The triangle chromatic index of Steiner triple systems

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Abstract

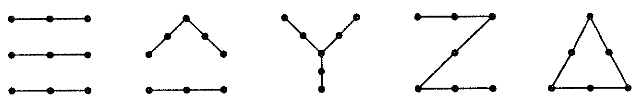
In a Steiner triple system of order v, STS(v), a set of three lines intersecting pairwise in three distinct points is called a *triangle*. A set of lines containing no triangle is called *triangle-free*. The minimum number of triangle-free sets required to partition the lines of a Steiner triple system S, is called the *triangle chromatic index of* S. We prove that for all admissible v, there exists an STS(v) with triangle chromatic index at most $8\sqrt{3v}$. In addition, by showing that the projective geometry PG$(n,3)$ may be partitioned into $O(6^{n/5})$ caps, we prove that the STS(v) formed from the points and lines of the affine geometry AG$(n,3)$ has triangle chromatic index at most Av^s, where $s = \log_e 6/(3\log_e 5) \approx 0.326186$, and A is a constant. We also determine the values of the index for STS(v) with $v \leq 13$.

1 Introduction

Recent papers [5, 7] have investigated generalised chromatic indices for Steiner triple systems. The former paper was concerned with the so-called 2-parallel chromatic index, and the latter with four of the five three-line chromatic indices. In the current paper we present results on the remaining three-line chromatic index of a Steiner triple system S, namely the triangle chromatic index $\chi(B_5, S)$. We shall show that this behaves very differently from the other two- and three-line chromatic indices.

For Steiner triple systems of order 3^n, our estimate of a lower bound for $\chi(B_5, S)$ is related to the question of partitioning the projective geometry $\mathrm{PG}(n,3)$ into the minimum number of caps and, in turn, to the chromatic number of the affine geometry $\mathrm{AG}(n,3)$.

A balanced incomplete block design $\mathrm{BIBD}(v,k,\lambda)$ is an ordered pair $(V, \mathcal{B})$, where V is a set of cardinality v (the *points*) and $\mathcal{B}$ is a collection of k-element subsets of V (the *blocks*) which has the property that every 2-element subset of V is contained in precisely λ blocks. A $\mathrm{BIBD}(v,3,1)$ is called a Steiner triple system of order v, $\mathrm{STS}(v)$, and the blocks are then also referred to as *triples* or *lines*. It is well-known that an $\mathrm{STS}(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$; such values of v are called *admissible*. If S is an $\mathrm{STS}(v)$ then its *chromatic index* $\chi'(S)$ is the smallest number of colours required to colour the lines of S, each with a single colour, so that no two intersecting lines receive the same colour. The generalisation of this concept given in [5] relates to colouring the lines of an $\mathrm{STS}(v)$ so as to avoid monochromatic copies of a configuration C. By a *configuration* C we simply mean a collection of lines of an $\mathrm{STS}(v)$. The resulting chromatic index is denoted by $\chi(C,S)$. The possible 2-line configurations are: (a) two lines intersecting in a point, and (b) two parallel (i.e. non-intersecting) lines. In the former case $\chi(C,S)$ is just the ordinary chromatic index $\chi'(S)$. The latter case gives rise to the 2-parallel chromatic index denoted by $\chi''(S)$. There are five 3-line configurations B_1, B_2, B_3, B_4, B_5 which appear in Steiner triple systems and these are shown in Figure 1 with their traditional names.



B_1	B_2	B_3	B_4	B_5
(3-ppc)	(Hut)	(3-star)	(3-path)	(Triangle)

Figure 1: The five 3-line configurations.

It is generally difficult to determine the precise value of $\chi(C,S)$ for given C and S. However, it is possible to obtain upper and lower bounds in some cases. For admissible v we may define

$$\begin{aligned}\overline{\chi}(C,v) &= \max\{\chi(C,S) : S \text{ is an } \mathrm{STS}(v)\} \text{ and}\\ \underline{\chi}(C,v) &= \min\{\chi(C,S) : S \text{ is an } \mathrm{STS}(v)\}.\end{aligned}$$

(And we can make similar definitions for $\chi'(S)$ and $\chi''(S)$.)

Regarding the ordinary chromatic index, for $v \equiv 3 \pmod 6$ we have $\underline{\chi}'(v) = (v-1)/2$, a result which is equivalent to the theorem of Ray-Chaudhuri and Wilson [14] concerning the existence of Kirkman triple systems. For $v \equiv 1 \pmod 6$ and $v \geq 19$ we have $\underline{\chi}'(v) = (v+1)/2$, which is equivalent to the existence of Hanani triple systems [16]. In the 2-parallel case, it is shown in [5] that if $v \geq 27$ then, for $v \equiv 3$ or $7 \pmod{12}$, $\underline{\chi}''(v) = (v-1)/2$, and for $v \equiv 1$ or $9 \pmod{12}$, $\underline{\chi}''(v) = (v+1)/2$. In [7], $\underline{\chi}(C, v)$ is precisely determined (for all sufficiently large v) for each of the three-line configurations $C = B_1, B_2$ or B_3, and an asymptotic estimate for $\underline{\chi}(B_4, v)$ is also given. In the case of each of these two- and three-line configurations, $\underline{\chi}(C, v) \sim kv$ as $v \to \infty$, for an appropriate constant k. In the current paper we prove that $\underline{\chi}(B_5, v) \leq 8\sqrt{3v}$.

