Magical coronations of graphs*

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Abstract

A (p,q) graph G is called edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(v) + f(uv)$ is constant for any edge uv of G. Moreover, G is said to be super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$. Every super edge-magic (p,q) graph is harmonious, sequential and felicitous whenever it is a tree or satisfies $q \geq p$. In this paper, we prove that the n-crown, a cycle with n pendant edges attached at each vertex, is super edge-magic for any positive integer n, and thus extend what was known about the harmoniousness, sequentialness and felicitousness of such graphs. We also present three results on attaching pendant edges to the vertices of certain super edge-magic graphs to obtain more super edge-magic graphs.

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1 Introduction

In this paper, the authors extend what is known in the literature about vertex labelings of unicyclic graphs. Specifically, in our main result, we show that a cycle with n pendant edges attached at each vertex, known as an n-crown, is super edge-magic, harmonious, sequential and felicitous. We also present three results that illustrate how attaching pendant edges to the vertices of certain super edge-magic graphs will result in more super edge-magic graphs.

For most of the graph theory terminology and notation used, we follow Chartrand and Lesniak [1] throughout this paper. In particular, we mean a graph to be finite and simple, that is, without allowing loops or multiple edges. Furthermore, the vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

For a (p,q) graph G, a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ is an *edge-magic labeling* of G if $f(u) + f(v) + f(uv) = k$ is a constant, which is independent on the choice of any edge uv of G. If such a labeling exists, then the constant k is called the *valence* of f, and G is said to be an *edge-magic graph.* Moreover, f is called a *super edge-magic labeling* if $f(V(G)) = \{1, 2, \ldots, p\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

The concepts presented above were first introduced and studied by Kotzig and Rosa [8, 9], and Enomoto, Lladó, Nakamigawa and Ringel [2]. The edge-magic labelings were previously named magic valuations by Kotzig and Rosa [8]. For other results on this subject, the reader is referred to [3, 4].

The following result found in [3] provides a necessary and sufficient condition for a graph to be super edge-magic.

Lemma 1.1 *A (p,q) graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) \,|uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and*

$$\begin{aligned} S &= \{f(u) + f(v) \,|uv \in E(G)\} \\ &= \{k - (p+1), k - (p+2), \ldots, k - (p+q)\}. \end{aligned}$$

In 1980, Graham and Sloane [7] initiated the study of harmonious labelings. A (p,q) graph G with $q \geq p$ is *harmonious* if there exists an injective function $f : V(G) \rightarrow \mathbb{Z}_q$ such that each edge uv of G is labeled $f(u) + f(v) \pmod q$ and the resulting edge labels are distinct. Such a function is called a *harmonious labeling.* If G is a tree (so that $q = p - 1$) exactly two vertices are labeled the same; otherwise, the definition is the same.

The notion of sequential labelings was introduced by Grace [6] who was inspired by the definition of harmonious labelings. For a (p,q) graph G, an injective function $f : V(G) \rightarrow \{0, 1, \ldots, q-1\}$ is a *sequential labeling* of G if each edge uv of G is labeled $f(u) + f(v)$ and the resulting edge labels are $\{m, m+1, \ldots, m+q-1\}$ for some positive integer m. If such a labeling exists, then G is said to be *sequential.* In the case of a tree, Grace permits the vertex labels to range from 0 to q with no vertex label used twice.

We now consider a graph labeling introduced by Shee [10] that is similar in nature to a harmonious labeling. A graph G of size q is *felicitous* if there exists an injective function $f : V(G) \to \mathbb{Z}_{q+1}$ such that each edge uv of G is labeled $f(u) + f(v) \pmod{q}$ and the resulting edge labels are distinct. Such a function is called a *felicitous labeling.*

In [3], the authors proved the following result.

Lemma 1.2 *Every super edge-magic (p, q) graph G is harmonious and sequential whenever G is a tree or satisfies $q \geq p$.*

Thus, we obtain the following relationship between super edge-magic labelings and felicitous labelings since every harmonious labeling is certainly a felicitous labeling.

