Greedy defining sets of graphs

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Abstract

For a graph G and an order σ on $V(G)$, we define a greedy defining set as a subset S of $V(G)$ with an assignment of colors to vertices in S, such that the pre-coloring can be extended to a $\chi(G)$-coloring of G by the greedy coloring of (G, σ). A greedy defining set of a $\chi(G)$-coloring C of G is a greedy defining set, which results in the coloring C (by the greedy procedure). We denote the size of a greedy defining set of C with minimum cardinality by $GDN(G, \sigma, C)$. In this paper we show that the problem of determining $GDN(G, \sigma, C)$, for an instance (G, σ, C) is an NP-complete problem.

1 Introduction and preliminaries

We begin with some terminology and notation which are used throughout the paper. For the other necessary definitions and notation, we refer the reader to texts such as [4].
A harmonious coloring of a graph G is a proper vertex coloring of G, such that the subgraph induced on any two different colors has at most one edge.
By a hypergraph we mean an ordinary hypergraph which consists of a vertex set V and a collection of subsets of V, called (hyper)edges. In a hypergraph $\mathcal{H}$ a blocking set is a subset of the vertex set which intersects every edge of $\mathcal{H}$. A minimum blocking set is one with the smallest cardinality. Note that when H is a graph then a blocking set is usually called a vertex cover.
In the sequel we shall use the following VERTEX COVER problem (VC), which is well known to be NP-complete [2].

Instance: A graph $G = (V, E)$ and a positive integer k.

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Question: Is there a vertex cover of size k or less for G?

The NP-completeness of VC and background for the theory of NP-completeness can be found in [2].

One simple procedure for coloring the vertices of a graph $G = (V, E)$ by positive integers, so that adjacent vertices receive distinct colors, is to define a linear order σ on $V(G)$ and to process the vertices with respect to this order, giving them the smallest admissible color. The coloring procedure described above is called the **greedy coloring** of G with respect to σ. Therefore, with respect to any ordering on the vertices of G, we have a unique greedy coloring. Hence, naturally, by the greedy coloring of (G, σ), we mean the greedy coloring of G with respect to the order σ on $V(G)$.

It is not always the case that greedy coloring uses $\chi(G)$ colors. For example, the vertex set of the graph $K_{n,n} \backslash nK_2$ (the graph obtained by deleting n pairwise non-adjacent edges from $K_{n,n}$) has an order σ such that the greedy coloring with respect to σ uses n colors.

On the other hand, it is a simple well known fact that for every graph G there exists an order σ such that the greedy coloring of (G, σ) uses exactly $\chi(G)$ colors. For convenience, we call such an order σ, an **optimal order** of G. Similarly, for a $\chi(G)$ coloring C of G, we call an order σ, a C-**optimal order** if the greedy coloring of (G, σ) results in the coloring C. Note that for some colorings such an order may not exist. For example in the graph $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{ab, bc, ca, ad\}$ if we color vertices c, d by 3 and vertex a by 1, then the resulting 3-coloring does not admit an optimal ordering.

It is not known in general, when a greedy procedure gives an optimal coloring. In other words, the characterization of optimal orders of a given graph is an unsolved problem. Chvátal [1] has studied orders having a strong condition. He calls an order σ on the vertex set of a given graph G, a **perfect order** if for each induced subgraph H of G, with the induced order, the greedy procedure gives an optimal coloring of H. Also a graph is called **perfectly orderable** if it admits a perfect order. Chvátal showed that an order σ of G is perfect if in the orientation of G followed by σ for no induced path $P = p_1p_2p_3p_4$ the edges p_1p_2 and p_3p_4 are oriented from p_1 to p_2 and from p_4 to p_3. He also showed that any perfectly orderable graph is a perfect graph.

Obviously an optimal order is not necessarily a perfect order. Hence, Chvátal's results do not provide complete knowledge about optimal orders. On the other hand, for a given non-optimal order we may want to know how 'close' the order is to an optimal order.

In the following we introduce a new concept which enables us to find out some more characteristics of different orders and their associated greedy colorings.

Definition 1. *For a graph G and an order σ on $V(G)$, a* **greedy defining set** *is a subset S of $V(G)$ with an assignment of colors to vertices in S, such that the pre-coloring can be extended to a $\chi(G)$-coloring of G by a greedy coloring of (G, σ). The*

greedy defining number *of G is the size of a greedy defining set which has minimum cardinality, and is denoted by $GDN(G, \sigma)$. A* **greedy defining set for a $\chi(G)$-coloring** *C is a greedy defining set of G which results in C. The size of a greedy defining set of C with the smallest cardinality is denoted by $GDN(G, \sigma, C)$.*

Greedy defining sets are in fact a generalization of ordinary defining sets in vertex coloring of graphs. These have been widely studied in the literature (e.g. [3]).

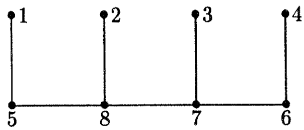

Fig.1

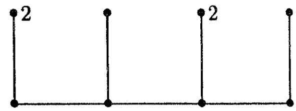

Fig.2

Example 1. In Figure 1, an ordered graph is shown in which the number next to a vertex is its order. The greedy coloring gives rise to a 4-coloring of the graph, where, if we color two vertices of degree one with number 2, as shown in Figure 2, and fix their colors, then the modified greedy coloring gives an optimal coloring. Hence it is easily seen that $GDN(G) = 2$.

2 Main Theorems

We still need some new concepts to interpret our previous concepts in terms of the theory of hypergraphs.