In the course of our investigations we need a few more items of basic terminology. A set of n parallel (i.e. mutually disjoint) lines of an STS(v) is called an (n-)partial parallel class, abbreviated to (n-)ppc. If $n = v/3$, the maximum possible value, then an n-ppc is called a (full) parallel class. An STS(v) whose lines may be partitioned into full parallel classes is said to be *resolvable*. Such a design together with its partition is called a Kirkman triple system, KTS(v). These exist if and only if $v \equiv 3 \pmod 6$ [14]. A BIBD(v, k, λ) is said to be *cyclic* if it has an automorphism of order v. The design may then be formed as a union of orbits of k-element subsets of the point set V under the action of the cyclic group generated by this automorphism.

Denote by F_3^n the vector space of dimension n over F_3, the field of order 3. We will take the elements of F_3 to be 0, 1 and 2 ($= -1$), and we will write elements of F_3^n without brackets or commas, for example $0120 \in F_3^4$. The *affine geometry* of dimension n, AG(n, 3), is the set of all cosets of subspaces of F_3^n. For $k = 0, 1, \ldots, n$, a k-flat of AG(n, 3) is a coset of a subspace of dimension k. The *projective geometry* of dimension n, PG(n, 3), is the set of equivalence classes of non-zero points from AG($n+1$, 3) under the equivalence relation $\sim$ given by $x \sim y$ if $x = \lambda y$ for $\lambda = 1$ or 2. For $k = 0, 1, \ldots, n$, a k-flat of PG(n, 3) is defined to be the image of a $(k+1)$-flat of AG($n+1$, 3). In both AG(n, 3) and PG(n, 3), the 0-flats are called points (in the former case identified with the elements of F_3^n), the 1-flats are called lines and the 2-flats are called planes. The lines of AG(n, 3) comprise triples of distinct points $\{x, y, z\}$ in F_3^n for which $x + y + z = 0$, and the points and lines of AG(n, 3) form an STS(3^n). Three distinct points x, y, z of PG(n, 3) are collinear if and only if $x \pm y \pm z = 0$ in F_3^{n+1}.

2 Decomposing PG(n, 3) into caps

A *cap* in AG(n, 3) or PG(n, 3) is a set of points, no three of which are collinear. A cap of cardinality k is called a k-cap. A cap is *maximal* if it is not properly contained in any other cap. We will here denote by $A(n)$ and $P(n)$ the sizes of the largest maximal caps in AG(n, 3) and in PG(n, 3) respectively. For $n \leq 5$, Table 1 gives precise values for $A(n)$ and $P(n)$ with the exception of $A(5)$ which Bruen, Haddad and Wehlau [2] have shown to lie between 45 and 48 (inclusive).

Both $(A(n))^{\frac{1}{n}}$ and $(P(n))^{\frac{1}{n}}$ have a common limiting value $c \leq 3$. It is shown by Calderbank and Fishburn [3] that $c > 2.210147$.

n	1	2	3	4	5
$A(n)$	2	4	9^*	20^{P}	$45 \leq A(5) \leq 48$
$P(n)$	2	4^{B}	10^{B}	20^{P}	56^{H}

(B: Bose [1]; H: Hill [10]; P: Pellegrino [13]; *: see, for example, [11]. The other values are easily checked.)

Table 1.

Denote the minimum number of caps required to partition AG$(n, 3)$ and PG$(n, 3)$, by $\alpha(n)$ and $\pi(n)$ respectively. The values of $A(n)$ and $P(n)$ may be used to provide bounds for $\alpha(n)$ and $\pi(n)$. The values $\pi(1) = 2$ and $\pi(2) = 4$ are easily verified. Ebert [6] proves that $\pi(4r-1) \leq (3^{2r}-1)/2$ giving, in particular, $\pi(3) \leq 4$. From consideration of $P(3)$ it follows that $\pi(3) = 4$. We show below that $\pi(4) = 7$ and that $7 \leq \pi(5) \leq 12$. The values $\alpha(1) = 2$ and $\alpha(2) = 3$ are also easily verified. The following is a partition of AG$(3,3)$ into three caps Q_1, Q_2 and Q_3:

$$\begin{aligned} Q_1 &= \{000, 200, 020, 220, 102, 012, 212, 122, 111\} \\ Q_2 &= \{001, 201, 021, 221, 100, 010, 210, 120, 112\} \\ Q_3 &= \{002, 202, 022, 222, 101, 011, 211, 121, 110\} \end{aligned}$$

By considering $A(3)$ it follows that $\alpha(3) = 3$. Haddad [8] shows that $\alpha(4) = 5$, and in [2] it is shown that $\alpha(5) = 6$. These results are summarised in Table 2.

n	1	2	3	4	5
$\alpha(n)$	2	3	3	5	6
$\pi(n)$	2	4	4	7	$7 \leq \pi(5) \leq 12$

Table 2.

Our results depend on recursive constructions given in Theorems 2.1 and 2.2.

Theorem 2.1 *Given a partition of* $AG(m, 3)$ *into* M *caps and a partition of* $AG(n, 3)$ *into* N *caps, we may form a partition of* $AG(m+n, 3)$ *into* MN *caps.*
Proof. The points of AG$(m+n, 3)$ may be considered as pairs (x, y) where $x \in F_3^m$ and $y \in F_3^n$. Denote by $P_1, P_2, \ldots, P_M$ the caps decomposing AG$(m, 3)$ and by $Q_1, Q_2, \ldots, Q_N$ the caps decomposing AG$(n, 3)$. Put $S_{i,j} = \{(x, y) : x \in P_i, y \in Q_j\}$. The sets $S_{i,j}$ decompose AG$(m+n, 3)$, the decomposition having MN parts. Suppose that $z_k = (x_k, y_k), k = 1, 2, 3$, are points of $S_{i,j}$ satisfying $z_1 + z_2 + z_3 = 0$. Then $x_1 + x_2 + x_3 = 0$ and so $x_1 = x_2 = x_3$. Likewise, $y_1 = y_2 = y_3$, and so $z_1 = z_2 = z_3$. Thus each $S_{i,j}$ is a cap in AG$(m+n, 3)$. $\square$

