\mathrm{Aut}(\mathrm{K}_p)=\mathrm{S}_q\times\mathrm{S}_p.$$

If K is solvable, then the Sylow p-subgroup P of K is normal in A. Then $A/C_A(P)$ is isomorphic to a subgroup of $\mathbb{Z}_{p-1}$. Since $q > 3$ and $\mathrm{Aut}(\Gamma) \geq \mathbb{Z}_p \times \mathrm{S}_q$, it follows that A is insolvable, and hence $C_A(P)$ is also insolvable. Let M be a p'-Hall subgroup of $C_A(P)$. Then $C_A(P) = P \times M$, and hence $M \trianglelefteq A$. So A has blocks of length q.

Now, we assume that A has a block $\mathcal{B}$ of length q. Let $\Sigma = \{\mathcal{B}_j \mid j \in \mathbb{Z}_p\}$ be the complete block system containing $\mathcal{B}$. We may assume, without loss of generality, that $\mathcal{B}_j = \{(i,j) \mid i \in \mathbb{Z}_q\}$. It follows that

$$A \leq \mathrm{Aut}(\mathrm{Y}_p^{(1)}[q\mathrm{K}_1]) \cap \mathrm{Aut}(\mathrm{Y}_p^{(2)}[q\mathrm{K}_1] - q\mathrm{Y}_p^{(2)}) \cap \mathrm{Aut}(q\mathrm{Y}_p^{(3)})$$

and that $A \leq \mathrm{S}_q \wr \mathrm{S}_p$. Suppose that $\alpha = (\pi_1, \pi_2, \cdots, \pi_p; \sigma) \in A$. We have

$$j_2 - j_1 \in \mathcal{S}_p^{(3)} \Rightarrow (i, j_1) \sim_{q\mathrm{Y}_p^{(3)}} (i, j_2) \Leftrightarrow (i^{\pi_{j_1}}, j_1^\sigma) \sim (i^{\pi_{j_2}}, j_2^\sigma) \Leftrightarrow \begin{cases} i^{\pi_{j_1}} = i^{\pi_{j_2}}, \\ j_2^\sigma - j_1^\sigma \in \mathcal{S}_p^{(3)} \end{cases};$$

$$j_2 - j_1 \in \mathcal{S}_p^{(2)}, i_1 \neq i_2 \Rightarrow (i_1, j_1) \sim_{\mathrm{Y}_p^{(2)}[q\mathrm{K}_1] - q\mathrm{Y}_p^{(2)}} (i_2, j_2) \Leftrightarrow (i_1^{\pi_{j_1}}, j_1^\sigma) \sim (i_2^{\pi_{j_2}}, j_2^\sigma)$$
$$\Rightarrow j_2^\sigma - j_1^\sigma \in \mathcal{S}_p^{(2)};$$

$$j_2 - j_1 \in \mathcal{S}_p^{(2)} \Rightarrow (i, j_1) \not\sim_{\mathrm{Y}_p^{(2)}[q\mathrm{K}_1] - q\mathrm{Y}_p^{(2)}} (i, j_2) \Leftrightarrow (i^{\pi_{j_1}}, j_1^\sigma) \not\sim (i^{\pi_{j_2}}, j_2^\sigma) \Rightarrow i^{\pi_{j_1}} = i^{\pi_{j_2}}.$$

It follows that $\alpha = (\pi, \pi, \cdots, \pi; \sigma)$ when $\mathcal{S}_p^{(2)} \neq \varnothing$ or $\mathcal{S}_p^{(3)} \neq \varnothing$. Additionally,

$$j_2 - j_1 \in \mathcal{S}_p^{(1)} \Rightarrow (i_1, j_1) \sim_{\mathrm{Y}_p^{(1)}[q\mathrm{K}_1]} (i_2, j_2) \Leftrightarrow (i_1^{\pi_{j_1}}, j_1^\sigma) \sim (i_2^{\pi_{j_2}}, j_2^\sigma) \Rightarrow j_2^\sigma - j_1^\sigma \in \mathcal{S}_p^{(1)}.$$

So we have $\sigma \in \bigcap_{t=1}^3 \mathrm{Aut}(\mathrm{Y}_p^{(t)})$. Then $\mathrm{Aut}(\Gamma(p, q; \mathcal{S}_p^{(t)}, 3)) = \bigcap_{t=1}^3 \mathrm{Aut}(\mathrm{Y}_p^{(t)}) \times \mathrm{S}_q$. $\square$

