Queens graphs for chessboards on the torus

A.P. Burger   C.M. Mynhardt

Department of Mathematics, University of South Africa
P.O. Box 392, Unisa, 0003 South Africa
alewynburger@mweb.co.za   mynhacm@unisa.ac.za

E.J. Cockayne

Department of Mathematics, University of Victoria,
P.O. Box 3045, Victoria BC, V8W 3P4, Canada
cockayne@smart.math.uvic.ca

Abstract

We consider the independence, domination and independent domination numbers of graphs obtained from the moves of queens on chessboards drawn on the torus, and determine exact values for each of these parameters in infinitely many cases.

1 Introduction

The study of combinatorial problems on chessboards dates back to 1848, when a German chess player, Max Bezzel [2], first posed the n-queens problem, that is, the problem of placing n queens on an $n \times n$ chessboard so that no two queens attack each other. As stated in [15], several mathematicians worked on this problem, but Ahrens [1] apparently was the first to prove that the n-queens problem has a solution for all $n \geq 4$. The study of chessboard domination problems dates back to 1862, when C.F. de Jaenisch [11] first considered the queens domination problem, that is, the problem of determining the minimum number of queens required to dominate every square on an $n \times n$ chessboard. Since then many papers concerning combinatorial problems on chessboards have appeared in the literature. Surveys of the topic are given in [12, 15]; more recent results can be found in [5 – 10, 13, 16, 17].

In this paper we begin the study of the n-queens problem and the queens domination problem for chessboards on the torus. The rows and columns of the chessboard are rings on the torus. We cut the torus along arbitrary lines separating two rows and two columns, and draw the $n \times n$ toroidal chessboard in the plane, numbering its rows and columns from 0 to $n-1$, beginning at the bottom left hand corner. Thus each square has *co-ordinates* (x, y), where x and y are the column and row numbers of the square, respectively. The *lines* of the board are the rows, columns, *sum diagonals*, abbreviated *s-diagonals* (*i.e.*, sets of squares such that $x+y \equiv k \pmod n$, where k is a constant) and *difference diagonals*, abbreviated *d-diagonals* (sets of squares such that $y-x \equiv k \pmod n$). Note that there are n s-diagonals and n d-diagonals, and each contains n squares.

The vertices of Q_n^t, the *queens graph obtained from an $n \times n$ chessboard on the torus*, are the n^2 squares of the chessboard, and two squares are adjacent if they are collinear. Note that Q_n^t is vertex-transitive. As is standard in domination theory (see [14]) we denote the domination, independent domination and independence numbers of Q_n^t by $\gamma(Q_n^t)$, $i(Q_n^t)$ and $\beta(Q_n^t)$, respectively. The n-queens problem on the torus is thus equivalent to determining whether $\beta(Q_n^t) = n$, and the queens domination problem on the torus is the problem of determining $\gamma(Q_n^t)$.

In Section 2 we show that there are infinitely many values of n such that $\beta(Q_n^t) = n$ and such that $\beta(Q_n^t) = n-1$. In the remaining cases we also list exact values of $\beta(Q_n^t)$ for small values of n. In Section 3 we show that if $\beta(Q_n^t) = n$, then β-sets (independent sets of cardinality β) of Q_n^t can be used to find γ-sets and i-sets (defined similarly) of Q_{3n}^t of cardinality n, and upper bounds for $\gamma(Q_{2n}^t)$ and $i(Q_{2n}^t)$. Moreover, if $\beta(Q_n^t) = n-1$, then β-sets of Q_n^t can be used to find γ-sets of Q_{3n}^t of cardinality $n+1$. We determine a general lower bound for $\gamma(Q_n^t)$ which is exact in infinitely many cases, and list exact values of $\gamma(Q_n^t)$ and $i(Q_n^t)$ for other small values of n. It turns out that not one of the functions β, γ and i is monotone. Some open problems are listed in Section 4.

2 Maximum independent sets of Q_n^t

We first consider the n-queens problem on the torus, that is, the problem to place n queens on an $n \times n$ board on the torus so that no two queens lie on the same row, column, s-diagonal or d-diagonal. Note that any independent set of Q_n^t is independent in Q_n (the queens graph of order n in the plane), hence $\beta(Q_n^t) \leq \beta(Q_n)$. We begin by showing that the existence of a solution to the n-queens problem on the torus is equivalent to the existence of a certain type of permutation of the set $N = \{0, 1, ..., n-1\}$.

Proposition 1 *The set S with $|S| = n$ is an independent set of Q_n^t if and only if $S = \{(x, f(x)) : x \in N\}$, where f is a permutation of N such that the functions $g, h : N \to N$ defined by $g(x) \equiv (f(x) + x) \pmod n$ and $h(x) \equiv (f(x) - x) \pmod n$ are permutations of N.*

Proof. Suppose $S = \{(x, f(x)) : x \in N\}$, where f, g and h are permutations of N. Clearly, no two elements of S are in the same column, nor, since f is a permutation, in the same row. The square $(x, f(x))$ lies on the s-diagonal $(f(x)+x) \pmod n$ and on the d-diagonal $(f(x)-x) \pmod n$, and since g and h are permutations, no two elements of S are in the same diagonal. Thus S is independent.