Corollary 2.1 *For* $m, n \geq 1$, $\alpha(m+n) \leq \alpha(m)\alpha(n)$. $\square$

Theorem 2.2 *Given a partition of $AG(m,3)$ into M caps, a partition of $PG(n,3)$ into N caps, and a partition of $PG(m-1,3)$ into L caps, we may form a partition of $PG(m+n,3)$ into $MN+L$ caps.*

Proof. The points of $\mathrm{PG}(r,3)$ may be taken as those of $F_3^{r+1} \setminus \{0\}$, with one representative point, say that with last non-zero coordinate equal to 1, selected from each of the two alternatives in each equivalence class. We may then consider the points of $\mathrm{PG}(m+n,3)$ as pairs (x,y) where either

(a) $x \in F_3^m$, $y \in \mathrm{PG}(n,3)$, or

(b) $x \in \mathrm{PG}(m-1,3)$, $y = 0 \in F_3^{n+1}$.

Denote by $P_1, P_2, \ldots, P_M$ the caps decomposing $\mathrm{AG}(m,3)$, by $Q_1, Q_2, \ldots, Q_N$ the caps decomposing $\mathrm{PG}(n,3)$, and by $R_1, R_2, \ldots, R_L$ the caps decomposing $\mathrm{PG}(m-1,3)$. Put $S_{i,j} = \{(x,y) : x \in P_i, y \in Q_j\}$ and $T_k = \{(x,y) : x \in R_k, y = 0 \in F_3^{n+1}\}$. The sets $S_{i,j}$ and T_k form a decomposition of the points of $\mathrm{PG}(m+n,3)$, the decomposition having $MN+L$ parts. Suppose that $z_k = (x_k, y_k), k = 1,2,3$, are points of $S_{i,j}$ satisfying $z_1 \pm z_2 \pm z_3 = 0$. Then $y_1 \pm y_2 \pm y_3 = 0$. But $y_1, y_2, y_3 \in Q_j$ and so the only solution of this is $y_1 = y_2 = y_3$, giving $y_1 + y_2 + y_3 = 0$. Consequently, $x_1 + x_2 + x_3 = 0$ and the only solution of this in P_i is $x_1 = x_2 = x_3$. Hence $z_1 = z_2 = z_3$. Thus each $S_{i,j}$ is a cap in $\mathrm{PG}(m+n,3)$. That each set T_k is a cap in $\mathrm{PG}(m+n,3)$ follows immediately from the fact that the corresponding R_k is a cap in $\mathrm{PG}(m-1,3)$. □

Corollary 2.2 *For $m, n \geq 1$, $\pi(m+n) \leq \alpha(m)\pi(n) + \pi(m-1)$.* □

Remark. It is sometimes possible to improve the previous estimate for $\pi(m+n)$ by amalgamating some, or even all, of the caps T_k with the caps $S_{i,j}$ described in Theorem 2.2. In general, T_k can be amalgamated with $S_{i,j}$ if, for each $z = (x,y)$ with $x \in P_i, y \in Q_j$, and for each $w = (u,0)$ with $u \in R_k$ (and $0 \in F_3^{n+1}$), we have $(x \pm u)$ distinct from every point of P_i. Note that this property is not dependent on j. Note also that it may therefore be advantageous to select the caps R_k not to form a *minimal* partition of $\mathrm{PG}(m-1,3)$ but, rather, to permit good amalgamation. One may further observe that it is not necessary to use the same cap partition $\{P_i\}$ for each separate cap Q_j. This may aid further amalgamations as in Lemma 2.2 below.

As an example, consider the case $m = 2$. In Theorem 2.2, we may take $P_1 = \{00, 10, 01, 11\}, P_2 = \{02, 20, 12\}, P_3 = \{21, 22\}$, and $R_1 = \{01\}, R_2 = \{21\}, R_3 = \{10, 11\}$. Here T_1 can be amalgamated with $S_{2,j}$ (for any j), T_2 with $S_{3,k}$ (for any k) and T_3 with $S_{3,k'}$ (for any $k' \neq k$). Noting that $\pi(n) \geq 2$ for $n \geq 1$, we obtain the following result.

Lemma 2.1 *For $n \geq 1$, $\pi(2+n) \leq 3\pi(n)$.* □

Corollary 2.3 $7 \leq \pi(5) \leq 12$.

Proof. The lower bound follows from $P(5) = 56$. The upper bound follows from the previous Lemma by taking $n = 3$. □

Lemma 2.2 $\pi(4) = 7$.
Proof. Take the partition of PG(3,3) into four caps Q_1, Q_2, Q_3, Q_4 given by

$$\begin{aligned}
Q_1 &= \{1000, 2100, 1110, 2001, 1201, 0211, 1010, 2121, 2021, 2111\} \\
Q_2 &= \{0010, 0021, 1211, 0110, 0201, 2011, 1210, 2211, 2210, 1011\} \\
Q_3 &= \{0001, 2101, 1021, 0011, 2110, 1101, 0121, 1121, 0221, 1001\} \\
Q_4 &= \{0100, 0210, 0111, 1100, 2010, 1221, 0101, 2221, 2201, 1111\}.
\end{aligned}$$

