A note on Brooks' theorem for triangle-free graphs

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Abstract

For the class of triangle-free graphs Brooks' Theorem can be restated in terms of forbidden induced subgraphs, i.e. *let G be a triangle-free and $K_{1,r+1}$-free graph. Then G is r-colourable unless G is isomorphic to an odd cycle or a complete graph with at most two vertices.* In this note we present an improvement of Brooks' Theorem for triangle-free and r-sunshade-free graphs. Here, an *r-sunshade* (with $r \geq 3$) is a star $K_{1,r}$ with one branch subdivided.

A classical result in graph colouring theory is the theorem of Brooks [2], asserting that every graph G is $(\Delta(G))$-colourable unless G is isomorphic to an odd cycle or a complete graph. Bryant [3] simplified this proof with the following characterization of cycles and complete graphs. Thereby he highlights the exceptional role of the cycles and complete graphs in Brooks' Theorem. Here we give a new elementary proof of this characterization.

Proposition 1 (Bryant [3]). *Let G be a 2-connected graph. Then G is a cycle or a complete graph if and only if $G - \{u, v\}$ is not connected for every pair (u, v) of vertices of distance two.*

Proof. Let G be a 2-connected graph of order n. If G is a cycle or a complete graph, then obviously $G - \{u, v\}$ is not connected for every pair (u, v) of vertices of distance two. Hence, assume that G is neither a cycle nor a complete graph and that $G - \{u, v\}$ is not connected for every pair (u, v) of vertices of distance two. Note that then there exists at least one vertex v of G with $2 < d_G(v) < n - 1$. Since G

is 2-connected, there exists at least one cycle in G. Now let C be a longest cycle in G. Assume C is not a Hamiltonian cycle of G. Since C is a longest cycle and G is connected, there exist vertices y, z of C and $x \in V(G) - V(C)$, such that z is adjacent to x and y and x is not adjacent to y, i.e. $\text{dist}_G(x, y) = 2$. Now $G - \{x, y\}$ is not connected and the 2-connectivity of G ensures besides the $x - y$-connecting path P_1 via the remaining vertices of C the existence of a second $x - y$-connecting path P_2, which is vertex disjoint from P_1. But then by gluing the common end-vertices of P_1 and P_2 together we obtain a cycle C' of length greater than C — a contradiction to the special choice of C. Thus $C = v_0 v_1 \dots v_{n-1} v_0$ is a Hamiltonian cycle. Now we consider a vertex v_i with $2 < d_G(v_i) < n - 1$. Then there exists $j \notin \{i-1, i+1\}$ such that v_i is without loss of generality adjacent to v_j but not adjacent to v_{j-1}. Since $G - \{v_i, v_{j-1}\}$ is not connected, we obtain that v_j is not adjacent to v_{j-2}. Therefore $G - \{v_j, v_{j-2}\}$ is not connected. Thus $d_G(v_{j-1}) = 2$, and in particular v_{j-1} is not adjacent to v_{j+1}. But now finally, since $G - \{v_{j-1}, v_{j+1}\}$ is connected, we immediately achieve a contradiction, which completes the proof of this proposition. ■

Theorem 2 (Brooks [2]). *Let G be neither a complete graph nor a cycle graph with an odd number of vertices. Then G is $\Delta(G)$-colourable.*

In the recent book of Jensen and Toft [5], (Problem 4.6, p. 83), the problem of improving Brooks' Theorem (in terms of the maximal degree Δ) for the class of triangle-free graphs is stated. The problem has its origin in a paper of Vizing [7]. The best known (non-asymptotic) improvement of Brooks' Theorem in terms of the maximal degree for the class of triangle-free graphs is due to Borodin and Kostochka [1], Catlin [4] and Kostochka (personal communication mentioned in [5]). The last author proved that $\chi(G) \leq 2/3(\Delta(G) + 3)$ for every triangle-free graph G. The remaining authors independently proved that $\chi(G) \leq 3/4(\Delta(G) + 2)$ for every triangle-free graph G. For the class of triangle-free graphs, Brooks' Theorem can be restated in terms of forbidden induced subgraphs, since triangle-free graphs G satisfy $G[N_G[x]] \cong K_{1,d_G(x)}$ for every vertex x of G.