Conversely, if S is independent, then each row, column, s- and d-diagonal of Q_n^t contains exactly one element of S. Therefore $S = \{(x, f(x)) : x \in N\}$, where for each

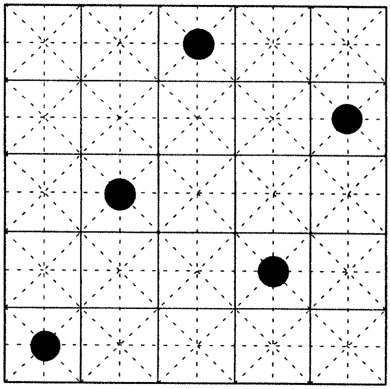

Figure 1

pair $x, x' \in N$ with $x \neq x'$, $f(x) \not\equiv f(x') \pmod{n}$, $f(x) + x \not\equiv (f(x') + x') \pmod{n}$ and $f(x) - x \not\equiv (f(x') - x') \pmod{n}$. Since N is finite it follows that f, g and h are permutations of N. ■

We now show that for certain values of n there do indeed exist permutations of $N = \{0, 1, ..., n-1\}$ that satisfy the conditions of Proposition 1.

Proposition 2 *Let $n \equiv 1$ or $5 \pmod{6}$ and define $f, g, h : N \to N$ by $f(x) \equiv 2x \pmod{n}$, $g(x) \equiv (f(x) + x) \equiv 3x \pmod{n}$ and $h(x) \equiv (f(x) - x) \equiv x \pmod{n}$ for each $x \in N$. Then f, g and h are permutations of N.*

Proof. Suppose $2x \equiv 2y \pmod{n}$ for some $x, y \in N$. Then $x \equiv y \pmod{n}$ since n is odd. But $x, y \in \{0, 1, ..., n-1\}$ and so it follows that $x = y$. Since f is one-to-one and N is finite, f is a permutation of N.

Now suppose $3x \equiv 3y \pmod{n}$ for some $x, y \in N$. Then $x \equiv y \pmod{n}$ since n is not a multiple of 3. As above we obtain that $x = y$ and that g is a permutation of N. The identity h is obviously a permutation. ■

Proposition 3 *If $n \equiv 1$ or $5 \pmod{6}$, then the set $S = \{(x, 2x) : x \in N\}$ (arithmetic modulo n) is an independent set of Q_n^t of cardinality n.*

Proof. The result follows immediately from Propositions 1 and 2. The set S is illustrated in Figure 1 for $n = 5$. ■

Corollary 4 *If $n \equiv 1$ or $5 \pmod{6}$, then $\beta(Q_n^t) = n$.*

Proof. The result follows from Proposition 3 and the obvious fact that $\beta(Q_n^t) \leq n$. ■

Our next result concerns the non-existence of permutations of N that satisfy the conditions of Proposition 1 in the case where n is even.

Theorem 5 *If n is even then there does not exist a permutation f of N such that the function $g : N \to N$ defined by $g(x) \equiv (f(x) + x) \pmod n$ is a permutation.*

Proof. Suppose to the contrary that the permutations f and g of N exist. Then

$$\begin{array}{rrcll} & \sum_{x=0}^{n-1}[x+f(x)] & \equiv & \sum_{x=0}^{n-1} g(x) \pmod n & \\ \therefore & 2\sum_{i=0}^{n-1} i & \equiv & \sum_{i=0}^{n-1} i \pmod n & \text{since } f, g \text{ are permutations} \\ \therefore & n(n-1) & \equiv & \frac{n}{2}(n-1) \pmod n & \\ \therefore & \frac{n}{2}(n-1) & = & kn, & \text{where } k \text{ is an integer} \\ \therefore & k & = & \frac{n-1}{2}, & \end{array}$$

a contradiction since $n-1$ is odd and k is an integer. ■

Corollary 6 *If n is even, then $\beta(Q_n^t) \leq n - 1$.*

We now give a configuration of $n-1$ independent queens on Q_n^t for $n \equiv 2$ or $10 \pmod{12}$, thus showing that the bound in Corollary 6 is exact if n is even but not a multiple of 3 or 4.

Theorem 7 *Let $n \geq 10$, $n \equiv 2$ or $10 \pmod{12}$ and define $f : N \to N$ by*

$$f(x) \equiv \begin{cases} 3x \pmod n & \text{if } x \in \{0, ..., \frac{n}{2} - 1\} \\ (n-3) \pmod n & \text{if } x = \frac{n}{2} \\ \frac{n}{2} & \text{if } x = \frac{n}{2} + 1 \\ 3(x-1) \pmod n & \text{if } x \in \{\frac{n}{2} + 2, ..., n-1\}. \end{cases}$$

Then f is a permutation of N and $S = \{(x, f(x)) : x \in N - \{\frac{n}{2}\}\}$ is an independent set of Q_n^t.

Proof. Clearly f is everywhere defined on N. We first prove that f is one-to-one. Suppose firstly that $x \in \{0, ..., \frac{n}{2} - 1\}$. If there exists $x' \in \{0, ..., \frac{n}{2} - 1\}$ such that $f(x) \equiv f(x') \pmod n$, then

$$\begin{array}{rcl} & 3x \equiv 3x' \pmod n, & \\ i.e., & x \equiv x' \pmod n & \text{since } 3 \nmid n. \end{array}$$

But then $x = x'$ since $x, x' \in \{0, ..., \frac{n}{2} - 1\}$. Now suppose $f(x) \equiv (n-3) \pmod n$. Then

$$\begin{array}{rcl} & n|(3x - n + 3), & \\ i.e., & n|(x+1) & \text{since } 3 \nmid n. \end{array}$$

Hence $x \equiv -1 \equiv (n-1) \pmod n$, contradicting the choice of x. Next, suppose $f(x) \equiv \frac{n}{2} \pmod n$, that is, $3x \equiv \frac{n}{2} \pmod n$. Since $3 \nmid \frac{n}{2}$ and $x \in \{0, ..., \frac{n}{2} - 1\}$, it follows that $3x = n + \frac{n}{2}$, *i.e.* $x = \frac{n}{2}$, which is not the case. Further, suppose there exists $x' \in \{\frac{n}{2} + 2, ..., n-1\}$ such that $f(x) \equiv f(x') \pmod{n}$. Then

$$\begin{aligned} 3x &\equiv 3(x'-1) \pmod{n}, \\ \textit{i.e.}, \quad x &\equiv (x'-1) \pmod{n} \qquad \text{since } 3 \nmid n. \end{aligned}$$

This is impossible by the choice of x and x'.