Next put

$$\begin{aligned}
C_1 &= \{(1,x), (2,x), x \in Q_1\} \\
C_2 &= \{(1,x), (2,x), x \in Q_2\} \\
C_3 &= \{(1,x), (2,x), x \in Q_3\} \\
C_4 &= \{(0,x), (2,x), x \in Q_4\} \\
C_5 &= \{(0,x), x \in Q_1\} \cup \{10000, 10100, 10111, 10101, 12010\} \\
C_6 &= \{(0,x), x \in Q_2\} \cup \{11100, 12201, 10210\} \\
C_7 &= \{(0,x), x \in Q_3\} \cup \{11221, 12221, 11111\}.
\end{aligned}$$

Then $\{C_i : i = 1, 2, \ldots, 7\}$ forms a partition of PG(4,3) into seven caps. Hence $\pi(4) \leq 7$. Since $P(4) = 20$, we also have $\pi(4) \geq 7$. $\square$

Theorem 2.3 *For $n \geq 6$, $\pi(n) \leq (67.6^r - 7)/5$, where $r = \lfloor \frac{n-1}{5} \rfloor$.*
Proof. Note that $\pi(i) \leq 12$ for $i = 1, 2, 3, 4, 5$. Put $n = 5r + i$, where $r \geq 1$ and $1 \leq i \leq 5$. Applying Corollary 2.2 r times, we have

$$\begin{aligned}
\pi(n) &\leq \pi(i)(\alpha(5))^r + \pi(4)\left[\frac{(\alpha(5))^r - 1}{\alpha(5) - 1}\right] \\
&\leq 12.6^r + \frac{7}{5}(6^r - 1) = (67.6^r - 7)/5.
\end{aligned}$$

$\square$

Theorem 2.4 *The limits $\lim_{n\to\infty}(\pi(n))^{\frac{1}{n}}$ and $\lim_{n\to\infty}(\alpha(n))^{\frac{1}{n}}$ both exist and have a common value $l \leq 6^{\frac{1}{5}} \approx 1.430969$. Furthermore, $l \geq 3/c$, where $c = \lim_{n\to\infty}(P(n))^{\frac{1}{n}}$.*
Proof. By Corollary 2.1, the function $\log_e \alpha$ is sub-additive and so $(\alpha(n))^{\frac{1}{n}}$ has a limiting value $l = \inf\{(\alpha(n))^{\frac{1}{n}}\}$. By inspecting the values of $\alpha(n)$ given earlier, $\inf\{(\alpha(n)^{\frac{1}{n}}\} \leq (\alpha(5))^{\frac{1}{5}} = 6^{\frac{1}{5}}$. Corollary 2.2 gives

$$\pi(rm + n) \leq \left(\pi(n) + \frac{\pi(m-1)}{\alpha(m) - 1}\right)(\alpha(m))^r,$$

from which we may deduce that $\limsup_{n\to\infty}(\pi(n))^{\frac{1}{n}} \leq l$.

Now consider a partition of PG$(n,3)$ into $\pi(n)$ caps. If C_i is one of these caps then, for each $x \in C_i$, take a representative of this point in F_3^{n+1}, say $\bar{x}$. Then define $\bar{C}_i = \{y \in F_3^{n+1} : y = \bar{x} \text{ or } y = 2\bar{x} \text{ for some } x \in C_i\}$. Each $\bar{C}_i$ forms a cap in $\mathrm{AG}(n+1,3)$ and these caps partition $\mathrm{AG}(n+1,3)\setminus\{0\}$. Consequently, $\alpha(n+1)\le\pi(n)+1$. From this inequality it easily follows that $\liminf_{n\to\infty}(\pi(n))^{\frac{1}{n}}\ge l$. Thus we deduce that $(\pi(n))^{\frac{1}{n}}$ also has limiting value l.

As noted earlier, both $(A(n))^{\frac{1}{n}}$ and $(P(n))^{\frac{1}{n}}$ have a common limiting value $c>2.210147$. Consequently, $\liminf_{n\to\infty}(\pi(n))^{\frac{1}{n}}$ and $\liminf_{n\to\infty}(\alpha(n))^{\frac{1}{n}}$ are both at least $3/c$. Thus $l\ge 3/c$. □

3 The triangle chromatic index

Theorem 3.1 *If v is of the form $v=3^n$, then $\underline{\chi}(B_5,v)\le Av^s$, where A is a constant and $s=\frac{\log_e 6}{5\log_e 3}\approx 0.326186$.*

Proof. Consider the Steiner triple system of order $v=3^n$ formed from the points and lines of $\mathrm{AG}(n,3)$. This is resolvable into $(3^n-1)/2$ parallel classes, each class being a coset of a line through the point 0. Thus each parallel class corresponds to a point of $\mathrm{PG}(n-1,3)$. If $P_1,P_2,\ldots,P_k$ denote k of these parallel classes and $p_1,p_2,\ldots,p_k$ are the corresponding points of $\mathrm{PG}(n-1,3)$, then $\bigcup_{i=1}^k P_i$ is triangle-free if the set $\{p_1,p_2,\ldots,p_k\}$ forms a cap in $\mathrm{PG}(n-1,3)$. It follows that $\underline{\chi}(B_5,3^n)$ is bounded above by $\pi(n-1)$. This gives $\underline{\chi}(B_5,3^n)\le(67.6^r-7)/5$, where $r=\lfloor\frac{n-2}{5}\rfloor$, provided $n\ge 7$. Replacing 3^n by v, we obtain $\underline{\chi}(B_5,v)\le Av^s$ where $s=\frac{\log_e 6}{5\log_e 3}$ and $A=67/(5\cdot 36^{1/5})$, and in fact this bound also holds for $n=1,2,3,4,5$ and 6. □

In a sense, Theorem 3.1 is our best result for the triangle chromatic index in that it has the lowest exponent s. However, we can make progress when v is not of the form 3^n by at least two distinct methods. One is to compute $\chi(B_5,S)$ for "small" systems S and to employ a product construction. A second is to obtain bounds on the analogous triangle chromatic index for transversal designs $\mathrm{TD}(3,n)$, i.e. Latin squares of side n. We firstly turn our attention to "small" systems S.