Theorem 3 (Triangle-free Version of Brooks' Theorem)
Let G be a triangle-free and $K_{1,r+1}$-free graph. Then G is r-colourable unless G is isomorphic to an odd cycle or a complete graph with at most two vertices.

Our main theorem will extend this triangle-free version of Brooks' Theorem. An *r-sunshade* (with $r \geq 3$) is a star $K_{1,r}$ with one branch subdivided. The 3-sunshade is sometimes called a *chair* and the 4-sunshade a *cross*.

Proposition 4 *Let G be a triangle-free and chair-free graph; then $\chi(G) \leq 3$. Moreover if G is connected, then equality holds if and only if G is an odd hole.*

Proof. Let G be a triangle-free and chair-free graph. Without loss of generality let G be a connected graph. If G is bipartite, then $\chi(G) \leq 2$ and we are done. So let G be a non-bipartite graph. With a result of König that every non-bipartite graph contains an odd cycle and the clique size constraint we deduce that G contains an odd hole C. If $G \cong C$, then $\chi(G) = 3$. If $G \ncong C$, then there exists a vertex

$y \in V(G)-V(C)$ adjacent to a nonempty subset I of $V(C)$. Since $\omega(G) \leq 2$, we have I is an independent set and $|I| \leq (n(C)-1)/2$. But then there exist four consecutive vertices $x_1, \ldots, x_4$ of C such that $I \cap \{x_1, \ldots, x_4\} = \{x_2\}$. Therefore $\{y, x_1, \ldots, x_4\}$ induces a chair in G, a contradiction. This completes the proof of the proposition. ■

Now let G be a connected triangle-free graph. For convenience we define for every $i \in \mathbb{N}$ and $x \in V(G)$ the sets $N_G^{(i)}(x) := \{y \in V(G) \mid \text{dist}_G(x,y) = i\}$. A vertex $y \in N_G^{(i)}(x)$ is also called an (i,x)-level vertex. Note that the triangle-freeness of G forces $N_G^{(1)}(x)$ to be independent for every $x \in V(G)$.

Proposition 5 *Let G be a triangle-free and cross-free graph. Then $\chi(G) \leq 3$.*

Proof. Let G be a triangle-free and cross-free graph. If $\Delta(G) = \Delta \leq 3$, then we are done with Brooks' Theorem and $\chi(G) \leq 3$. Now let $v \in V(G)$ be a vertex of maximal degree Δ with $\Delta \geq 4$ and suppose without loss of generality that G is connected and we have $N_G^{(2)}(v) \neq \emptyset$. We can also assume that the $N(u)/N(v)$-*argument* holds, (i.e. G contains no pair of non-adjacent vertices u and v, such that $N_G(u) \subset N_G(v)$). Note that the cross-freeness of G forces every vertex of $N_G^{(2)}(v)$ to be adjacent to at least $|N_G^{(1)}(v)| - 2 = \Delta - 2 \geq 2$ vertices of $N_G^{(1)}(v)$.