If $\frac{n}{2} \equiv (n-3) \pmod{n}$, then $n|(3-\frac{n}{2})$. The only possibility is $\frac{n}{2} = 3$, contradicting the conditions on n.

Now let $x \in \{\frac{n}{2}+2, ..., n-1\}$. If there exists $x' \in \{\frac{n}{2}+2, ..., n-1\}$ such that $f(x) \equiv f(x') \pmod{n}$, we obtain a contradiction as in the case where $x, x' \in \{0, ..., \frac{n}{2}-1\}$. Suppose $f(x) \equiv (n-3) \pmod{n}$. Then

$$\begin{aligned} &n|(3(x-1)-(n-3)), \\ \textit{i.e.}, \quad &n|3x \end{aligned}$$

and so $n|x$. Thus $x = 0$, contradicting the choice of x. Finally, suppose $f(x) \equiv \frac{n}{2} \pmod{n}$, that is, $3(x-1) \equiv \frac{n}{2} \pmod{n}$. By the choice of x, we have that $3(x-1) = \frac{n}{2} + 2n = 5\frac{n}{2}$ and so $3|\frac{n}{2}$, which contradicts the conditions on n. We have therefore shown that f is one-to-one and thus a permutation of the finite set N.

Now consider the set $S = \{(x, f(x)) : x \in N - \{\frac{n}{2}\}\}$ and the functions g and h as defined before. Since f is one-to-one, no two elements of S are in the same row or column of Q_n^t. To show that no two elements of S are on the same s- or d-diagonal, we must show that g and h restricted to $N - \{\frac{n}{2}\}$ are one-to-one.

Let $x, x' \in \{0, ..., \frac{n}{2}-1\}$. If $g(x) \equiv g(x') \pmod{n}$, then

$$\begin{aligned} 4x &\equiv 4x' \pmod{n}, \\ \textit{i.e.}, \quad n&|4(x-x'). \end{aligned}$$

Since $n \equiv 2 \pmod{4}$ it follows that

$$\tfrac{n}{2}|(x-x'),$$

and so, by the choice of x and x', $x = x'$. If $h(x) \equiv h(x') \pmod{n}$, then

$$2x \equiv 2x' \pmod{n}$$

and we obtain $x = x'$ as above. Now let $x \in \{0, ..., \frac{n}{2}-1\}$ and $x' = \frac{n}{2}+1$. If $g(x) \equiv g(x') \pmod{n}$, then

$$4x \equiv n+1 \equiv 1 \pmod{n}.$$

This is clearly impossible since $4x$ and n are both even. A similar argument shows that $h(x) \not\equiv h(x') \pmod{n}$. Next, let $x \in \{0, ..., \frac{n}{2}-1\}$ and $x' \in \{\frac{n}{2}+2, ..., n-1\}$. If $g(x) \equiv g(x') \pmod{n}$, then

$$4x \equiv (4x'-3) \pmod{n}$$

and the parity of these numbers shows that this is impossible. Similarly, $h(x) \not\equiv h(x') \pmod{n}$. Further, let $x = \frac{n}{2} + 1$ and $x' \in \{\frac{n}{2} + 2, ..., n - 1\}$. If $g(x) \equiv g(x') \pmod{n}$, then

$$\begin{aligned} & (n+1) \equiv (4x' - 3) \pmod{n}, \\ i.e., \quad & n | 4(x' - 1) \\ i.e., \quad & \tfrac{n}{2} | (x' - 1), \end{aligned}$$

which is clearly impossible by the choice of x'. Similarly, $h(x) \not\equiv h(x') \pmod{n}$. Finally, if $x, x' \in \{\frac{n}{2} + 2, ..., n - 1\}$ we obtain a contradiction as in the case where $x, x' \in \{0, ..., \frac{n}{2} - 1\}$. It follows that $g, h \upharpoonright (N - \{\frac{n}{2}\})$ are one-to-one and so S is independent. ■

The following corollary is immediate from Corollary 5 and Theorem 7.

Corollary 8 *If* $n \equiv 2$ *or* $10 \pmod{12}$, *then* $\beta(Q_n^t) = n - 1$.

We now prove a result similar to Theorem 5 concerning the non-existence of permutations of N that satisfy the conditions of Proposition 1 in the case where $n \equiv 3$ or $6 \pmod 9$. As is common practice, we denote the domain and range of a function f by $\mathrm{Dom}(f)$ and $\mathrm{Ran}(f)$, respectively. For easy reference we first state two results about the equivalence relation congruency modulo a given integer; the proofs are simple and omitted.

Lemma 9 *Let* $n \equiv 0 \pmod{p}$, $r \equiv i \pmod{p}$ *and* $s \equiv j \pmod{p}$, *where* $r + s \equiv k \pmod{n}$. *Then* $k \equiv (i + j) \pmod{p}$.

Lemma 10 *If* $n = pm$, *where* p *and* m *are positive integers, then for each* $i \in \{0, ..., p - 1\}$, N *has exactly* m *elements congruent to* $i \pmod{p}$.