Lemma 1.3 *If a (p, q) graph G is super edge-magic, then G is felicitous whenever it is a tree or satisfies $q \geq p$.*

Throughout this paper, we will utilize the *corona product* $G_1 \odot G_2$ of two graphs G_1 and G_2 defined as the graph obtained by taking one copy of G_1 (which has order p_1) and p_1 copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2. The graph $C_m \odot \overline{K}_n$ is called the *n-crown* with cycle length m. We will refer to the 1-crown as the *crown* with cycle length m. For the sake of brevity, we will refer to the n-crown with cycle length m simply as the n-crown if its cycle length is clear from the context.

2 Coronations of Some Super Edge-Magic Graphs

In this section, we first provide a construction that shows that $G \odot \overline{K}_n$ is super edge-magic whenever G is a graph of odd order at least 3 and admits certain super edge-magic labelings.

Theorem 2.1 *Let G be a graph of odd order $p \geq 3$ for which there exists a super edge-magic labeling f with the property that*

$$\max\{\, f(u) + f(v) |\, uv \in E(G)\} = \frac{3p+1}{2}.$$

Then $G \odot \overline{K}_n$ is super edge-magic for every positive integer n.

Proof. Let f be a super edge-magic labeling of G with valence k, and assume that f has the property that $f(v_i) = i$ for every integer i with $1 \leq i \leq p$, where $V(G) = \{v_i \,|1 \leq i \leq p\}$. Further, let

$$V(G \odot \overline{K}_n) = V(G) \cup \left\{ w_i^j \middle| \, 1 \leq i \leq p \text{ and } 1 \leq j \leq n \right\}$$

and

$$E(G \odot \overline{K}_n) = E(G) \cup \left\{ v_i w_i^j \middle| \, 1 \leq i \leq p \text{ and } 1 \leq j \leq n \right\}.$$

Now, define the vertex labeling $g : V(G \odot \overline{K}_n) \to \{1, 2, \ldots, p(n+1)\}$ such that $g(v) = f(v)$ for every vertex v of G, and

$$g\left(w_i^j\right) = \begin{cases} p + i + \frac{p(2j-1)+1}{2}, & \text{if } 1 \leq i \leq \frac{p-1}{2} \text{ and } 1 \leq j \leq n; \\ i + \frac{p(2j-1)+1}{2}, & \text{if } \frac{p+1}{2} \leq i \leq p \text{ and } 1 \leq j \leq n. \end{cases}$$

To show that g extends to a super edge-magic labeling of $G \odot \overline{K}_n$, consider the set

$$S_i^j = \left\{ f(v_i) + f\left(w_i^j\right) \middle| \, 1 \leq i \leq p \text{ and } 1 \leq j \leq n \right\},$$

and let

$$m_j = \min\left\{ S_i^j \middle| \, 1 \leq i \leq p \right\} = p + 1 + \frac{p(2j-1)+1}{2},$$

$$M_j = \min\left\{ S_i^j \middle| \, 1 \leq i \leq p \right\} = 2p + \frac{p(2j-1)+1}{2}.$$

Finally, observe that $m_1 = (3p+1)/2 + 1$, $M_j + 1 = m_{j+1}$ $(1 \leq j \leq n-1)$, and S_i^j is a set of consecutive integers $(1 \leq i \leq p$ and $1 \leq j \leq n)$, which implies that g extends to a super edge-magic labeling of $G \odot \overline{K}_n$ with valence $k + 2np$. ■

By Lemmas 1.2 and 1.3, we have the following corollary.

Corollary 2.2 *Let G be a graph of odd order $p \geq 3$ for which there exists a super edge-magic labeling f with the property that*

$$\max\{ f(u) + f(v) |\, uv \in E(G)\} = \frac{3p+1}{2}.$$

Then $G \odot \overline{K}_n$ is harmonious, sequential and felicitous for every positive integer n.

We now provide a similar, though in a sense weaker, result for graphs with even order at least 4.