Case 1: Suppose we have $d_G(v) \geq 5$.
Then $N_G^{(2)}(v)$ is an independent set. Otherwise, if there exist two adjacent $(2,v)$-level vertices u_1 and u_2, then because of $|N_G(u_j) \cap N_G^{(1)}(v)| \geq \Delta(G) - 2$ for $j = 1, 2$, there exists at least one $(1,v)$-level vertex u_3 being adjacent to both vertices. But then $\{u_1, u_2, u_3\}$ induces a triangle — a contradiction.
Suppose now there exists $u_3 \in N_G^{(3)}(v)$, such that u_3 is adjacent to a vertex $u_4 \in (N_G^{(3)}(v) \cup N_G^{(4)}(v))$. Since $u_3 \in N_G^{(3)}(v)$, there exists $u_2 \in (N_G^{(2)}(v) \cap N_G(u_3))$. Note that the triangle-freeness of G forces that u_2 is not adjacent to u_4. Since u_2 is adjacent to at least $\Delta(G) - 2$ vertices of $N_G^{(1)}(v)$, there exists $\{u_1^{(1)}, u_1^{(2)}, u_1^{(3)}\} \subseteq (N_G^{(1)}(v) \cap N_G(u_2))$. Recall that because of the definition of the sets $N_G^{(i)}(v)$, each vertex of $\{u_1^{(1)}, u_1^{(2)}, u_1^{(3)}\}$ is non-adjacent to each vertex of $\{u_3, u_4\}$.
But then $\{u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_2, u_3, u_4\}$ induces a cross — a contradiction. Hence $N_G^{(3)}(v)$ is independent and $N_G^{(i)}(v) = \emptyset$ for every $i \geq 4$. Since for any $x \in V(G)$ the set $N_G^{(i)}(x)$ for $i \in \{1,2,3\}$ is independent, we obtain that G is bipartite in Case 1.

Case 2: $d_G(v) = \Delta(G) = 4$.
In the following we will examine the structure of $G[N_G^{(2)}(v)]$. Firstly, recall that the cross-freeness of G forces every $(2,v)$-level vertex u to be adjacent to at least two $(1,v)$-level vertices. On the other hand the $N(u)/N(v)$-argument forces every $(2,v)$-level vertex u to be adjacent to at most three $(1,v)$-level vertices.

Case 2.1: Suppose there exists a $(2,v)$-level vertex u_1, adjacent to at least two further $(2,v)$-level vertices u_2 and u_3.
Note that the triangle-freeness of G forces that u_2 and u_3 are not adjacent. Furthermore, u_1 is adjacent to exactly two $(1,v)$-level vertices v_1 and v_2 and $N_G(u_2) \cap$

$N_G(v) = N_G(u_3) \cap N_G(v) = N_G(v) - N_G(u_1) = \{v_3, v_4\}$. Since the $N(u)/N(v)$-argument holds, there exist $u_4 \in (N_G(u_2) - N_G(u_3))$ and $u_5 \in (N_G(u_3) - N_G(u_2))$. Note that u_4 and u_5 are contained in $N_G^{(2)}(v) \cup N_G^{(3)}(v)$. Firstly, suppose that u_4 is a $(3, v)$-level vertex. Then $\{v_1, u_1, u_2, v_3, v_4, u_4\}$ induces a cross — a contradiction. Thus $\{u_4, u_5\} \subseteq N_G^{(2)}(v)$. Observe that $N_G(u_4) \cap N_G(v) = N_G(u_5) \cap N_G(v) = \{v_1, v_2\}$ and therefore $\{u_1, u_4, u_5\}$ forms an independent set. Because of the $N(u)/N(v)$-argument there exist $u_6 \in (N_G(u_4) - N_G(u_1))$ and $u_7 \in (N_G(u_5) - N_G(u_1))$. Then $u_6 = u_7$, since otherwise $\{v, v_1, u_1, u_4, u_5, u_7\}$ induces a cross and we obtain a contradiction. Analogously to the previous consideration we obtain that u_6 is a $(2, v)$-level vertex, $N_G(u_6) \cap N_G(v) = \{v_3, v_4\}$ and $\{u_2, u_3, u_6\}$ forms an independent set. Hence $\{v, v_1, \dots, v_4, u_1, \dots, u_6\}$ obviously induces a 4-regular graph G''. But then, since $\Delta(G) = 4$ and G is connected, we deduce that $G = G''$. Note that since $\{u_1, u_2, \dots, u_6\}$ induces a 6-cycle, G is easily 3-colourable.