Proposition 11 *If* $n \equiv 3$ *or* $6 \pmod 9$, *then there does not exist a permutation* f *of* N *such that the functions* $g, h : N \to N$ *defined by* $g(x) \equiv (f(x) + x) \pmod{n}$ *and* $h(x) \equiv (f(x) - x) \pmod{n}$ *are both permutations.*

Proof. Suppose to the contrary that such a permutation f of N exists. Let $n = 3m$, where m is not a multiple of 3, and for $i \in \{0, 1, 2\}$, define $[i] = \{x \in N : x \equiv i \pmod 3\}$. By Lemma 10, $|[i]| = m$ for each i. Since g is a permutation, $\mathrm{Ran}(g) = \cup_{i=0}^{2}[i] = N$. By Lemma 9 the m integers in $\mathrm{Ran}(g) \cap [0]$ result when

$$(x, f(x)) \in ([0] \times [0]) \cup ([1] \times [2]) \cup ([2] \times [1]). \tag{1}$$

For $N', N'' \subseteq N$, let $N' \times_f N'' = \{(x, f(x)) : x \in N' \text{ and } f(x) \in N''\}$. Suppose $|[0] \times_f [0]| = p$. Then, since $|[0]| = m$,

$$|([0] \times_f [1]) \cup ([0] \times_f [2])| = m - p,$$

and since $|[1] \cup [2]| = 2m$,

$$|([1] \cup [2]) \times_f ([1] \cup [2])| = 2m - (m - p) = m + p. \tag{2}$$

However, since $|[0]| = m$ and $|[0] \times_f [0]| = p$, it follows from (1) that

$$|([1] \times_f [2]) \cup ([2] \times_f [1])| = m - p, \tag{3}$$

and so, by (2) and (3),

$$\begin{aligned} |([1] \times_f [1])| + |([2] \times_f [2])| &= |([1] \times_f [1]) \cup ([2] \times_f [2])| \\ &= (m + p) - (m - p) = 2p. \end{aligned}$$

However, for each $i \in \{0, 1, 2\}$,

$$(x, f(x)) \in [i] \times_f [i] \Leftrightarrow h(x) \in [0].$$

Since h is a permutation, $\mathrm{Ran}(h) \cap [0] = [0]$, whence

$$\begin{aligned} m &= |[0]| \\ &= |([0] \times_f [0])| + |([1] \times_f [1])| + |([2] \times_f [2])| \\ &= 3p, \\ \textit{i.e.}, \quad p &= \tfrac{m}{3}. \end{aligned}$$

This contradicts the fact that m is not a multiple of 3 and the result follows. ■

Corollary 12 *If* $n \equiv 3$ *or* $6 \pmod 9$, *then* $\beta(Q_n^t) \leq n - 1$.

Although Corollaries 6, 8 and 12 show that the natural upper bound for $\beta(Q_n^t)$ is often not exact, we have not yet established a general lower bound for $n \equiv i \pmod{12}$, where $i \in \{0, 3, 4, 6, 8, 9\}$. However, we have established $\beta(Q_n^t)$ by computer for the small values of n given in Table 1; the solution given in each case lists the values of $f(x)$ for $x = 0, 1, ..., n-2$ and sometimes $n-1$, in that order. In the case where $n \equiv 1$ or $5 \pmod 6$, the solution is provided by Proposition 3, while if $n \equiv 2$ or $10 \pmod{12}$, an alternative solution is given by the set S defined in Theorem 7. Note that $\beta(Q_n^t)$ is not monotone!

n	$\beta(Q_n^t)$	Solutions
6	4	0,2,5,1
7	7	
8	6	0,2,5,1,6,4
9	7	0,2,4,7,1,3,5
10	9	0,2,7,5,3,1,9,4,6
11	11	
12	10	0,2,4,6,3,9,11,1,5,7
13	13	
14	13	0,2,5,8,11,13,3,6,4,1,12,9,7
15	13	0,2,4,7,9,12,5,13,1,3,6,8,10
16	14	0,2,4,6,8,10,5,13,15,1,3,7,9,11
17	17	
18	16	0,2,4,6,12,10,5,1,14,16,7,3,8,17,9,13

Table 1: $\beta(Q_n^t)$ for small values of n

3 Dominating sets of Q_n^t

We now show that there is a correspondence between β-sets of Q_n^t of size n and dominating sets of Q_{3n}^t of size n. Let r_i, c_i, d_i and s_i, $i \in N = \{0, 1, ..., n-1\}$, denote the rows, columns, difference diagonals and sum diagonals, respectively, of Q_n^t.

Theorem 13 *For any n, $\beta(Q_n^t) = n$ if and only if there is a dominating set of Q_{3n}^t of cardinality n.*

Proof. Suppose $\beta(Q_n^t) = n$ and let S be a β-set of Q_n^t. By Proposition 1 there is a permutation f of N such that $S = \{(x, f(x)) : x \in N\}$, where f satisfies the conditions on g and h as stated in Proposition 1. We show that $T = \{(3x, 3f(x)) : x \in N\}$ (arithmetic modulo $3n$) is an independent dominating set of Q_{3n}^t. Consider any square (x', y') of Q_{3n}^t. If $x' = 3x$ or $y' = 3y$ for some $x, y \in N$, then (since $y \in \mathrm{Ran}(f)$) one of $(3x, 3f(x)), (3f^{-1}(y), 3y) \in T$ dominates (x', y') by column or row. Hence suppose neither x' nor y' is a multiple of 3. Say $x' = 3p + r$, $y' = 3q + t$, where $p, q \in N$ and $r, t \in \{1, 2\}$.