Theorem 2.3 *Let G be a graph of even order $p \geq 4$ having a super edge-magic labeling f with the property that*

$$\max\{ f(u) + f(v) |\, uv \in E(G)\} = \frac{3p}{2}.$$

Then the graph H obtained by attaching n pendant edges to each vertex of G except the vertex v with $f(v) = p$ is super edge-magic for every positive integer n.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$, then take a super edge-magic labeling f of G with valence k satisfying the property that $f(v_i) = i$ for $i = 1, 2, \ldots, p$. Now, define the graph H as follows:

$$V(H) = V(G) \cup \left\{ w_i^j \middle| \, 1 \leq i \leq p-1 \text{ and } 1 \leq j \leq n \right\}$$

and

$$E(H) = E(G) \cup \left\{ v_i w_i^j \middle| 1 \le i \le p-1 \text{ and } 1 \le j \le n \right\}.$$

Consequently, through an analogous argument to the one used in the proof of the previous theorem, the vertex labeling $g : V(H) \to \{1, 2, \ldots, p(n+1) - n\}$ such that $g(v) = f(v)$ for every vertex v of G, and

$$g\left(w_i^j\right) = \begin{cases} i + \frac{p}{2} + (p-1)j + 1, & \text{if } 1 \le i \le \frac{p}{2} - 1 \text{ and } 1 \le j \le n \\ i + \frac{p}{2} + (p-1)(j-1) + 1, & \text{if } \frac{p}{2} \le i \le p-1 \text{ and } 1 \le j \le n \end{cases}$$

extends to a super edge-magic labeling of H with valence $k + 2n(p-1)$. ■

Again, by Lemmas 1.2 and 1.3, we have the following corollary.

Corollary 2.4 *Let G be a graph of even order $p \ge 4$ having a super edge-magic labeling f with the property that*

$$\max\{ f(u) + f(v) |\, uv \in E(G)\} = \frac{3p}{2}.$$

Then the graph H obtained by attaching n pendant edges to each vertex of G except the vertex v with $f(v) = p$ is harmonious, sequential and felicitous for every positive integer n.

3 Super Edge-Magic Labelings of n-Crowns

We now proceed to study the super edge-magicness of n-crowns. To do this, we start with a result pertaining 2-regular graphs.

Theorem 3.1 *If G is a (super) edge-magic 2-regular graph, then $G \odot \overline{K}_n$ is (super) edge-magic for every positive integer n.*

Proof. Let f be a (super) edge-magic labeling of G with valence k. Assume that H is a component of $G \odot \overline{K}_n$. Then $H \cong C_r \odot \overline{K}_n$ for some integer $r \ge 3$. Let

$$V(H) = \{v_i \,|i \in \mathbb{Z}_r\} \cup \{u_{i,j} \,|i \in \mathbb{Z}_r \text{ and } 1 \le j \le n\}$$

and

$$E(H) = \{v_i v_{i+1} \,|i \in \mathbb{Z}_r\} \cup \{v_i u_{i,j} \,|i \in \mathbb{Z}_r \text{ and } 1 \le j \le n\},$$

where $\mathbb{Z}_r$ denotes the set of integers modulo r.

Then $f|_H$ extends to a labeling g of H as follows:

$$\begin{aligned} g(v_i) &= (n+1)f(v_i) - n, \\ g(v_{i-1}v_i) &= nf(v_{i-1}v_i), \\ g(u_{i,j}) &= (n+1)f(v_{i-1}) - n + j, \\ g(v_i u_{i,j}) &= nf(v_{i-1}v_i) - j, \end{aligned}$$

where $i \in \mathbb{Z}_r$ and $1 \leq j \leq n$. Therefore, f extends likewise in every component of $G \odot \overline{K}_n$, and a (super) edge-magic labeling of $G \odot \overline{K}_n$ is obtained with valence $n(k-2)+2$. ■

Now, recall the following super edge-magic characterization of the n-cycle C_n found by Enomoto, Lladó, Nakamigawa and Ringel [2].

Theorem 3.2 *The n-cycle C_n is super edge-magic if and only if $n \geq 3$ is odd.*

Hence, by Theorem 3.1, we know that the n-crowns with cycle length m are super edge-magic when $m \geq 3$ is odd. In the following result, we show with considerable more effort that the n-crowns with cycle length m are also super edge-magic when $m \geq 4$ is even.

Theorem 3.3 *For every two integers $m \geq 3$ and $n \geq 1$, the n-crown $G \cong C_m \odot \overline{K}_n$ is super edge-magic.*

Proof. Let $G \cong C_m \odot \overline{K}_n$ be the n-crown with

$$V(G) = \{u_i | 1 \leq i \leq m\} \cup \{v_{i,j} | 1 \leq i \leq m \text{ and } 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1 u_m\} \cup \{u_i u_{i+1} | 1 \leq i \leq m-1\} \cup \{u_i v_{i,j} | 1 \leq i \leq m \text{ and } 1 \leq i \leq n\}.$$

Now, notice that if $m \geq 3$ is odd, then the result follows from Theorems 3.1 and 3.2. Thus, assume that $m \geq 4$ is even for the remainder of the proof, and proceed by cases by means of Lemma 1.1.

