Kernel operators and improved inclusion-exclusion bounds

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Abstract

We present a new and elementary proof of some recent improvements of the classical inclusion-exclusion bounds. The key idea is to use an injective mapping, similar to the bijective mapping in Garsia and Milne's "bijective" proof of the classical inclusion-exclusion principle.

1 Introduction

Probabilists and statisticians frequently use the classical inclusion-exclusion truncation bounds to approximate the probability of a union of finitely many events. The general result, first discovered by Ch. Jordan [8] and later by Bonferroni [1], states that for any finite family of sets $\{A_v\}_{v\in V}$ and any $r \in \mathbb{N}$,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\leq\; \sum_{\substack{I\in\mathcal{P}^*(V)\\|I|\leq r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}), \tag{1}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\geq\; \sum_{\substack{I\in\mathcal{P}^*(V)\\|I|\leq r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}), \tag{2}$$

where $\mathcal{P}^*(V)$ denotes the set of non-empty subsets of V, and where for any set A, $\chi(A)$ denotes the indicator function of A, that is, $\chi(A)(\omega) = 1$ if $\omega \in A$, and $\chi(A)(\omega) = 0$ if $\omega \notin A$. These bounds are usually referred to as *Bonferroni bounds* or *inclusion-exclusion bounds*. Note that there is no real restriction in using indicator functions rather than measures, since both sides of the inequalities can be integrated with respect to any finite measure μ (e.g., a probability measure) on any σ-field containing the sets A_v, $v \in V$.

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In recent years, a lot of work has been done on improving these inclusion-exclusion bounds. These improvements are usually of the form

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{\substack{I\in\mathcal{S}\\|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}), \tag{3}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\ge\; \sum_{\substack{I\in\mathcal{S}\\|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}), \tag{4}$$

where $\mathcal{S}$ is a restricted set of non-empty subsets of V, and where (3) and (4) are at least as sharp as their classical counterparts (1) and (2), see e.g., [2, 3, 6, 7, 9, 10, 11]. Usually, these improved inclusion-exclusion bounds require the collection of sets to satisfy some structural restrictions. Examples of such well-structured collections of sets arise in some problems of statistical inference [9, 10], reliability theory [2, 6, 7], and chromatic graph theory [2].

2 Improved bounds via kernel operators

The results in this section require the concept of a kernel operator (cf. [4]).

Definition 2.1 Let V be a set. A *kernel operator* on V is a mapping k from the power set of V into itself such that for all subsets X and Y of V,

$$\begin{array}{rl} \text{(i)} & k(X)\subseteq X \quad \text{(intensionality)},\\ \text{(ii)} & X\subseteq Y \Rightarrow k(X)\subseteq k(Y) \quad \text{(monotonicity)},\\ \text{(iii)} & k(k(X)) = k(X) \quad \text{(idempotence)}. \end{array}$$

A subset X of V is called *k-open* if $k(X) = X$.

There is a well-known correspondence between kernel operators on V and union-closed subsets of the power set of V. Namely, if k is a kernel operator on V, then the set of k-open subsets of V is union-closed. On the other hand, if a set $\mathcal{X}$ of non-empty subsets of V is union-closed, then

$$k(I) := \bigcup\{X\in\mathcal{X}\,|\,X\subseteq I\} \quad (I\subseteq V)$$

defines a kernel operator on V such that X is k-open if and only if $X\in\mathcal{X}$. Thus, the following results may be formulated in terms of union-closed sets as in [3]. As mentioned in [3], the following theorem subsumes several known results in the area and has applications to chromatic graph theory and reliability theory. Note that when $k(I)=\emptyset$ for every subset I of V, the theorem agrees with the classical inclusion-exclusion bounds (1) and (2).

Theorem 2.2 [3] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k be a kernel operator on V such that for any non-empty and k-open subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v .$$

Then, for any $r \in \mathbb{N}$,

$$\chi\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r\ \textit{even}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r\ \textit{odd}).$$

The results in [3] immediately imply that the inclusion-exclusion bounds associated with k' are at least as sharp as those associated with k if both k and k' are as required in Theorem 2.2 and $k' \leq k$, where $\leq$ is defined by

$$k' \leq k \quad :\Leftrightarrow \quad k(I) \subseteq k'(I) \text{ for any subset } I \text{ of } V \tag{5}$$

or equivalently,

$$k' \leq k \quad :\Leftrightarrow \quad \text{all } k\text{-open subsets of } V \text{ are } k'\text{-open}. \tag{6}$$

In particular, since the kernel operator $I \mapsto \emptyset$ on V is largest with respect to this partial order, the improved bounds are at least as sharp as their classical counterparts. The following theorem makes this precise for general k and k'.

Theorem 2.3 *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k and k' be kernel operators on V such that $k' \leq k$ with respect to (5) or (6) and such that for any non-empty and k'-open subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v .$$

Then, for any $r \in \mathbb{N}$,

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \geq \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r\ \textit{even}),$$

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \leq \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r\ \textit{odd}).$$

3 Proofs

In this section, we present new proofs for the results of the preceding section. In contrast to the original proofs in [3], the proofs presented here are elementary and do not require knowledge of abstract tube theory or combinatorial topology. The key ingredient in these proofs is an injective mapping similar to the bijective mapping in Garsia and Milne's proof of the classical inclusion-exclusion principle [5, 12]. In the literature (see e.g., [12]) Garsia and Milne's proof is often referred to as a "bijective" proof as the key idea in their proof rests upon bijective mapping. We adopt this terminology and refer to our new proofs as "injective" proofs.