If $r = t$, then by Lemma 9, $y' - x' = 3(q - p) \equiv 3a \pmod{3n}$ for some $a \in N$. But by Proposition 1 the function $h : N \to N$ defined by $h(x) \equiv (f(x) - x) \pmod n$ is a permutation, and so there exists $x \in N$ such that $f(x) - x \equiv a \pmod n$. Thus $(3x, 3f(x)) \in T$ and (x', y') lies on the same d-diagonal of Q_{3n}^t as $(3x, 3f(x))$; that is, T dominates (x', y').

Now assume without loss of generality that $r = 1, t = 2$. Then by Lemma 9, $x' + y' = 3(p + q + 1) \equiv 3b \pmod{3n}$ for some $b \in N$. Since $g : N \to N$ defined by $g(x) \equiv (f(x) + x) \pmod n$ is a permutation, there exists $x \in N$ such that $f(x) + x \equiv b \pmod n$. Then $(3x, 3f(x)) \in T$ and lies on the same s-diagonal as (x', y'). We have thus shown that T dominates Q_{3n}^t. Note that the elements of T

occur in every third row, column, s-diagonal and d-diagonal of Q_{3n}^t. Also note that T is independent – this follows directly from the fact than S is independent.

Conversely, suppose T is a dominating set of Q_{3n}^t of cardinality n. Let $\mathcal{R} = \{r_i : r_i \cap T = \phi,\ i = 0, 1, ..., 3n-1\}$ and $\mathcal{C} = \{c_i : c_i \cap T = \phi,\ i = 0, 1, ..., 3n-1\}$. We prove the following three lemmas.

Lemma 13.1 *For each $r_i \in \mathcal{R}$, T dominates every square in r_i exactly once.*

Proof of Lemma 13.1. Each element of T dominates each row in $\mathcal{R}$ exactly three times (by column, s-diagonal and d-diagonal). There are $3n$ squares in each row, each of which is dominated, and $|T| = n$. □

Lemma 13.2 *$\mathcal{R}$ contains at most two consecutive rows of Q_{3n}^t.*

Proof of Lemma 13.2. Suppose without loss of generality that $r_0, r_1, r_2 \in \mathcal{R}$. There obviously exist two adjacent columns c and c' such that $c \in \mathcal{C}$ and $c' \notin \mathcal{C}$; say without loss of generality that $c_0 \notin \mathcal{C}$ and $c_1 \in \mathcal{C}$. Since $r_1 \in \mathcal{R}$ ($c_1 \in \mathcal{C}$, respectively), there is no element of T on the same row (column) as the square (c_1, r_1). Since $c_0 \notin \mathcal{C}$, the squares (c_0, r_0) and (c_0, r_2) are on the same column as an element of T (which is not in rows r_0, r_1, r_2). By Lemma 13.1, T dominates each of (c_0, r_0) and (c_0, r_2) exactly once, and so there is no element of T on the d-diagonal $d = 0$ or on the s-diagonal $s = 2$. But these are the diagonals containing (c_1, r_1), and so T does not dominate (c_1, r_1), a contradiction. □

Lemma 13.3 *T is independent and its elements occur in every third row and third column of Q_{3n}^t.*

Proof of Lemma 13.3. Note that results similar to Lemmas 13.1 and 13.2 hold for the columns of Q_{3n}^t. If two elements of T are on the same row, column or diagonal, then some square in a row in $\mathcal{R}$ or a column in $\mathcal{C}$ is dominated more than once, contradicting Lemma 13.1. The latter part of the lemma follows directly from Lemma 13.2. □

We may therefore assume without loss of generality that the n elements of T occur in rows $3y$ and columns $3x$, $x, y \in N$. That is, there is a permutation f of N such that $T = \{(3x, 3f(x)) : x \in N\}$. Moreover, the elements of T occur in the diagonals $s \equiv 3(x + f(x))$ and $d \equiv 3(f(x) - x)$, $x, y \in N$. Since T is independent, this means that $x + f(x)$ and $f(x) - x$ take on n distinct values each. Therefore the functions g and h as defined above are permutations of N. By Proposition 1, $S = \{(x, f(x)) : x \in N\}$ is an independent set and hence a β-set of Q_n^t. ■

The proof of Theorem 13 also gives the following result.

Corollary 14 *For any n and any permutation f of N, $S = \{(x, f(x)) : x \in N\}$ is a β-set of Q_n^t if and only if $T = \{(3x, 3f(x)) : x \in N\}$ (arithmetic modulo $3n$) is a dominating set of Q_{3n}^t of cardinality n.*

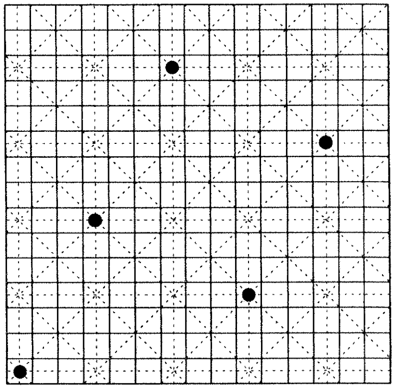

Figure 2

Note that Lemma 13.3 holds for any dominating set of Q_{3n}^t, that is,

Corollary 15 *Any dominating set of Q_{3n}^t of cardinality n is independent and its elements occur in every third row and third column of Q_{3n}^t.*

Corollary 16 *If $n \equiv 3$ or $15 \pmod{18}$, then $\gamma(Q_n^t) \leq i(Q_n^t) \leq \frac{n}{3}$.*

Proof. If $n \equiv 3$ or $15 \pmod{18}$, then $\frac{n}{3} \equiv 1$ or $5 \pmod 6$ and the result follows from Proposition 3. ■

The dominating set of Q_{15}^t obtained as described in the proof of Theorem 13 from the β-set of Q_5^t of Figure 1 is illustrated in Figure 2. The structure of this dominating set suggests that such dominating sets may well be optimal, and we now show that this is indeed the case.