There is, up to isomorphism, precisely one STS(7), one STS(9), two STS(13)s and 80 STS(15)s. We denote the unique STS(7) by S_7 and the unique STS(9) by S_9. Of the two STS(13)s, one is cyclic and one is not. The cyclic one we here denote by C, and the non-cyclic one by N. We may then state the following result.

Lemma 3.1 $\chi(B_5,S_7)=3$, $\chi(B_5,S_9)=2$, $\chi(B_5,C)=4$ *and* $\chi(B_5,N)=3$.

Proof. In each case the triangle chromatic index is at least two. For S_7, it is easily seen that any set of four lines contains a triangle, so that $\chi(B_5,S_7)\ge 3$. On the other hand, the following partition of the lines of S_7 provides a triangle-free 3-colouring: $C_1=\{\{0,1,2\},\{0,3,4\},\{0,5,6\}\}$, $C_2=\{\{1,3,5\},\{2,4,5\}\}$, $C_3=\{\{1,4,6\},\{2,3,6\}\}$. Hence $\chi(B_5,S_7)=3$.

The system S_9 is resolvable into four parallel classes P_1,P_2,P_3 and P_4. The sets $P_1\cup P_2$ and $P_3\cup P_4$ provide a partition of the lines of S_9 giving a triangle-free 2-colouring. Hence $\chi(B_5,S_9)=2$.

For the systems of order 13 we have recourse to the computer. In both cases we find that there are no sets of ten or more lines which are triangle-free. Consequently, the only possible partitions of the 26 lines into three disjoint colour classes must comprise sets of cardinalities nine, nine and eight.

In the case of the cyclic system C, we find that there are 169 sets of nine lines and 2964 sets of eight lines which are triangle-free. However, it is not possible to select three of these sets which partition the 26 lines. (In fact the maximum coverage by three triangle-free sets is 25 lines.) Consequently $\chi(B_5, C) \geq 4$. The ordinary chromatic index of C, $\chi'(C)$ equals eight and, by combining the corresponding colour classes in pairs, we obtain a triangle-free 4-colouring. Thus $\chi(B_5, C) = 4$.

In the case of the non-cyclic system N, we find that there are 178 sets of nine lines and 3233 sets of eight lines which are triangle-free. An exhaustive computer search produced twelve sets of three from these 3411 sets, each of which partition the 26 lines of N into three triangle-free colour classes. One of these solutions corresponds to the following partition. (For clarity, blocks are listed without brackets and commas.)

$$\begin{aligned} C_1 &= \{2\ 3\ 6,\ 4\ 5\ 8,\ 0\ 2\ 7,\ 4\ 6\ 12,\ 1\ 7\ 9,\ 2\ 8\ 10,\ 3\ 9\ 11,\ 1\ 10\ 12,\ 0\ 5\ 11\} \\ C_2 &= \{1\ 2\ 5,\ 3\ 4\ 7,\ 6\ 7\ 10,\ 8\ 9\ 12,\ 0\ 9\ 10,\ 1\ 3\ 8,\ 2\ 4\ 9,\ 5\ 7\ 12,\ 1\ 6\ 11\} \\ C_3 &= \{0\ 1\ 4,\ 5\ 6\ 9,\ 7\ 8\ 11,\ 4\ 10\ 11,\ 2\ 11\ 12,\ 0\ 3\ 12,\ 3\ 5\ 10,\ 0\ 6\ 8\} \end{aligned}$$

It follows that $\chi(B_4, N) = 3$. □

In the standard listing of the 80 non-isomorphic STS(15)s [12], #1 is the point-line design obtained from PG(3,2). Denoting this design by D, we may state the following result.

Lemma 3.2 $\chi(B_5, D) = 3$.

Proof. The design D is cyclic and may be obtained from two full orbits and a one-third orbit with starters $\{0,1,12\}, \{0,2,9\}$ and $\{0,5,10\}$ respectively. We number the blocks as follows. The block $\{0,1,12\}$ is numbered 1, $\{1,2,13\}$ is numbered 2, and so on until $\{14,0,11\}$ is numbered 15. Then $\{0,2,9\}$ is numbered 16, $\{1,3,10\}$ is numbered 17, and so on until $\{14,1,8\}$ is numbered 30. Finally, the blocks of the one-third orbit are numbered from 31 to 35 starting with $\{0,5,10\}$ and ending with $\{4,9,14\}$. Computer analysis gives several sets of 15 blocks which are triangle-free. One of these is $\{1,2,3,7,9,11,15,19,20,22,23,24,26,27,35\}$. A second may be obtained from this by multiplying each block by 2 (mod 15). This intersects the original set in five blocks; discarding these we obtain a set of ten blocks which is triangle-free, namely $\{5,10,12,16,17,18,21,28,29,34\}$. The remaining set of ten blocks, $\{4,6,8,13,14,25,30,31,32,33\}$, is also triangle-free. Hence $\chi(B_5, D) \leq 3$. However, D contains an STS(7) subsystem, and so $\chi(B_5, D) \geq \chi(B_5, S_7) = 3$. □

The following Theorem is similar to Theorems 2.1 and 2.2.