Case 2.2: Every $(2, v)$-level vertex u is adjacent to at most one $(2, v)$-level vertex. Now assume that a $(2, v)$-level vertex u_1 is adjacent to another $(2, v)$-level vertex u_2. Then, as already mentioned, u_1 is adjacent to, say the $(1, v)$-level vertices v_1 and v_2 and u_2 is adjacent to the left $(1, v)$-level vertices v_3 and v_4. If, say, u_1 is adjacent to a fourth vertex u_3, then u_3 is a $(3, v)$-level vertex and u_3 is not adjacent to u_2. But then $\{v_1, v_2, u_3, u_1, u_2, v_3\}$ induces a cross — a contradiction. Hence, we obtain that $d_G(u_1) = d_G(u_2) = 3$, if the $(2, v)$-level vertices u_1 and u_2 are adjacent. Note that then neither u_1 nor u_2 is adjacent to a $(3, v)$-level vertex.
For convenience, we divide the $(2, v)$-level vertices into three subsets: A_1 contains all $(2, v)$-level vertices, which are each adjacent to exactly one other $(2, v)$-level vertex, $A_2 \subseteq (N_G^{(2)}(v) - A_1)$ contains all remaining $(2, v)$-level vertices, which are each adjacent to exactly two other $(1, v)$-level vertices and finally $A_3 = (N_G^{(2)}(v) - A_1) - A_2$ contains all remaining $(2, v)$-level vertices. Note that each vertex of A_3 is adjacent to exactly one $(3, v)$-level vertex, each vertex of A_2 is adjacent to at least one and at most two $(3, v)$-level vertices and also recall that each vertex of A_1 is not adjacent to any $(3, v)$-level vertex. Suppose that $u \in A_2$ is adjacent to exactly two $(3, v)$-level vertices w_1 and w_2. With $G''' := G - N_G[N_G(v)]$ we then have $N_{G'''}(w_1) = N_{G'''}(w_2)$. Otherwise, if there exists, say, $w_3 \in N_{G'''}(w_1) - N_{G'''}(w_2)$, then $N_G[u] \cup \{w_3\}$ induces a cross — a contradiction.
In the final part of the proof, we 3-colour G. Now let $x_1, x_2, \dots, x_{n-5}$ be the vertices of $G - N_G[v]$, listed so that we have $\text{dist}_G(v, x_i) \geq \text{dist}_G(v, x_j)$ for $1 \leq i \leq j \leq n-5$. Furthermore, let $G_i := G[\{x_1, \dots, x_i\}]$ for every $1 \leq i \leq n-5$. Suppose that there exists $i_0 \in \{1, \dots, n-5\}$ with $d_{G_{i_0}}(x_{i_0}) \geq 3$. Note that there exist vertices $y_1 \in (V(G) - \{v\})$ and $y_2 \in V(G)$, such that $\text{dist}_G(v, y_2) = \text{dist}_G(v, y_1) - 1 = \text{dist}_G(v, x_{i_0}) - 2$ and y_1 is adjacent to both vertices x_{i_0} and y_2. But then with $\Delta(G) = 4$ we have $d_{G_{i_0}}(x_{i_0}) = 3$. Observe that because of the definition of G_{i_0} we have $N_G(y_2) \cap N_G[x_{i_0}] = \{y_1\}$. But then $G[N_G[x_{i_0}] \cup \{y_2\}]$ contains an induced cross — a contradiction. Thus we have $d_{G_i}(x_i) \leq 2$ for every $i \in \{1, \dots, (n - \Delta(G) - 2)\}$. Hence we can easily 3-colour the graph $G' := G - N_G[v]$ along the sequence $x_1, x_2, \dots, x_{n-\Delta(G)-2}$. We modify this 3-colouring procedure with the following additional rule: Suppose that we have already 3-coloured all vertices of $x_1, \dots, x_{i-1}$ and we will colour the vertex x_i. If there