Case 1: For $m = 4$, define the vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, 4(n+1)\}$ such that

$$\begin{aligned} f(u_{2i-1}) &= i; & f(u_{2i}) &= 3i; \\ f(v_{2i-1,1}) &= 2i+3; & f(v_{2i,1}) &= 12-4i \end{aligned}$$

when $i = 1$ or 2; and $f(v_{i,j}) = 4j - i + 5$ when $1 \leq i \leq 4$ and $2 \leq j \leq n$.

Case 2: For $m = 6$, define the vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, 6(n+1)\}$ such that

$$\begin{aligned} f(u_1) &= 9; & f(u_2) &= 1; & f(u_3) &= 4; \\ f(u_4) &= 2; & f(u_5) &= 5; & f(u_6) &= 3; \\ f(v_{1,1}) &= 6; & f(v_{2,1}) &= 8; & f(v_{3,1}) &= 7; \\ f(v_{4,1}) &= 12; & f(v_{5,1}) &= 11; & f(v_{6,1}) &= 10; \end{aligned}$$

and

$$f(v_{i,j}) = \begin{cases} 5i + 6j - 4, & \text{if } 1 \leq i \leq 2 \text{ and } 2 \leq j \leq n; \\ i + 6j - 1, & \text{if } 3 \leq i \leq 6 \text{ and } 2 \leq j \leq n. \end{cases}$$

Case 3: For $m = 8$, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 8(n+1)\}$ such that

$$
\begin{array}{rclrclrcl}
f(u_1) &=& 1; & f(u_2) &=& 5; & f(u_3) &=& 2;\\
f(u_4) &=& 6; & f(u_5) &=& 3; & f(u_6) &=& 7;\\
f(u_7) &=& 4; & f(u_8) &=& 12; & & &\\
f(v_{1,1}) &=& 11; & f(v_{2,1}) &=& 13; & f(v_{3,1}) &=& 15;\\
f(v_{4,1}) &=& 14; & f(v_{5,1}) &=& 16; & f(v_{6,1}) &=& 8;\\
f(v_{7,1}) &=& 10; & f(v_{8,1}) &=& 9; & & &
\end{array}
$$

and $f(v_{i,j}) = 8j - i + 9$, if $1 \le i \le 8$ and $2 \le j \le n$.

Case 4: Let $m = 8k+2$, where k is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, (8k+2)(n+1)\}$$

such that

$$f(u_l) = \begin{cases} 12k+3, & \text{if } l = 1;\\ 4k+i, & \text{if } l = 2i-1 \text{ and } 2 \le i \le 4k+1;\\ i, & \text{if } l = 2i \text{ and } 1 \le i \le 4k+1; \end{cases}$$

$$f(v_{l,1}) = \begin{cases} 8k+i+1, & \text{if } l = 2i-1 \text{ and } 1 \le i \le 2k+2;\\ 12k+2, & \text{if } l = 2;\\ 12k+i+2, & \text{if } l = 2i \text{ and } 2 \le i \le 2k;\\ 14k+2i+4, & \text{if } l = 4k+4i-2 \text{ and } 1 \le i \le k;\\ 14k-i+5, & \text{if } l = 4k+i+3 \text{ and } 1 \le i \le 2;\\ 10k+2i+2, & \text{if } l = 4k+4i+3 \text{ and } 1 \le i \le k-1;\\ 14k+2i+3, & \text{if } l = 4k+4i+4 \text{ and } 1 \le i \le k-1;\\ 10k+2i+3, & \text{if } l = 4k+4i+5 \text{ and } 1 \le i \le k-1;\\ 16k+3, & \text{if } l = 8k+2; \end{cases}$$

and for $2 \le j \le n$, we have that

$$f(v_{2i-1,j}) = \begin{cases} 2(4k+1)j+i, & \text{if } 1 \le i \le 2k+1;\\ 2(4k+1)j+i+1, & \text{if } 2k+2 \le i \le 4k+1; \end{cases}$$

$$f(v_{2i,j}) = \begin{cases} (4k+1)(2j+1)+i, & \text{if } 2 \le i \le 2k;\\ (4k+1)(2j+1)+i-1, & \text{if } 2k+2 \le i \le 4k+1; \end{cases}$$

$f(v_{2,j}) = 2(4k+1)(j+1)$ and $f(v_{4k+2,j}) = 2(k+1) + 2(4k+1)j$.