Theorem 3.2 *Given a triangle-free colouring of an STS(u) in M colours and a triangle-free colouring of an STS(v) in N colours, we may obtain a triangle-free colouring of an STS(uv) in $MN + M + N$ colours.*

Proof. Suppose that the point sets of the given systems are U and V, and that the sets of blocks are $\mathcal{B}_u$ and $\mathcal{B}_v$ respectively. We define an STS(uv) on $U \times V$ by taking as blocks all triples of the forms

(a) $\{(a,x),(b,y),(c,z)\}$ where $\{a,b,c\} \in \mathcal{B}_u$ and $\{x,y,z\} \in \mathcal{B}_v$,

(b) $\{(a,x),(a,y),(a,z)\}$ where $a \in U$ and $\{x,y,z\} \in \mathcal{B}_v$,

(c) $\{(a,x),(b,x),(c,x)\}$ where $\{a,b,c\} \in \mathcal{B}_u$ and $x \in V$.

It is easy to see that the resulting blocks form an STS(uv).

Denote the colour classes partitioning the STS(u) by $P_1, P_2, \ldots, P_M$ and the colour classes partitioning the STS(v) by $Q_1, Q_2, \ldots, Q_N$. Put

$$\begin{aligned} R_{i,j} &= \{\{(a,x),(b,y),(c,z)\} : \{a,b,c\} \in P_i, \{x,y,z\} \in Q_j\} \\ S_i &= \{\{(a,x),(a,y),(a,z)\} : a \in U, \{x,y,z\} \in Q_i\} \\ T_i &= \{\{(a,x),(b,x),(c,x)\} : \{a,b,c\} \in P_i, x \in V\}. \end{aligned}$$

Then the sets $R_{i,j}, S_i$ and T_i form a decomposition of the blocks of the STS(uv) into $MN + M + N$ parts.

Suppose that L_1 and L_2 are intersecting triples in $R_{i,j}$. We may write $L_1 = \{(a_1,x_1),(b_1,y_1),(c_1,z_1)\}$ and $L_2 = \{(a_1,x_1),(b_2,y_2),(c_2,z_2)\}$. If $\{b_1,c_1\} \neq \{b_2,c_2\}$ then, without loss of generality, any third block which intersects L_1 and L_2 (but not in (a_1,x_1)) has the form $L_3 = \{(b_1,y_1),(c_2,z_2),(d,w)\}$, and if this also lies in $R_{i,j}$ then $\{a_1,b_1,c_1\}, \{a_1,b_2,c_2\}$ and $\{b_1,c_2,d\}$ form a triangle in P_i, a contradiction. So suppose that $\{b_1,c_1\} = \{b_2,c_2\}$ so that L_2 has the form $L_2 = \{(a_1,x_1),(b_1,y_2),(c_1,z_2)\}$ and any third block intersecting L_1 and L_2 (but not in (a_1,x_1)) and lying in $R_{i,j}$ must, without loss of generality, have the form $L_3 = \{(b_1,y_1),(c_1,z_2),(a_1,w)\}$. But then either (a) $\{x_1,y_1,z_1\}, \{x_1,y_2,z_2\}$ and $\{y_1,z_2,w\}$ form a triangle in Q_j, or (noting that $L_3 \in R_{i,j}$ and so $y_1 \neq z_2$) (b) $y_1 = y_2, z_1 = z_2$ and $w = x_1$, giving $L_1 = L_2 = L_3$. Thus in either case (a) or (b) we have a contradiction. It follows that $R_{i,j}$ is triangle-free.

That each set S_i and T_i is triangle-free follows immediately from the fact that the corresponding Q_i and P_i are triangle-free. □

We may employ Theorem 3.2 together with the results of Lemmas 3.1 and 3.2 to establish, for example, the following result.

Corollary 3.1 *If v is of the form $v = 7^l \cdot 13^m \cdot 15^n$, where $l, m, n \geq 0$, then $\underline{\chi}(B_5, v) \leq v^s - 1$, where $s = \frac{\log_e 4}{\log_e 7} \approx 0.712414$.* □

An alternative approach to estimating $\underline{\chi}(B_5, v)$ involves consideration of the analogous problem for *transversal designs* TD$(3,n)$. Such a design comprises three disjoint sets (called *groups*) each containing n points, together with a set of triples which collectively cover every pair of elements from distinct groups precisely once and which do not cover any pair of elements from a common group. A TD$(3,n)$ is equivalent to a Latin square of side n. We will represent a TD$(3,n)$ by taking a single set of n points, $T = \{0, 1, \ldots, n-1\}$ and considering the triples to be points of T^3. Three triples are then said to form a triangle if they have the form

$$(i,j,k), (i,j',k'), (i',j,k')$$

where i, i', j, j', k, k' are six distinct points of T.