Lemma 2.6 *Suppose that* $\Gamma = \mathrm{Cay}(\mathbb{Z}_{pq}, \mathcal{S})$ *is a Cayley graph of* $\mathbb{Z}_{pq}$. *We set* $\mathbb{Z}_{pq} = \{(i,j) | i \in \mathbb{Z}_q, j \in \mathbb{Z}_p\}$. *Then*

(1) $\mathrm{Aut}(\Gamma)$ *is solvable if and only if either* Γ *is normal, or* $\Gamma \cong Y_n[Y_m]$ *and* $\mathrm{Aut}(Y_m)$, $\mathrm{Aut}(Y_n)$ *are solvable.*

(2) *If* $q \mid p-1$, *and* $R(\mathbb{Z}_{pq}) \trianglelefteq \mathrm{Aut}(\Gamma)$, *then* Γ *is a Cayley graph of* $\mathbb{F}_{pq}$ *if and only if* $q^2 \mid |\mathrm{Aut}(\Gamma)|$.

(3) *If* $R(\mathbb{Z}_{pq}) \trianglelefteq \mathrm{Aut}(\Gamma)$ *and* $q^2 \mid |\mathrm{Aut}(\Gamma)|$, *then* Γ *is nonnormal for* $\mathbb{F}_{pq}$ *if and only if* $(\mathrm{Aut}(\Gamma))' = R(\mathbb{Z}_{pq})$.

Proof (1) If Γ is normal then $\mathrm{Aut}(\Gamma) = R(\mathbb{Z}_{pq})\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})$. It follows that $\mathrm{Aut}(\Gamma)$ is solvable.

Conversely, if $\mathrm{Aut}(\Gamma)$ is solvable, then $\mathrm{Aut}(\Gamma)$ acts imprimitively on $V(\Gamma)$. Let $\Sigma = \{\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{n-1}\}$ be a complete block system and that K be the kernel of the action of $\mathrm{Aut}(\Gamma)$ on Σ. By Lemma 2.1 and Lemma 2.3, we may assume that K is faithful on every $\mathcal{B}_i$. Then the Sylow p-subgroup P of $\mathrm{Aut}(\Gamma)$ is normal in $\mathrm{Aut}(\Gamma)$. (In fact, if $|\mathcal{B}_i| = q$ and $n = p$, then $KP \trianglelefteq \mathrm{Aut}(\Gamma)$, and hence $P \trianglelefteq \mathrm{Aut}(\Gamma)$; if $|\mathcal{B}_i| = p$, then P is the Sylow p-subgroup of K, so $P \trianglelefteq \mathrm{Aut}(\Gamma)$.) Then the orbits of P on $V(\Gamma)$

are blocks of $\mathrm{Aut}(\Gamma)$ of length p. So we may assume, without loss of generality, that $\mathcal{B}_i = \{(i,j) | j \in \mathbb{Z}_p\}$ and that $P = \langle R((0,1))\rangle$. Then $P \le K$. Set $H = \langle R((1,0))\rangle$; we have

$$C_A(P) \cap KH = H(C_A(P) \cap K) = HP = R(\mathbb{Z}_{pq}) \Rightarrow R(\mathbb{Z}_{pq}) \trianglelefteq \mathrm{Aut}(\Gamma).$$

(2) Suppose that $q \mid p-1$ and $R(\mathbb{Z}_{pq}) \trianglelefteq \mathrm{Aut}(\Gamma)$. Then $\mathrm{Aut}(\Gamma)$ has a regular subgroup isomorphic to $\mathbb{F}_{pq}$ if and only if $q \mid |\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})|$ if and only if $q^2 \mid |\mathrm{Aut}(\Gamma)|$. Obviously, if $\mathrm{Aut}(\Gamma)$ has noncyclic regular subgroup, then $q^2 \mid |\mathrm{Aut}(\Gamma)|$. Conversely, we can construct a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to $\mathbb{F}_{pq}$. Let P, K be as in (1). Then $q \mid |K|$. Let γ be an element of order q in K, and let α, β be elements of order p and q respectively in $R(\mathbb{Z}_{pq})$. Then $\langle \alpha, \beta\gamma\rangle$ is a regular subgroup isomorphic to $\mathbb{F}_{pq}$.