exists a vertex $x_j \in \{x_1, \ldots, x_{i-1}\}$ with $N_{G_i}(x_i) \subseteq N_{G_i}(x_j)$, then x_i should receive the same colour as x_j. Now there exists a $j^* \in \{1, \ldots, n-5\}$, such that $G_{j*} = G'''$. In the following we will extend the achieved 3-colouring ϕ of G''' to a 3-colouring of G. Now we colour the vertex v with the first colour α and every $(1, v)$-level vertex with the colour β. Then we colour the (blocking-set-) vertices of A_1 with the colours α and γ. Every vertex of A_3 is adjacent to three (β)-coloured $(1, v)$-level vertices and one vertex of G'''. Hence the neighbours of an A_3 vertex consume at most two colours. Thus there exists for each A_3 vertex a colour, which was not used in the neighbourhood. Analogously we can colour each vertex of A_2, which is adjacent to exactly one $(3, v)$-level vertex. Therefore, suppose that there exists an A_2 vertex u which is adjacent to exactly two $(3, v)$-level vertices w_1 and w_2. But then as already mentioned we have $N_{G'''}(w_1) = N_{G'''}(w_2)$. Thus, because of ϕ's special choice, we have $\phi(w_1) = \phi(w_2)$. But then again the neighbours of u consume at most two colours and there exists a colour, which was not used in the neighbourhood. Thus G is 3-colourable. This completes the proof of the theorem. ■

Theorem 6 *Let G be a connected, triangle-free and r-sunshade-free graph with $r \geq 3$, which is not an odd cycle. Then*

(i) G is r-colourable;

(ii) G is bipartite, if $\Delta(G) \geq 2r - 3$;

(iii) G is $(r-1)$-colourable, if $r = 3, 4$ or if $\Delta(G) \leq r - 1$.

Proof. If $3 \leq r \leq 4$, then the theorem holds because of the last propositions. So let $r \geq 5$. Let G^* be a connected, triangle-free and r-sunshade-free graph. If $\Delta(G^*) = \Delta \leq r - 1$, then we are done with Brooks' Theorem and $\chi(G^*) \leq r - 1$. We prove inductively on the order $n(G)$ of a connected, triangle-free and r-sunshade-free graph G with $\Delta(G) \leq r$ that for every vertex v of maximal degree there exists an r-colouring c of G, such that all vertices of $N_G(v)$ consume the same colour of c. Now suppose the statement holds for every (connected) triangle-free and r-sunshade-free graph of order less than $n(G^*) = n$. Now let $v \in V(G^*)$ be a vertex of maximal degree Δ with $\Delta \geq r$ and suppose we have $N_{G^*}^{(2)}(v) \neq \emptyset$. Note that the r-sunshade-freeness of G^* forces every vertex of $N_{G^*}^{(2)}(v)$ to be adjacent to at least $|N_{G^*}^{(1)}(v)| - (r-2) = \Delta - (r-2) \geq 2$ vertices of $N_{G^*}^{(1)}(v)$.

(i): Now let $x_1, x_2, \ldots, x_{n-(\Delta+1)}$ be the vertices of $G - N_{G^*}[v]$, listed so that we have $\text{dist}_{G^*}(v, x_i) \geq \text{dist}_{G^*}(v, x_j)$ for $1 \leq i \leq j \leq (n - (\Delta + 1))$. Furthermore let $G_i^* := G^*[\{x_1, \ldots, x_i\}]$ for every $1 \leq i \leq (n-(\Delta+1))$. Suppose there exists $i_0 \in \{1, \ldots, (n-(\Delta+1))\}$ with $d_{G_{i_0}^*}(x_{i_0}) \geq (r-1)$. Note that there exist vertices $y_1 \in (V(G^*) - \{v\})$ and $y_2 \in V(G^*)$, such that $\text{dist}_{G^*}(v, y_2) = \text{dist}_{G^*}(v, y_1) - 1 = \text{dist}_{G^*}(v, x_{i_0}) - 2$ and y_1 is adjacent to both vertices x_{i_0} and y_2. Observe that because of the definition of $G_{i_0}^*$ we have $N_{G^*}(y_2) \cap N_{G^*}[x_{i_0}] = \{y_1\}$. But then $N_{G^*}[x_{i_0}] \cup \{y_2\}$ induces a supergraph of the r-sunshade — a contradiction. Thus we have $d_{G_i^*}(x_i) \leq (r-2)$ for every $i \in \{1, \ldots, (n - (\Delta + 1))\}$. Hence we easily can colour the graph $G' := G^* - N_{G^*}[v]$ along the sequence $x_1, x_2, \ldots, x_{n-(\Delta+1)}$ with $(r-1)$ colours $\alpha_1, \alpha_2, \ldots, \alpha_{r-1}$. Since