Lemma 3.3 *Suppose that n is odd. Then the set of triples $\{(i, j, (i+j)/2) : 0 \leq i, j \leq n-1\}$ (where arithmetic is modulo n) forms a $TD(3, n)$. Also, for each fixed $a \in \{0, 1, \ldots, n-1\}$ the set of triples $\{(i, a+i, (a+2i)/2) : 0 \leq i \leq n-1\}$ forms a parallel class of the TD (that is, a set of triples covering each point in each group precisely once). Furthermore, if $A \subseteq \{0, 1, \ldots, n-1\}$ has the property that for every triple of distinct elements $a_1, a_2, a_3 \in A$, $a_2+a_3 \not\equiv 2a_1 \pmod n$, then the set of triples $\{(i, a+i, (a+2i)/2) : 0 \leq i \leq n-1, a \in A\}$ is triangle-free.*

Proof. If n is odd then, working modulo n, any two elements of a triple $(i, j, (i+j)/2)$ uniquely determines the third, and hence the set of all such triples forms a $\mathrm{TD}(3, n)$. For each fixed a, no two distinct triples of the form $(i, a+i, (a+2i)/2)$ can intersect, and each point in each group occurs precisely once in such a triple. Therefore $\{(i, a+i, (a+2i)/2) : 0 \leq i \leq n-1\}$ forms a parallel class of triples of the TD. Suppose next that a_1, a_2, a_3 are three distinct residues modulo n and that the set $\{(i, a+i, (a+2i)/2) : 0 \leq i \leq n-1, a \in \{a_1, a_2, a_3\}\}$ contains a triangle. Then, without loss of generality, we may assume that two triples of the triangle are $(i, a_1+i, (a_1+2i)/2)$ and $(i, a_2+i, (a_2+2i)/2)$, and that the third, again without loss of generality, is $(j, a_3+j, (a_3+2j)/2)$ where $a_3+j \equiv a_1+i \pmod n$ and $(a_3+2j)/2 \equiv (a_2+2i)/2 \pmod n$. But these congruencies yield $2a_1 \equiv a_2+a_3 \pmod n$. □

We can make use of known properties of $\mathrm{PG}(2, q)$ to obtain the following result.

Lemma 3.4 *If $n = q^2+q+1$, where q is a prime power, then the triples of the $TD(3, n)$ described in Lemma 3.3 may be partitioned into $(q+1)$ triangle-free classes.*

Proof. If q is a prime power then the point-line design of $\mathrm{PG}(2, q)$ is a cyclic $\mathrm{BIBD}(q^2+q+1, q+1, 1)$ formed from a single cyclic orbit. (This is a consequence of a result of Singer [15].) We may represent this design on the points $0, 1, \ldots, q^2+q$ with cyclic automorphism generated by the mapping $i \mapsto i+1 \pmod{q^2+q+1}$. Consider an orbit which forms this design and let the blocks containing the point 0 be $\{0, x_1^1, x_2^1, \ldots, x_q^1\}$, $\{0, x_1^2, x_2^2, \ldots, x_q^2\}$, $\ldots$, $\{0, x_1^{q+1}, x_2^{q+1}, \ldots, x_q^{q+1}\}$. Then the elements x_j^i are all distinct and they cover all the points $1, 2, \ldots, q^2+q$ precisely once. Within each of the blocks listed, no difference is repeated and so none of these blocks contain three distinct elements a_1, a_2, a_3 for which $a_2+a_3 \equiv 2a_1 \pmod{q^2+q+1}$. Discarding 0 from all but the first block, we arrive at a partition of the residue classes modulo q^2+q+1 into sets A_i $(1 \leq i \leq q+1)$ with the property that for every triple of distinct elements $a_1, a_2, a_3 \in A_i$, $a_2+a_3 \not\equiv 2a_1 \pmod{q^2+q+1}$. But then, by the preceding Lemma, the $\mathrm{TD}(3, q^2+q+1)$ having triples $\{(i, j, (i+j)/2) : 0 \leq i, j \leq n-1\}$ may have these triples partitioned into $q+1$ triangle-free classes $\{(i, a+i, (a+2i)/2) : 0 \leq i \leq n-1, a \in A_j\}$. □

The result of the previous Lemma shows that there are transversal designs $\mathrm{TD}(3, n)$ whose triples may be partitioned into $O(\sqrt{n})$ triangle-free classes. This is likely to be far from best possible since the partition is based on avoiding repeated differences within sets which themselves partition the residue classes modulo n. Whilst this is sufficient to avoid triples $\{a_1, a_2, a_3\}$ for which $a_2+a_3 \equiv 2a_1$ (mod n), and hence leads to a partition of the triples of a particular TD, it is by no means necessary.

Nevertheless, the growth rate of the cardinality of our partition is sublinear. In fact we can achieve $O(\sqrt{n})$ even when n is not of the form q^2+q+1 for q a prime power. We show this in the following three Lemmas. First, it is convenient to introduce some terminology.

A set of residues modulo n containing no repeated difference will be called a *d-set* (modulo n). A collection of d-sets which partition the complete set of distinct residues modulo n will be called a *complete family* of d-sets (modulo n). The sets A_i $(1 \le i \le q+1)$ given in the proof of Lemma 3.4 form a complete family of d-sets modulo q^2+q+1.

Lemma 3.5 *Suppose that $S = \{x_1, x_2, \ldots, x_k\}$ (with $0 \le x_i < n$ for $i = 1, 2, \ldots, k$) forms a d-set modulo n and that $2 \le m < n$. If $S_1 = \{x \in S : 0 \le x \le \lfloor \frac{m-1}{2} \rfloor\}$ and $S_2 = \{x \in S : \lfloor \frac{m+1}{2} \rfloor \le x \le m-1\}$, then both S_1 and S_2 form d-sets modulo m.*

Proof. Suppose that α, β are distinct elements of S_1. Then $0 \le \alpha, \beta < m/2$ and so the difference between α and β satisfies $0 < |\alpha - \beta| < m/2$. Thus, if S_1 contains two equal differences modulo m, then the two differences must be equal as positive integers. But this contradicts the fact that that S forms a d-set modulo n. Hence S_1 forms a d-set modulo m, and likewise for S_2. □