(3) Suppose that G is a noncyclic regular subgroup of $\mathrm{Aut}(\Gamma)$ of order pq. Then $P \le G = PQ$, where Q is a Sylow q-subgroup of G. We may, without loss of generality, assume that $Q \le \langle\beta\rangle\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})$. Let H be a Hall subgroup of $\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})$ such that the prime divisors of $|H|$ are not less than q and that all prime divisors of $|\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S}) : H|$ are less than q. Then $q \mid |H|$, $H \le K$ and $Q \le \langle\beta\rangle H$. It is easy to see that $[\langle\beta\rangle, H] = 1$ and $C_{\mathrm{Aut}(\Gamma)}(P) \cap H = 1$. Then $G \trianglelefteq \mathrm{Aut}(\Gamma)$ if and only if $Q \trianglelefteq \langle\beta\rangle\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})$. Assume that $Q = \langle \beta^i\gamma^j\rangle$ for some $\gamma \in H$. Then $i, j \not\equiv 1 (\mathrm{mod}\ q)$. If $(\mathrm{Aut}(\Gamma))' \ne R(\mathbb{Z}_{pq})$, then $(\mathrm{Aut}(\Gamma))' = P = \langle\alpha\rangle$ or $\langle\beta\rangle$. In the latter case, $H \le C_{\mathrm{Aut}(\Gamma)}(P)$, a contradiction. So $\langle\beta\rangle\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})$ is an abelian group. It implies that $G \trianglelefteq \mathrm{Aut}(\Gamma)$. Conversely, $(\mathrm{Aut}(\Gamma))' = R(\mathbb{Z}_{pq})$ implies that $\langle\beta\rangle\mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S})$ is nonabelian. Let $\eta \in \mathrm{Aut}(\mathbb{Z}_{pq}, \mathcal{S}) \setminus C_{\mathrm{Aut}(\mathbb{Z}_{pq},\mathcal{S})}(\beta)$. Suppose that $\beta^\eta = \beta^l$. Then $\eta \notin N_{\langle\beta\rangle\mathrm{Aut}(\mathbb{Z}_{pq},\mathcal{S})}(Q)$. In fact,

$$Q^\eta = \langle \beta^{li}\gamma^j\rangle = Q \Rightarrow \beta^{li}\gamma^j = \beta^{ki}\gamma^{kj} \Rightarrow \beta^{(l-k)i} = \gamma^{(k-1)j} = 1$$

$$\Rightarrow q \mid (l-k)i, q \mid (k-1)j \Rightarrow q \mid (l-1),$$

a contradiction. So G is not normal in $\mathrm{Aut}(\Gamma)$. □

Example 2.1. Let $T = PSL(2,11)$. We consider an imprimitive action of T on a set Ω of 55 elements. By the proof of [10, Lemma 4.6], we know the following facts:

(1) T has suborbits of length $1, 4, 4, 4, 6, 12, 12, 12$.

(2) The orbital graphs, say $\Gamma_4^{(1)}$ and $\Gamma_4^{(2)}$, associated with two suborbits of length 4 are disconnected.

(3) Two suborbits of length 12 are not self-paired. We use $\Gamma_{12}^{(1)}$ and $\Gamma_{12'}^{(1)}$ to denote the corresponding orbital graphs.

(4) The rest of the suborbits of length 4, 6 and 12 are suborbits of $PGL(2,11)$ acting on Ω. We denote three corresponding orbital graphs as Γ_4, Γ_6 and Γ_{12}.

(5) $\Gamma_8 = \Gamma_4^{(1)} \cup \Gamma_4^{(2)}$, $\Gamma_{24} = \Gamma_{12}^{(1)} \cup \Gamma_{12'}^{(1)}$ are orbital graphs of $PGL(2,11)$.

Since $PGL(2,11)$ is primitive on Ω, it follows that an imprimitive vertex-transitive graph admitting T is necessarily one of $\Gamma_4^{(1)} \cup \Gamma$, $(\Gamma_4^{(1)} \cup \Gamma)^c$, where Γ is an edge-disjoint union of at least one of Γ_4, Γ_6, Γ_{12}, and Γ_{24}. Since T has a noncyclic subgroup of order 55, $\Gamma_4^{(1)} \cup \Gamma$, $(\Gamma_4^{(1)} \cup \Gamma)^c$ are Cayley graphs of $\mathbb{F}_{5\cdot 11}$. Their full automorphism groups are $PSL(2,11)$. So they are nonnormal. $\square$