Proof of Theorem 2.2. It suffices to prove that

$$\chi\left(\bigcup_{v\in V} A_v\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I| \text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\ge\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I| \text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r \text{ even}), \tag{7}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I| \text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\le\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I| \text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r \text{ odd}). \tag{8}$$

For any $\omega \in \bigcup_{v\in V} A_v$ and any $r \in \mathbb{N}$ define

$$\begin{aligned}
\mathcal{E}_r(\omega) &:= \left\{ I \in \mathcal{P}(V_\omega) \,\middle|\, k(I)=\emptyset,\, |I|\le r,\, |I| \text{ even} \right\},\\
\mathcal{O}_r(\omega) &:= \left\{ I \in \mathcal{P}(V_\omega) \,\middle|\, k(I)=\emptyset,\, |I|\le r,\, |I| \text{ odd} \right\},
\end{aligned}$$

where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$. Obviously, (7) and (8) are equivalent to

$$|\mathcal{E}_r(\omega)| \;\ge\; |\mathcal{O}_r(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v \quad (r \text{ even}), \tag{9}$$

$$|\mathcal{E}_r(\omega)| \;\le\; |\mathcal{O}_r(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v \quad (r \text{ odd}). \tag{10}$$

To prove (9) and (10), fix $\omega \in \bigcup_{v\in V} A_v$. The definition of V_ω and the requirements of the theorem imply that V_ω is not k-open. Thus, some $v \in V_\omega \setminus k(V_\omega)$ can be chosen. It follows that for any subset I of V_ω, $v \notin k(I\cup\{v\})$ since otherwise $v \in k(I\cup\{v\}) \subseteq k(V_\omega\cup\{v\}) = k(V_\omega)$, contradicting $v \notin k(V_\omega)$. Since $v \notin k(I\cup\{v\})$ and $k(I\cup\{v\}) \subseteq I\cup\{v\}$ we obtain $k(I\cup\{v\}) \subseteq I$ and hence, $k(I\cup\{v\}) \subseteq k(I)$. From the latter we conclude that for any subset I of V_ω, $k(I)=\emptyset \Rightarrow k(I\cup\{v\})=\emptyset$. Hence, $I \mapsto I \,\triangle\, \{v\}$, where $\triangle$ denotes symmetric difference, is an injective mapping from $\mathcal{O}_r(\omega)$ into $\mathcal{E}_r(\omega)$ if r is even, and an injective mapping from $\mathcal{E}_r(\omega)$ into $\mathcal{O}_r(\omega)$ if r is odd. $\square$

Remark. Note that our proof of Theorem 2.2 is new even in the traditional case where $k(I)=\emptyset$ for any subset I of V.

We finally present our new proof of Theorem 2.3:

Proof of Theorem 2.3. It suffices to prove that

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\ge\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right),$$

if r is even, and

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\le\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right),$$

if r is odd. Since $k' \le k$ these inequalities are equivalent to

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\ge\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r \text{ even}), \tag{11}$$

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\le\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r \text{ odd}). \tag{12}$$

For any $\omega \in \bigcup_{v\in V} A_v$ and any $r \in \mathbb{N}$ define

$$\begin{aligned} \mathcal{E}_r^*(\omega) &:= \left\{ I \in \mathcal{P}^*(V_\omega) \,\middle|\, k(I)=\emptyset,\, k'(I)\neq\emptyset,\, |I|\le r,\, |I| \text{ even} \right\}, \\ \mathcal{O}_r^*(\omega) &:= \left\{ I \in \mathcal{P}^*(V_\omega) \,\middle|\, k(I)=\emptyset,\, k'(I)\neq\emptyset,\, |I|\le r,\, |I| \text{ odd} \right\}, \end{aligned}$$

where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$. Evidently, (11) and (12) are equivalent to

$$|\mathcal{E}_r^*(\omega)| \;\ge\; |\mathcal{O}_r^*(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v \quad (r \text{ even}), \tag{13}$$

$$|\mathcal{E}_r^*(\omega)| \;\le\; |\mathcal{O}_r^*(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v \quad (r \text{ odd}). \tag{14}$$

Now, in order to establish (13) and (14), fix some $\omega \in \bigcup_{v\in V} A_v$ and choose some arbitrary $v \in V_\omega \setminus k'(V_\omega)$. Since $k' \le k$ it follows that $v \in V_\omega \setminus k(V_\omega)$. By similar arguments as in the second proof of Theorem 2.2 it follows that for any subset I of V_ω, $k(I) = \emptyset \Rightarrow k(I \cup \{v\}) = \emptyset$ as well as $k'(I) \neq \emptyset \Rightarrow k'(I \setminus \{v\}) \neq \emptyset$. Therefore, $I \mapsto I \vartriangle \{v\}$, where $\vartriangle$ denotes symmetric difference, is an injective mapping from $\mathcal{O}_r^*(\omega)$ into $\mathcal{E}_r^*(\omega)$ if r is even, and an injective mapping from $\mathcal{E}_r^*(\omega)$ into $\mathcal{O}_r^*(\omega)$ if r is odd. Hence, the result. $\square$

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