$N^{(1)}_{G^*}(v)$ forms an independent set, we can easily extend the partial $(r-1)$-colouring of $G^*_{n-(\Delta+1)}$ to an r-colouring of G^*.

(ii): Suppose we have $d_G(v) \geq (2r-3)$.
Then $N^{(2)}_G(v)$ is an independent set. Otherwise, if there exist two adjacent $(2, v)$-level vertices u_1 and u_2, then because of $|N_G(u_j) \cap N^{(1)}_G(v)| \geq \Delta - (r-2)$ for $j = 1, 2$ there exists at least one $(1, v)$-level vertex u_3 being adjacent to both vertices. But then $\{u_1, u_2, u_3\}$ induces a triangle — a contradiction. Suppose now there exists $u_3 \in N^{(3)}_G(v)$, such that u_3 is adjacent to a vertex $u_4 \in (N^{(3)}_G(v) \cup N^{(4)}_G(v))$. Since $u_3 \in N^{(3)}_G(v)$, there exists $u_2 \in (N^{(2)}_G(v) \cap N_G(u_3))$. Note that the triangle-freeness of G forces that u_2 is not adjacent to u_4. Because u_2 is adjacent to at least $\Delta - (r-2)$ vertices of $N^{(1)}_G(v)$, there exist $\{u^{(1)}_1, \ldots, u^{(r-1)}_1\} \subset (N^{(1)}_G(v) \cap N_G(u_2))$. Recall that because of the definition of the sets $N^{(i)}_G(v)$, each vertex of $\{u^{(1)}_1, \ldots, u^{(r-1)}_1\}$ is non-adjacent to each vertex of $\{u_3, u_4\}$. But then $\{u^{(1)}_1, \ldots, u^{(r-1)}_1, u_2, u_3, u_4\}$ induces an r-sunshade — a contradiction. Hence $N^{(3)}_G(v)$ is independent and $N^{(i)}_G(v) = \emptyset$ for every $i \geq 4$. Since every set $N^{(i)}_G(x)$ for $i \in \{1, 2, 3\}$ is independent, we obtain that G is bipartite. ■

Problem 7 *Let $\mathcal{G}$ be the class of all connected, triangle-free and r-sunshade-free graphs with* $5 \leq r \leq \Delta(G) \leq 2r-4$. *Does there exist an r-chromatic member $G^* \in \mathcal{G}$?*

Using Kostochka's result that $\chi(G) \leq 2/3(\Delta(G)+3)$ for every triangle-free graph G, it is not very difficult for $r \geq 9$ to reduce the above problem to the range $3/2(r-3) \leq \Delta(G) \leq 2r-4$.
An intriguing improvement of Brooks' Theorem by bounding the chromatic number of a graph by a convex combination of its clique number ω and its maximum degree Δ plus 1 is given by Reed [6] and he conjectured that every graph G can be colored with at most $\lceil(\omega(G) + \Delta(G) + 1)/2\rceil$ colors? If Reeds conjecture is true for the special case of triangle-free graphs (*'every triangle-free graph G satisfies* $\chi(G) \leq \lceil(\Delta(G)+3)/2\rceil$'), then it is not very difficult to reduce the above problem to the unique value $\Delta(G) = 2r-4$, which seems to be not intractable. Moreover, an affirmative answer to this special case of Reeds conjecture on triangle-free graphs, would imply that there exists no 5-regular, 5-chromatic or 6-regular, 6-chromatic triangle-free graph. This negative results would settle the remaining cases of Grünbaums girth problem (see [5]).

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