Example 2.2. Consider the imprimitive action of $T = PSL(3,2) \cong PSL(2,7)$ on a set Ω of 21 elements. Then T has only blocks of length 3. Let $\mathcal{B}$ be a block of length 3, and let $x \in \mathcal{B}$. Then $T_{\mathcal{B}} = \mathrm{S}_4$ and $T_x = \mathrm{D}_8$. By the proof of [12, Lemma 2.3], we know that T has suborbits of lengths $1, 2, 2, 4, 4$ and 8. The orbital graphs associated with two suborbits of length 2 are disconnected. The union of two suborbits of length 2 is a suborbit of $PGL(2,7)$ on Ω. We denote this two orbital graphs as $\Gamma_2^{(1)}$ and $\Gamma_2^{(2)}$. Additionally, the suborbits of length of 4 are not self-paired, and the union of them is a suborbit of $PGL(2,7)$ acting on Ω. Since $PGL(2,7)$ acts primitively on Ω, so an imprimitive vertex-transitive graph is necessarily one of $\Gamma_2^{(i)} \cup L_3(2)_{21}^{8}$, $\Gamma_2^{(i)} \cup L_3(2)_{21}^{8'}$ and $\Gamma_2^{(i)} \cup L_3(2)_{21}^{8} \cup L_3(2)_{21}^{8'}$, $i = 1, 2$, where $L_3(2)_{21}^{8}$, $L_3(2)_{21}^{8'}$ are as in [10]. Their full automorphism groups are $PSL(3,2)$, and hence they are nonnormal Cayley graphs of $\mathbb{F}_{3\cdot 11}$. Since $PSL(3,2)$ has no elements of order 21, they are not Cayley graphs of $\mathbb{Z}_{21}$. $\square$

3 Main Result

In this section, we shall prove the following theorem:

Theorem 3.1 *Suppose that p, q are distinct primes such that $p > q \geq 3$. Let Γ be a nonnormal Cayley graph of some group G of order pq. Suppose further that $A = \mathrm{Aut}(\Gamma)$ acts imprimitively on $V(\Gamma)$. Then Γ is one of the following graphs:*

(1) *$Y_p[Y_q]$, $A = \mathrm{Aut}(Y_q) \wr \mathrm{Aut}(Y_p)$, for $\mathbb{Z}_{pq}$, and for $\mathbb{F}_{pq}$ when $q \mid p-1$ and $q \mid |\mathrm{Aut}(Y_p)|$;*

(1') *$Y_q[Y_p]$, $A = \mathrm{Aut}(Y_p) \wr \mathrm{Aut}(Y_q)$, for $\mathbb{Z}_{pq}$, and for $\mathbb{F}_{pq}$ when $q \mid p-1$;*

(2) *$\Gamma(q, p; \mathcal{S}_q^{(t)}, 3)$ and $\Gamma(q, p; \mathcal{S}_q^{(t)}, 3)^c$, $A \geq \mathbb{Z}_q \times S_p$, for $\mathbb{Z}_{pq}$, and for $\mathbb{F}_{pq}$ when $q \mid p-1$;*

(2') *$\Gamma(p, q; \mathcal{S}_p^{(t)}, 3)$ and $\Gamma(p, q; \mathcal{S}_p^{(t)}, 3)^c, q > 3$,*

$$\mathrm{Aut}(\Gamma(p, q; \mathcal{S}_p^{(t)}, 3)) = \bigcap_{t=1}^{3} \mathrm{Aut}(Y_p^{(t)}) \times S_q,$$

for $\mathbb{Z}_{pq}$, and for $\mathbb{F}_{pq}$ when $q \mid p-1$ and $q \mid |\bigcap_{t=1}^{3} \mathrm{Aut}(Y_p^{(t)})|$;

(3) *Γ is a normal Cayley graph of $\mathbb{Z}_{pq}$, $q^2 \mid |\mathrm{Aut}(\Gamma)|$, $(\mathrm{Aut}(\Gamma))' = R(\mathbb{Z}_{pq})$, for $\mathbb{F}_{pq}$;*

(4) *graphs in* Example 2.1, *$A = PSL(2,11)$, for $\mathbb{F}_{5\cdot 11}$;*

(5) *graphs in* Example 2.2, $A = PSL(3,2)$, *for* $\mathbb{F}_{3\cdot 7}$;

where Y_p, Y_q *are as in* Lemma 2.3(2), $\mathcal{S}_q^{(t)}, \mathcal{S}_p^{(t)}$, $Y_q^{(t)}$ *and* $Y_p^{(t)}$ *are as in* Lemma 2.5.