Proposition 17 *For all $n \geq 1$, $\gamma(Q_n^t) \geq \lceil n/3 \rceil$.*

Proof. The result is obvious for $n \leq 3$ and we assume that $n \geq 4$. Consider any γ-set S of Q_n^t. If $S \cap r \neq \phi$ for each row r of Q_n^t, then $|S| \geq n > n/3$. Hence let r be any row such that $r \cap S = \phi$. Each element of S dominates r exactly three times (by column and s- and d-diagonals), hence in at most three squares (since if n is even, the s- and d-diagonals may intersect in r). Thus, since $|S|$ is integral,

$$|S| \geq \left\lceil \frac{n}{3} \right\rceil .$$

■

Proposition 3, Corollary 14 and Proposition 17 immediately yield the solution of the queens domination problem on the torus in the following cases:

Corollary 18 *Let $n \equiv 1$ or $5 \pmod 6$. The set $S = \{(3i, 6i) : i \in N\}$ (arithmetic modulo $3n$) is an independent dominating set of Q_{3n}^t of cardinality n.*

Corollary 19 *If $n \equiv 3$ or $15 \pmod{18}$, then $\gamma(Q_n^t) = i(Q_n^t) = \frac{n}{3}$.*

By Theorem 13 and Proposition 17, if $\beta(Q_n^t) < n$, then $\gamma(Q_{3n}^t) \geq n+1$. We now use the β-sets of Q_n^t, $n \equiv 2$ or $10 \pmod{12}$, defined in Theorem 7 to find dominating sets of Q_{3n}^t of cardinality $n+1$ for these values of n, thus determining exact values of $\gamma(Q_{3n}^t)$.

Theorem 20 *Let $n \geq 10$, $n \equiv 2$ or $10 \pmod{12}$ and define $f : N \to N$ as in Theorem 7. Let $S' = \{(x, f(x)) : x \in N\}$, $T' = \{(3x, 3f(x)) : x \in N\}$ (arithmetic modulo $3n$) and $T = T' \cup \{(0, 3n-9)\}$. Then T is a dominating set of Q_{3n}^t.*

Proof. As was shown in the proof of Theorem 7, f is a permutation of N. Thus the elements of T' occur in every third row and column of Q_{3n}^t. Also by Theorem 7, the set $S = S' - \{(\frac{n}{2}, n-3)\}$ is an independent set of Q_n^t. Hence $T' - \{(\frac{3n}{2}, 3n-9)\}$ is independent and its elements occur in every third s- and d-diagonal, except for one in each case, of Q_{3n}^t. We determine the missing diagonals.

By the congruency of n and the definitions of f and T', the elements $x \in \{0, ..., \frac{n}{2} - 1\}$ account for the s-diagonals $s \equiv 12x \pmod{3n}$ and for the d-diagonals $d \equiv 6x \pmod{3n}$, that is, *all* $\frac{n}{2}$ s-diagonals and *all* $\frac{n}{2}$ d-diagonals of Q_{3n}^t where $s, d \equiv 0 \pmod 6$. The elements $x \in \{\frac{n}{2}+2, ..., n-1\}$ account for the s-diagonals $s \equiv (12x - 9) \pmod{3n}$ and the d-diagonals $d \equiv (6x - 9) \pmod{3n}$, that is, all s- and d-diagonals of Q_{3n}^t, where $s, d \equiv 3 \pmod 6$, except $s \in \{3, 3n-9\}$ and $d \in \{3n-3, 3n-9\}$. The element $(\frac{3n}{2}+3, \frac{3n}{2}) \in T'$ accounts for the s-diagonal $s = 3$ and the d-diagonal $d = 3n-3$. Thus the missing diagonals are $s = d = 3n-9$. However, $(0, 3n-9) \in T$ lies on these diagonals, and so the elements of T occur in every third row, column, s-diagonal and d-diagonal of Q_{3n}^t. (Some lines of Q_{3n}^t contain more than one element of T.)

It can now be shown as in the proof of Theorem 13 that T dominates Q_{3n}^t. ∎

Corollary 21 *If $n \equiv 6$ or $30 \pmod{36}$, then $\gamma(Q_n^t) = \frac{n}{3} + 1$ and $i(Q_n^t) \leq \frac{n}{3} + 3$.*

Proof. That $\gamma(Q_n^t) = \frac{n}{3} + 1$ follows directly from Theorems 13 and 20 and Corollary 6. Let $n = 3m$ and consider the sets $S = \{(x, f(x)) : x \in \{0, 1, ..., m-1\} - \{\frac{m}{2}\}\}$ as defined in Theorem 7 and $X = \{(3x, 3f(x)) : x \in \{0, 1, ..., m-1\} - \{\frac{m}{2}\}\}$. Since S is an independent set of Q_m^t, it follows that X is an independent set of Q_n^t. As in the proof of Theorem 20, X dominates all squares of Q_n^t except squares in row $3m - 9 = n - 9$, column $\frac{3m}{2} = \frac{n}{2}$ and the diagonals $s = d = 3m - 9 = n - 9$. Thus it is easy to find four mutually independent squares a, b, c, d of Q_n^t, one on each of these lines (and thus non-adjacent to any of the elements of X), such that $X \cup \{a, b, c, d\}$ is an independent dominating set of Q_n^t) of cardinality $\frac{n}{3} + 3$. ∎

Consider Q_n, the queens graph of order n in the plane, and lable the rows and columns of Q_n from 0 to $n-1$. If S is any dominating set of Q_n, then $S \cup \{(n, n)\}$ is a dominating set of Q_{n+1}. Hence if $\gamma(Q_n) = k$, then $\gamma(Q_{n+m}) \leq k + m$ for any $m \geq 1$. However, since the s-diagonal s_i (d-diagonal d_i) of Q_n^t is not contained in the s-diagonal s_i (d-diagonal d_i) of Q_{n+1}^t ($i = 0, ..., n-1$), the same result does not hold for queens graphs on the torus. Therefore the exact values of $\gamma(Q_n^t)$ and $i(Q_n^t)$ given in Corollary 19 do not give upper bounds for the other cases. Moreover, although it is easy to see that any dominating set of Q_n also dominates Q_n^t, and hence that $\gamma(Q_n^t) \leq \gamma(Q_n)$, the corresponding inequality does not hold for the independent domination number i, as an independent set of Q_n is not necessarily independent on Q_n^t.