Definition 2. *Let (G, σ) be an ordered graph and C be a $\chi(G)$-coloring of G. We denote by G_{ij} the subgraph of G induced by two colors i and j. We also denote by $H_{ij}(v)$ the subgraph of G_{ij} induced by a vertex v and its neighbors. For each i and j, $i < j$, and any vertex v of G_{ij} with $C(v) = j$ we call $H_{ij}(v)$ a* **descending source** *if the order of every vertex in $H_{ij}(v) \backslash \{v\}$ is greater than the order of v. In this case the set of vertices of $H_{ij}(v)$ is called a* **descending set**. *We denote the set of all* **descending sets** *by $\mathcal{H}_C$. We also consider $\mathcal{H}_C$ as a hypergraph whose vertex set is $V(G)$.*

Definition 3. *We define $M_C(G, \sigma)$ to be the size of a minimum blocking set of $\mathcal{H}_C$ and $M(G, \sigma) = \min \{ M_C(G, \sigma) \mid C \text{ is a } \chi(G)\text{-coloring of } G\}$.*

Our first theorem establishes a connection between greedy defining sets and blocking sets.

Theorem 1. *The vertices of any greedy defining set of a graph G is a blocking set of $\mathcal{H}_C$ for some coloring C, and for a coloring C of G any blocking set of $\mathcal{H}_C$ introduces a greedy defining set of C.*

Proof. Let S be a greedy defining set of G, and C be the $\chi(G)$–coloring of G which is the extension of S. Let $V(H_{ij}(v))$ be any hyperedge of $\mathcal{H}_C$. If $S \cap V(H_{ij}(v)) = \emptyset$ then when the precoloring is going to be extended to the coloring C, the vertex v may take the color i rather than j, but by the definition of $H_{ij}(v)$ we have $C(v) = j$. So we conclude that S is a blocking set for $\mathcal{H}_C$.
Conversely, we show that any blocking set B of $\mathcal{H}_C$ is a greedy defining set for C. Consider the induced coloring of C on B. Suppose we extend the coloring of B induced by C, greedily, and we obtain the coloring C'. We claim that $C = C'$. We prove by induction on the order of the vertex v that $C'(v) = C(v)$. Suppose $v \in V(G) \setminus B$ is the first vertex which is to be colored by the greedy procedure. Let $C(v) = j$ and $i < j$. It is clear that $H_{ij}(v)$ is a desending source and it has to intersect the greedy defining set B. Therefore, v has a neighbor with color i in B, which implies that $C'(v) \geq i$. On the other hand, v has no neighbor with color j in B. So the greedy procedure colors v by j, and consequently $C'(v) = j$. Now suppose $v \in V(G) \setminus B$ is an arbitrary vertex with $C(v) = j$ and let $i < j$. If there exists a vertex u in $H_{ij}(v)$ which is lower than v, then $C'(u) = i$ by the induction hypothesis $C'(v) \geq j$. Otherwise, $H_{ij}(v)$ is a descending source and hence some neighbor of v in $H_{ij}(v)$ belongs to B and therefore the color i appears in $H_{ij}(v)$. So $C'(v) \geq j$. On the other hand, the vertex v can't have a neighbor with color j (in the coloring C), and because in every stage of the pre-coloring extention of the greedy defining set B, we have a partial coloring of the whole coloring C, we conclude that $C'(v) = j$. □

As a corollary we obtain $M(G, \sigma) = GDN(G, \sigma)$.

We now pose the problem of the complexity status of finding the greedy defining number of a coloring. We consider the following problem which we call GREEDY DEFINING NUMBER, or GDN for short:

Instance: An ordered graph (G, σ), a $\chi(G)$-coloring C and an integer k.

Question: Does C have a greedy defining set of size less than or equal to k?

Theorem 2. *GDN is an* NP*-complete problem.*

Proof. It is easy to see that GDN $\in$ NP, since a nondeterministic algorithm needs only guess a subset of vertices with the appropriate size and check in polynomial time whether these colored vertices can be extended greedily to C.
We transform the vertex cover problem VC to GDN.
Let (F, k) be an instance of VC where F is a simple graph and k an integer. In order to transform this instance to an instance of GDN we require a harmonious coloring of F. For this we consider a vertex coloring of F with exactly $n = |V(F)|$ colors which is trivially a harmonious coloring. Let the set of colors be {1,2,...,n}. Now we construct an instance of GDN as follows: Let G be the graph obtained by attaching

a complete graph K_n to the vertex of F whose color in the harmonious coloring of F is n. There are $n-1$ vertices of K_n which are not colored yet. We color these vertices by $1, 2, ..., n-1$. We obtain an n-coloring of the whole graph G and denote it by C. Now we define an order σ on $V(G)$ as follows:
Any vertex $v \in V(G)$ either belongs to $V(K_n)$ or to $V(F)$. If $v \in V(K_n)$ and v is colored by i, $1 \leq i \leq n$, then we define the order of v to be i. If $v \in V(F)$ and its color is j, then we define its order to be $2n - j$, where $1 \leq j \leq n$. Our instance of GDN has been now constructed. It can be easily checked that $\mathcal{H}_C$ is isomorphic to F by a hypergraph isomorphism. So clearly a blocking set for $\mathcal{H}_C$ is a vertex cover of F and vice versa. The proof is complete by the previous theorem. □

We end the paper with an open problem.

Problem. *Is GREEDY DEFINING NUMBER for uncolored graphs an* NP*-complete problem ?*

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References

[1] V. Chvátal, *Perfectly ordered graphs*, in C. Berge and V. Chvátal, Eds., Topics on Perfect Graphs, North-Holland, Amesterdam, 1984, 63–65.

[2] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of* NP*-completness*, Freeman, New York, 1979.

[3] E.S. Mahmoodian, R. Naserasr, M. Zaker, *Defining sets in vertex colorings of graphs and latin rectangles*, Discrete Math. **167/168** (1997), 451–460.

[4] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 1996.

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