Case 5: Let $m = 8k+4$, where k is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, (8k+4)(n+1)\}$$

such that

$$f(u_l) = \begin{cases} i, & \text{if } l = 2i-1 \text{ and } 1 \le i \le 4k+2;\\ 4k+i+2, & \text{if } l = 2i \text{ and } 1 \le i \le 4k+1;\\ 12k+6, & \text{if } l = 8k+4; \end{cases}$$

$$f(v_{l,1}) = \begin{cases} 12k+5, & \text{if } l=1; \\ 16k-4i+8, & \text{if } l=4i-2 \text{ and } 1\le i\le k; \\ 16k-4i+9, & \text{if } l=4i-1 \text{ and } 1\le i\le k; \\ 16k-4i+10, & \text{if } l=4i \text{ and } 1\le i\le k; \\ 16k-4i+7, & \text{if } l=4i+1 \text{ and } 1\le i\le k; \\ 8k+4, & \text{if } l=4k+2; \\ 16k+8, & \text{if } l=4k+3; \\ 16k+7, & \text{if } l=8k+3; \\ 8k+5, & \text{if } l=8k+4; \\ 12k-i+5, & \text{if } l=4k+i+3 \text{ and } 1\le i\le 4k-1; \end{cases}$$

and $f(v_{i,j}) = 4(2k+1)(j+1)-i+1$, if $1\le i\le 8k+4$ and $2\le j\le n$.

Case 6: Let $m=8k+6$, where k is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1,2,\ldots,(8k+6)(n+1)\}$$

such that

$$f(u_l) = \begin{cases} 12k+9, & \text{if } l=1; \\ 4k+i+2, & \text{if } l=2i-1 \text{ and } 2\le i\le 4k+3; \\ i, & \text{if } l=2i \text{ and } 1\le i\le 4k+3; \end{cases}$$

$$f(v_{l,1}) = \begin{cases} 8k+i+5, & \text{if } l=2i-1 \text{ and } 1\le i\le 2k+3; \\ 12k+8, & \text{if } l=2; \\ 12k+i+8, & \text{if } l=2i \text{ and } 2\le i\le 2k+1; \\ 14k-2i+14, & \text{if } l=4k+3i+1 \text{ and } 1\le i\le 2; \\ 14k+2i+12, & \text{if } l=4k+4i+2 \text{ and } 1\le i\le k; \\ 14k+2i+9, & \text{if } l=4k+4i+4 \text{ and } 1\le i\le k; \\ 10k+2i+7, & \text{if } l=4k+4i+5 \text{ and } 1\le i\le k; \\ 10k+2i+8, & \text{if } l=4k+4i+7 \text{ and } 1\le i\le k-1; \\ 16k+11, & \text{if } l=8k+6; \end{cases}$$

and for $2\le j\le n$, we have that

$$f(v_{l,j}) = \begin{cases} 2(4k+3)j+i, & \text{if } l=2i-1,\ 1\le i\le 4k+3; \\ (4k+3)(2j+1)+i, & \text{if } l=2i,\ 1\le i\le 4k+3. \end{cases}$$

Case 7: Let $m=16k$, where k is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1,2,\ldots,16k(n+1)\}$$

such that

$$f(u_l) = \begin{cases} i, & \text{if } l=2i-1 \text{ and } 1\le i\le 8k; \\ 8k+i, & \text{if } l=2i \text{ and } 1\le i\le 8k-1; \\ 24k, & \text{if } l=16k; \end{cases}$$