Proof If Γ is disconnected, then Γ is one of the graphs described in Corollary 2.4 when $\mathrm{Y}_n = n\mathrm{K}_1$. Suppose that Γ is connected. Then we have two cases:

Case 1. A has a block $\mathcal{B}$ of length p.

Let Σ be the complete block system containing $\mathcal{B}$, and let K be the kernel of A acting on Σ. If K acts unfaithfully on some block or K is insolvable, then Γ is one of the graphs described in Corollary 2.4 and Lemma 2.5. So we may assume that K is solvable and that K acts faithfully on each block $\mathcal{B} \in \Sigma$. Let P be the Sylow p-subgroup of K. Then $\langle R(a)\rangle \leq P \trianglelefteq A$, and we have $A/C_A(P) \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$. It follows that $A' \leq C_A(P)$.

First, we assume that A is solvable. If $G \cong \mathbb{Z}_{pq}$, then Γ is one of the graphs described in (1) and (1') by Lemma 2.6(1) and Corollary 2.4. So, we suppose that $G \ncong \mathbb{Z}_{pq}$. Let M/K be the normal Sylow q-subgroup of A/K. Then $M \trianglelefteq A$ and $M' \leq K \cap C_A(P) = P$. It follows that M/P is abelian. If q is not a divisor of $|K|$, then $R(G) \trianglelefteq A$, a contradiction. So $q \mid |K|$. Let Q be a Sylow q-subgroup of K such that $Q\langle R(b)\rangle$ is a Sylow subgroup of A. Since $A/C_A(P)$ is a cyclic group and $Q\langle R(b)\rangle$ is noncyclic, there exists an element $\alpha \in C_A(P)$ of order q, such that $P \times \langle \alpha\rangle$ is a transitive subgroup of A. So Γ is a normal Cayley graph of $\mathbb{Z}_{pq}$. By Lemma 2.6, $q^2 \mid |\mathrm{Aut}(\Gamma)|$, $(\mathrm{Aut}(\Gamma))' = R(\mathbb{Z}_{pq})$ and Γ is nonnormal for $\mathbb{F}_{pq}$.

Now, we suppose that A is insolvable. Then A/K is an insolvable permutation group of degree q. So $q > 3$. Obviously, $C_A(P)$ is insolvable and $K \cap C_A(P) = P$. It follows that $C_A(P) = P \times M$, where M is the p'-Hall subgroup of $C_A(P)$ and hence $M \trianglelefteq A$. Then M is insolvable and M acts faithfully on Σ. It is easy to know that M is not transitive on $V(\Gamma)$. It follows that the set of orbits of M makes a complete block system. Then Γ is one of the graphs described in Lemma 2.4.

Case 2. A has only blocks of length q.

As in *Case 1*, we may assume that K, the kernel of A acting on a complete block system Σ, acts faithfully on each block of length q and that K is solvable. Suppose that A is solvable. Let KP/K be the normal Sylow p-subgroup of $\overline{A} = A/K$. Then KP is normal in A. It follows that P is a normal subgroup of A. So A has blocks of length p, a contradiction. Therefore, A and hence A/K is insolvable. Then $\overline{A}$ is a 2-transitive permutation of degree p. We shall prove that A has an insolvable subgroup M such that $M \trianglelefteq A$, $M \cap K = 1$ and M acts 2-transitively on Σ.

If $K = 1$, then it is trivial. Suppose that $K \neq 1$. Let Q be the normal Sylow q-subgroup of K. Then $Q \trianglelefteq A$. Let K_x be the stabilizer of $x \in G$ in K. Then $K_x = A_x \cap K$, and $K_{xa^i} = K_x^{R(a^i)} = (K \cap A_x)^{R(a^i)} = K_x^{R(a^i)}$. Note that $p > q$ and all K_{xa^i} are conjugate in K. then $R(a) \in N_A(K_x)$. It implies that $R(a) \in C_A(K_x)$. Since $A/C_A(Q)$ is abelian, $R(a) \in C_A(Q)$. So $R(a) \in C_A(K)$. It follows that $C_A(K)$ is transitive on $V(\Gamma)$ and that $\mathrm{Soc}(\overline{A}) \leq KC_A(K)/K$. If $K_x \neq 1$, then $K \cap C_A(K) = 1$. Then $M = C_A(K)$ satisfies the conditions requested. If $K_x = 1$, then $K = Q$. Let $\mathcal{B}$ be the block containing x. Then $A_{\mathcal{B}} = KA_x$, $K \cap A_x = 1$. Since $(|A : A_{\mathcal{B}}|, |K|) = 1$, it follows from [7, I, Theorem 17.4] that K has a complement H in A. Let $M = \mathrm{Soc}(H)$. Then M is nonabelian simple group, $M \leq C_A(K)$, $M \trianglelefteq A$ and $M \cap K = 1$.