We have established exact values of $\gamma(Q_n^t)$ and $i(Q_n^t)$ for some small values of n, some by human brain and some by computer; these are listed in Tables 2 and 3 together with the corresponding values for Q_n for comparison. Two interesting facts emerge: all three possibilities $i(Q_n^t) = i(Q_n)$, $i(Q_n^t) < i(Q_n)$ and $i(Q_n^t) > i(Q_n)$ occur, and neither $\gamma(Q_n^t)$ nor $i(Q_n^t)$ is monotone! (Recall that in the case of Q_n the corresponding questions of monotonicity remain unresolved.) Also note that while $\gamma(Q_5) = i(Q_5) = \gamma(Q_5^t) = 3$, $i(Q_5^t) = \beta(Q_5^t) = 5$ – is this an exceptional case or are there other values of n for which $i(Q_n^t) = \beta(Q_n^t)$? (We believe the former.)

As can be seen from Tables 2 and 3, $\gamma(Q_n^t) = i(Q_n^t) = \lceil n/2 \rceil$ for some small values of n and we now show that $\gamma(Q_n^t) \leq i(Q_n^t) \leq n/2$ for infinitely many even values of n. Interestingly, the n-queens problem on the torus is again involved.

n	$\gamma(Q_n)$	$\gamma(Q_n^t)$	Solutions
3	1	1	
4	2	2	(0,0),(1,2)
5	3	3	(0,0),(2,2),(4,4)
6	3	3	(0,0),(2,4),(4,2)
7	4	4	(0,0),(1,1),(3,3),(5,5)
8	5	4	(0,0),(3,7),(4,3),(7,4)
9	5	5	(0,0),(1,3),(3,7),(5,1),(7,5)
10	5	5	(0,0),(2,4),(4,8),(6,2),(8,6)
11	5	5	(0,0),((2,6),(4,2),(6,9),(8,4)
12	6	**6**	(0,0),(2,2),(4,4),(6,10),(8,8),(10,6)
13	7	≤ 7	
14	8*	≤ 7	Corollary 23
15	9*	**5**	Fig. 2

*See [13]

Table 2: $\gamma(Q_n^t)$ for small values of n

n	$i(Q_n)$	$i(Q_n^t)$	Solutions
3	1	1	
4	3	2	See γ
5	3	5	Fig. 1
6	4	4	(0,0),(2,3),(3,5),(5,2)
7	4	5	(0,0),(1,4),(2,6),(3,1),(4,5)
8	5	4	See γ
9	5	5	See γ
10	5	5	See γ
11	5	5	See γ
12	7	6	(0,0),(1,2),(2,11),(6,5),(7,3),(8,6)
13	7	7	(0,0),(1,2),(2,4),(3,12),(4,1),(5,3),(6,11)
14	8	7	(0,0),(2,4),(4,8),(6,12),(8,2),(10,6),(12,10)
15	9	5	Fig. 2
16	9	8	(0,0),(1,4),(2,8),(3,5),(4,9),(5,13),(8,3),(13,10)

Table 3: $i(Q_n^t)$ for small values of n

Proposition 22 *If* $S = \{(x, f(x)) : x \in N\}$ *is an independent set of* Q_n^t*, then* $T = \{(2x, 2f(x)) : x \in N\}$ *(arithmetic modulo* $2n$*) is an independent dominating set of* Q_{2n}^t *of cardinality* n.

Proof. Consider any square (x', y') of Q_{2n}^t. If $x' = 2x$ or $y' = 2y$ for some $x, y \in N$, then (since $y \in \text{Ran}(f)$) T dominates (x', y') by column or row. So suppose x' and y' are odd. Then $x' + y'$ is even, hence by Lemma 9, $x' + y' \equiv 2p \pmod{2n}$ for some $p \in N$. But by Proposition 1 the function g as defined there is a permutation, and hence there exists $a \in N$ such that $g(a) = p$, *i.e.*, $(a, f(a)) \in S$ and lies on the s-diagonal $s = p$ of Q_n^t. Therefore $(2a, 2f(a)) \in T$ and lies on the s-diagonal $s = 2p$ of Q_{2n}^t, which is also the s-diagonal of Q_{2n}^t which contains (x', y'). Hence T dominates Q_{2n}^t. The independence of T follows directly from the independence of S. ■

Corollary 23 *If* $n \equiv 2$ *or* $10 \pmod{12}$*, then* $\gamma(Q_n^t) \le i(Q_n^t) \le \frac{n}{2}$.

Proof. If $n \equiv 2$ or $10 \pmod{12}$, then $\frac{n}{2} \equiv 1$ or $5 \pmod 6$ and the result follows from Propositions 3 and 22. ■

4 Problems

The n-queens problem

We have shown that if $n \equiv 1$ or $5 \pmod 6$, that is, $n \equiv 1, 5, 7, 11 \pmod{12}$, then $\beta(Q_n^t) = n$, and if $n \equiv 2$ or $10 \pmod{12}$, then $\beta(Q_n^t) = n-1$. For all other congruence classes modulo 12, that is, when n is divisible by 3 or 4, the values of β listed in Table 1 show that $\beta(Q_n^t) = n - 2$ if n is small.