$f(v_{1,1}) = 24k - 1$; $f(v_{2,1}) = 16k + 3$; $f(v_{3,1}) = 32k - 1$;

$$f(v_{l,1}) = \begin{cases} 32k - 2i + 1, & \text{if } l = 2i - 1 \text{ and } 3 \leq i \leq 4k; \\ 32k - 2i + 2, & \text{if } l = 2i \text{ and } 2 \leq i \leq 4k; \\ 32k - 3i + 3, & \text{if } l = 8k + 2i - 1 \text{ and } 1 \leq i \leq 2; \\ 24k - 8i + 5, & \text{if } l = 8k + 8i - 6 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 6, & \text{if } l = 8k + 8i - 4 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 4, & \text{if } l = 8k + 8i - 3 \text{ and } 1 \leq i \leq k; \\ 24k - 8i, & \text{if } l = 8k + 8i - 2 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 2, & \text{if } l = 8k + 8i - 1 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 1, & \text{if } l = 8k + 8i \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 3, & \text{if } l = 8k + 8i + 1 \text{ and } 1 \leq i \leq k - 1; \\ 24k - 8i - 1, & \text{if } l = 8k + 8i + 3 \text{ and } 1 \leq i \leq k - 1; \end{cases}$$

and $f(v_{i,j}) = 16k(j + 1) - i + 1$, if $1 \leq i \leq 16k$ and $2 \leq j \leq n$.

Case 8: Let $m = 16k + 8$, where k is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, (16k + 8)(n + 1)\}$$

such that

$$f(u_l) = \begin{cases} i, & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 8k + 4; \\ 8k + i + 4, & \text{if } l = 2i \text{ and } 1 \leq i \leq 8k + 3; \\ 24k + 12, & \text{if } l = 16k + 8; \end{cases}$$

$f(v_{1,1}) = 24k + 11$; $f(v_{2,1}) = 16k + 11$; $f(v_{3,1}) = 32k + 15$;

$$f(v_{l,1}) = \begin{cases} 32k - 2i + 17, & \text{if } l = 2i - 1 \text{ and } 3 \leq i \leq 4k + 2; \\ 32k - 2i + 18, & \text{if } l = 2i \text{ and } 2 \leq i \leq 4k + 2; \\ 32k - 3i + 19, & \text{if } l = 8k + 2i + 3 \text{ and } 1 \leq i \leq 2; \\ 24k - 8i + 16, & \text{if } l = 8k + 8i - 2 \text{ and } 1 \leq i \leq k + 1; \\ 24k - i + 11, & \text{if } l = 8k + i + 7 \text{ and } 1 \leq i \leq 2; \\ 24k - 8i + 13, & \text{if } l = 8k + 8i + 2 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 15, & \text{if } l = 8k + 8i + 3 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 14, & \text{if } l = 8k + 8i + 4 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 12, & \text{if } l = 8k + 8i + 5 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 10, & \text{if } l = 8k + 8i + 7 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 9, & \text{if } l = 8k + 8i + 8 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 11, & \text{if } l = 8k + 8i + 9 \text{ and } 1 \leq i \leq k - 1; \end{cases}$$

and $f(v_{i,j}) = (16k + 8)(j + 1) - i + 1$, if $1 \leq i \leq 16k + 8$ and $2 \leq j \leq n$.

Therefore, by Lemma 1.1, f extends to a super edge-magic labeling of G with valence $m(4n + 5)/2 + 2$. ■

Using the relationships between super edge-magic labelings and other labelings mentioned in the introduction, we finish with the following corollary, which settles a conjecture by Yegnanarayanan [11].

Corollary 3.4 *For every two integers $m \geq 3$ and $n \geq 1$, the n-crown $G \cong C_m \odot \overline{K}_n$ is harmonious, sequential and felicitous.*

4 Conclusions

The constructions in the second section are of particular interest to the authors as before no binary graph operations were known to preserve the felicitous, harmonious, sequential and super edge-magic properties of graphs.

The results in the last section of this paper lead us to visualize, as a topic for further exploration, the question of which connected unicyclic graphs are super edge-magic, and which are not. This question is related to the conjecture by Enomoto, Lladó, Nakamigawa and Ringel [2] that all trees are super edge-magic; however, it provides more variety as not all such graphs are super edge-magic as illustrated by the even cycles.

We also remark that Theorem 3.3 significantly expands what was known about the harmoniousness and sequentialness of n-crowns with even cycle length as before only crowns were known to be harmonious and sequential; see [5] for a detailed discussion of this.

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