Let $T = \mathrm{Soc}(M)$. Then $T \trianglelefteq A$ and T acts transitively on $V(\Gamma)$. It follows that $|T : T_{\mathcal{B}}| = p, |T_{\mathcal{B}} : T_x| = q$. Since $\mathrm{A}_p (p > 5)$, M_{11}, M_{23}, and $PSL(2, 11)$ (when $p = 11, q = 3$) have no such subgroups, so, by the classification of 2-transitive permutation group of degree p (see [2]), T is one of the following groups:

(1) $PSL(2, 11)$, $p = 11, q = 5$;

(2) $PSL(2, 2^{2^s})$, $p = 2^{2^s} + 1$;

(3) $PSL(d, k)$, $d \geq 3$, k is a prime power, $p = (k^d - 1)/(k - 1)$, d is an odd prime, $(d, k - 1) = 1$.

We first assume that $T = PSL(2, 11)$. Then $T_{\mathcal{B}} = \mathrm{A}_5$ and $A_x = \mathrm{A}_4$. Since A_4 is self-normalizing in T, T_x fixes only one point x, so the centralizer of T in A is trivial, whence $K = 1$. Then Γ is one of graphs described in Example 2.1.

Now we begin to deal with (2) and (3). Let $V = (d, k)$ be the underlying n-dimensional vector space over $GF(k)$, so that T acts on Σ as on the set of one-dimensional subspaces. We identify Σ with this set. Let $\mathcal{B}$ be the block identified with $\langle v \rangle$, where $v = (1, 0, \ldots, 0)'$.

Suppose that $T = PSL(2, 2^{2^s})$. Since $q \geq 3$, q is not a divisor of $p - 1$. So $G \cong \mathbb{Z}_{pq}$. It follows that $K \neq 1$. Since $|A_{\mathcal{B}}| = 2^{2^s}(2^{2^s} - 1)$, $q \mid p - 2$. It follows from [9, Lemma 2.3, Proposition 4.1] that T has q suborbits of length 1 and q suborbits of length $p - 1$, and that $\Gamma \cong \mathrm{X}(2^s, q, \mathcal{S}_1, \mathcal{S}_2)$, where $\mathcal{S}_1$ is a Cayley subset of $\mathbb{Z}_q \backslash \{0\}$ and $\mathcal{S}_2 \subseteq \mathbb{Z}_q$. By Proposition 1 of [7], $\Gamma \cong \mathrm{X}(2^s, q, \mathcal{S}_1, \varnothing)$, or $\mathrm{X}(2^s, q, \mathcal{S}_1, \varnothing)$. This implies that $\Gamma \cong p\mathrm{Y}_q$, or $\mathrm{K}_p[\mathrm{Y}_q]$.

Finally, we assume that $T = PSL(d, k)$. Then $T_{\mathcal{B}} = O_k \cdot GL(d - 1, k)$, where O_k is an elementary abelian group of order k^{d-1}. Let $M = O_k \cdot SL(d - 1, k)$ be a normal subgroup of $T_{\mathcal{B}}$ with quotient group $\mathbb{Z}_{k-1}$.

Subcase 1. $SL(d - 1, k)$ is insolvable.

Suppose that M is transitive on $\mathcal{B}$. Since no proper normal subgroup of M involves $PSL(d - 1, k)$, then $M^{\mathcal{B}}$ has $PSL(d - 1, k)$ as a composition factor. So $M^{\mathcal{B}}$ is an insolvable transitive group of prime degree q. It follows that either $d = 3, k = 5$, $M^{\mathcal{B}} \cong \mathrm{A}_5$ and $q = 5$, or $M^{\mathcal{B}} \cong PSL(d - 1, k)$ and $q = (k^{d-1} - 1)/(k - 1)$, whence both d and $d - 1$ are primes, so $d = 3$, $q = k + 1$, and $q > 3$, since $PSL(d - 1, k)$ is insolvable. In the former case, T_x is transitive on $V(\Gamma) \backslash \mathcal{B}$. It follows that Γ is a lexicographic product. In the latter case, $k = 2^{2^s}$ for some $s \geq 1$. But $p = k^2 + k + 1$ is not a prime, a contradiction.