1. In determining the above values of β, we have also shown that if n is even or $n \equiv 3$ or $6 \pmod 9$, then $\beta(Q_n^t) < n$. Can we extend these results to all multiples of 3?

2. Is it true in general that $\beta(Q_n^t) \geq n - 2$, and that $\beta(Q_n^t) = n - 2$ if $n \equiv 0, 3, 4, 6, 8, 9 \pmod{12}$?

3. In the cases where $\beta(Q_n^t) = n$, how many non-isomorphic solutions are there to the n-queens problem? We can also ask this question if $\beta(Q_n^t) < n$, but suspect that this will be a difficult problem to solve in general.

The queens domination problem

We have shown that if $n \equiv 3$ or $15 \pmod{18}$, that is, if $n \equiv 3, 15, 21, 33 \pmod{36}$, then $\gamma(Q_n^t) = i(Q_n^t) = \frac{n}{3}$, and if $n \equiv 6$ or $30 \pmod{36}$, then $\gamma(Q_n^t) = \frac{n}{3} + 1$. In each case optimal solutions were obtained from solutions to the m-queens problem, where $m = \frac{n}{3}$. The following questions arise:

1. If we can show that $\beta(Q_n^t) = n - 2$, can we use such a solution to show that $\gamma(Q_{3n}^t) = n + 2$?

2. Can we in any way use optimal solutions for $\gamma(Q_{3n}^t)$ to get bounds or solutions for $\gamma(Q_{3n+1}^t)$ and $\gamma(Q_{3n+2}^t)$?

3. Is the bound $i(Q_{3n}^t) \leq n + 3$, $n \equiv 2$ or $10 \pmod{12}$, exact?

4. As in (1), can we use a β-set of Q_n^t of cardinality $n - 2$ to show that $i(Q_{3n}^t) \leq n + 6$, or to obtain other upper bounds?

Some of the questions mentioned above are addressed in [3, 4].

Other domination parameters of Q_n^t

We have only considered the independence, domination and independent domination numbers of Q_n^t. The irredundance number, upper domination and irredundance numbers, total domination number and other domination type parameters still need to be investigated.

Graphs for other pieces on chessboards on the torus

For chessboards in the plane, domination related problems for the queens graph are considered to be to more difficult than the corresponding problems for the graphs of the other chess pieces. It thus seems reasonable to expect that many of these problems could be solved for the graphs of other pieces on chessboards on the torus.

Acknowledgement

This paper was written while E.J. Cockayne was visiting the Department of Mathematics, Applied Mathematics and Astronomy of the University of South Africa in the year 2000. Research support from the University, the South African National Research Foundation (NRF) and the Canadian National Science and Engineering Research Council (NSERC) is gratefully acknowledged.

References

[1] W. Ahrens, Mathematische Unterhalten und Spiele, *Leipzig-Berlin*, 1910.

[2] M. Bezzel, Schachfreund, *Berliner Schachzeitung* **3**(1848), 363.

[3] A.P. Burger, E.J. Cockayne, O. Favaron and C.M. Mynhardt, Permutations and the n-queens problem on the torus, submitted.

[4] A.P. Burger, E.J. Cockayne and C.M. Mynhardt, On the number of solutions for the n-queens problem on the torus, submitted.

[5] A.P. Burger and C.M. Mynhardt, Queens on hexagonal boards, *J. Combin. Math. Combin. Comput.* **31**(1999), 97–111.

[6] A.P. Burger and C.M. Mynhardt, Symmetry and domination in queens graphs, *Bulletin of the ICA* **29**(2000), 11–24.

[7] A.P. Burger and C.M. Mynhardt, Small irredundance numbers for queens graphs, *J. Combin. Math. Combin. Comput.* **33**(2000), 33–43.

[8] A.P. Burger and C.M. Mynhardt, Properties of dominating sets of the queens graph Q_{4k+3}, *Utilitas Math.* **57**(2000), 237–253.

[9] A.P. Burger and C.M. Mynhardt, An upper bound for the minimum number of queens covering the $n \times n$ chessboard, submitted.

[10] E.J. Cockayne and C.M. Mynhardt, Properties of queens graphs and the irredundance number of Q_7, *Australas. J. Combin.* **23**(2001), 285–299.

[11] C.F. de Jaenisch. *Applications de l'Analyse Mathematique au Jeu des Echecs.* Petrograd, 1862.

[12] G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi, A.A. McRae, C.K. Wallis, M.S. Jacobson, H.W. Martin and W.D. Weakley, Combinatorial Problems on Chessboards: A Brief Survey. In Y. Alavi and A.J. Schwenk, editors, *Graph Theory, Combinatorics, and Algorithms*, Volume 1, pages 507–528. Wiley-Interscience, New York, 1995.

[13] M.D. Kearse and P.B. Gibbons, Computational methods and new results for chessboard problems, *Australas. J. Combin.* **23**(2001), 253–284.

[14] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[15] S.M. Hedetniemi, S.T. Hedetniemi and R. Reynolds, Combinatorial problems on chessboards: II. In T.W. Haynes, S.T. Hedetniemi and P.J. Slater, editors, *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[16] W.D. Weakley, A lower bound for domination numbers of the queen's graph, preprint.

[17] W.D. Weakley, Upper bounds for domination numbers of the queen's graph, preprint.

(Received 30/8/2000)