Suppose that M fixes $\mathcal{B}$ pointwise; then $q \mid k - 1$. For $x \in \mathcal{B}$, we have $T_x = O_k \cdot SL(d - 1, k) \cdot \mathbb{Z}_{(k-1)/q}$. It follows that T_x is trivial on $\mathcal{B}$. By the proof of [10, Lemma 4.7], we know that T_x acts transitively on $V(\Gamma) \backslash \mathcal{B}$. It follows that Γ is a lexicographic product.

Subcase 2. $SL(d - 1, k)$ is solvable. Then $d = 3$ and $k = 2$, or 3, and hence $T = PSL(3, 2)$, or $PSL(3, 3)$.

If $T = PSL(3, 2)$, then $p = 7, q = 3$. It follows that $T_{\mathcal{B}} = \mathrm{S}_4$, $T_x = \mathrm{D}_8$. Since D_8 is self-normalizing in T, so T_x fixes only point x. It follows that $C_A(T) = 1$, and hence $K = 1$. Then Γ is one of the graphs described in Example 2.2.

If $T = PSL(3,3)$, then $p = 13, q = 3$. By [12, Lemma 4.9], we know that T has suborbits of lengths $1, 2, 36$. It is easy to see that the suborbit of length 2 corresponds to the graph $13\mathrm{K}_3$. It follows that $\Gamma \cong \mathrm{K}_{13}[3\mathrm{K}_1]$. But in this case, $\mathrm{Aut}(\Gamma) = \mathrm{S}_3 \wr \mathrm{S}_{13}$ has no normal subgroup isomorphic to $PSL(3,3)$. So $T \ncong PSL(3,3)$. This completes the proof of the theorem. □

Remark: Sometime, it is not necessary to know the full automorphism group A to prove a Cayley graph Γ to be nonnormal. For example, in Theorem 3.1(2), it is enough for us to know $A \geq \mathbb{Z}_q \times \mathrm{S}_p$. However, we can calculate the full automorphism group of $\Gamma(q, p; \mathcal{S}_q^{(t)}, 3)$. In fact, a similar argument as in the proof of Lemma 2.5(2) leads to $\mathrm{Aut}(\Gamma(q, p; \mathcal{S}_q^{(t)}, 3)) = \bigcap_{t=1}^{3} \mathrm{Aut}(\mathrm{Y}_q^{(t)}) \times \mathrm{S}_p$. □

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References

[1] B. Alspach and M. Y. Xu, $\frac{1}{2}$-transitive graphs of order $3p$, *J. Algebraic Combin.* **3** (1994), 347–355.

[2] P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.

[3] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, *J. Combin. Theory ser. B* **4** (1987), 196–211.

[4] J. D. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, New York/Berlin, 1996.

[5] Shao-Fei Du, Ru-Ji Wang and Ming-Yao Xu, On the normality of Cayley digraphs of order twice a prime, *Australas. J. Combin.* **18** (1998), 227–234.

[6] B. D. Mckay and C. E. Praeger, Vertex-transitive graphs that are not Cayley graphs, II, *J. Graph Theory*, **22** (1996), 321–334.

[7] D. Marusic and R. Scapellato, Imprimitive representation of $SL(2, 2^k)$, *J. Combin. Theory ser. B* **58** (1993), 46–57.

[8] C. E. Praeger, R. J. Wang and M. Y. Xu, Symmetric graphs of order a product of two disdinct primes, *J. Combin. Theory ser. B* **58** (1993), 299–318.

[9] C. E. Praeger and M. Y. Xu, Vertex primitive graphs of order a product of two distinct primes, *J. Combin. Theory ser. B* **59** (1993), 245–266.

[10] R. J. Wang and M. Y. Xu, A classification of symmetric graphs of order $3p$, *J. Combin. Theory ser. B* **58** (1993), 179–216.

[11] Ming-Yao Xu, Automorphism groups and isomorphisms of Cayley graphs, *Discrete Mathematics* **182** (1998), 309–319.

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