Lemma 3.6 *Suppose $m \ge 4$. Then there exists a complete family, $\mathcal{F}$, of d-sets modulo m for which $|\mathcal{F}| \le 4\lfloor\sqrt{m}\rfloor - 4$.*

Proof. Suppose initially that $m \ge 16$, so that $\lfloor\sqrt{m}\rfloor > 3$. Then Bertrand's postulate [9] asserts the existence of a prime p satisfying $\lfloor\sqrt{m}\rfloor < p \le 2\lfloor\sqrt{m}\rfloor - 3$. But then $\sqrt{m} < p$ and so $m < p^2+p+1$. From the proof of Lemma 3.4, we can find a complete family $\mathcal{G}$ of d-sets modulo p^2+p+1, with $|\mathcal{G}| = p+1$. But then, by Lemma 3.5, there is a complete family, $\mathcal{F}$, of d-sets modulo m with $|\mathcal{F}| = 2|\mathcal{G}| \le 4\lfloor\sqrt{m}\rfloor - 4$. In the cases $4 \le m \le 15$, we have $4\lfloor\sqrt{m}\rfloor - 4 \ge \lceil \frac{m}{2} \rceil$ and we can establish the result by forming $\mathcal{F}$ from $\lfloor \frac{m}{2} \rfloor$ pairs $\{0,1\}, \{2,3\}, \ldots$, together with the singleton $\{m-1\}$ in the case when m is odd. □

Lemma 3.7 *Suppose $n \ge 5$ is odd. Then the triples of the $TD(3,n)$ defined in Lemma 3.3 may be partitioned into at most $4\lfloor\sqrt{n}\rfloor - 4$ triangle-free classes.*

Proof. The result follows by combining the previous Lemma with Lemma 3.3. □

Theorem 3.3 *If $v \equiv 3 \pmod 6$ and $v \ge 15$ then $\chi(B_5, v) \le 12\lfloor\sqrt{\frac{v}{3}}\rfloor - 11$.*

Proof. We apply the Bose construction (see [4], p25) for an STS$(6s+3)$ using the TD$(3, 2s+1)$ described in Lemma 3.3. The point set of the STS$(6s+3)$ is $\{0,1,\ldots,2s\} \times \{a,b,c\}$. The triples of the STS$(6s+3)$ are as follows:

$$\{(x,a),(x,b),(x,c)\} : x \in \{0,1,\ldots,2s\}$$
$$\{(x,a),(y,a),((x+y)/2,b)\} : x,y \in \{0,1,\ldots,2s\}, x \ne y$$
$$\{(x,b),(y,b),((x+y)/2,c)\} : x,y \in \{0,1,\ldots,2s\}, x \ne y$$
$$\{(x,c),(y,c),((x+y)/2,a)\} : x,y \in \{0,1,\ldots,2s\}, x \ne y.$$

By Lemma 3.7, provided $s \ge 2$, the triples of the TD$(3, 2s+1)$ may be partitioned into at most $4\lfloor\sqrt{2s+1}\rfloor - 4$ triangle-free classes. If A is one of these classes then the set $\{\{(x,a),(y,a),((x+y)/2,b)\} : (x,y,(x+y)/2) \in A\}$ will be a triangle-free set of triples of the STS$(6s+3)$. The same argument applies to the sets of triples $\{\{(x,b),(y,b),((x+y)/2,c)\} : (x,y,(x+y)/2) \in A\}$ and $\{\{(x,c),(y,c),((x+y)/2,a)\} : (x,y,(x+y)/2) \in A\}$. Finally, the set of triples $\{\{(x,a),(x,b),(x,c)\} : x \in \{0,1,\ldots,2s\}\}$ is also triangle-free. Thus the triples of the STS$(6s+3)$ are partitioned into at most $12\lfloor\sqrt{2s+1}\rfloor - 11$ triangle-free classes. With $v = 6s+3$, the result follows. □

Lemma 3.8 *Suppose that n is even. Then the set of triples $\{(i,j,i+j) : 0 \le i,j \le n-1\}$ (where arithmetic is modulo n) forms a TD$(3,n)$. Also, for each fixed $a \in \{0,1,\ldots,n-1\}$ the set of triples $\{(i,a+i,a+2i) : 0 \le i < n/2\}$ forms a partial parallel class of the TD (that is, a set of triples covering each point in each group at most once). Similarly, $\{(i,a+i,a+2i) : n/2 \le i \le n-1\}$ forms a partial parallel class. Furthermore, if $A \subseteq \{0,1,\ldots,n-1\}$ has the property that for every triple of distinct elements $a_1,a_2,a_3 \in A$, $a_2+a_3 \not\equiv 2a_1 \pmod n$, then the two sets of triples $\{(i,a+i,a+2i) : 0 \le i < n/2, a \in A\}$ and $\{(i,a+i,a+2i) : n/2 \le i \le n-1, a \in A\}$ are both triangle-free.*

Proof. This is similar to the proof of Lemma 3.3. □

Lemma 3.9 *Suppose $n \ge 4$ is even. Then the triples of the TD$(3,n)$ defined in Lemma 3.8 may be partitioned into at most $8\lfloor\sqrt{n}\rfloor - 8$ triangle-free classes.*

Proof. The result follows by combining the previous Lemma with Lemma 3.6. □

Theorem 3.4 *If $v \equiv 1 \pmod 6$ and $v \ge 13$ then $\chi(B_5,v) \le 24\lfloor\sqrt{\frac{v-1}{3}}\rfloor - 22$.*

Proof. We apply the generalisation of the Bose construction due to Skolem (see [4], p26) for an STS$(6s+1)$. This requires a commutative half-idempotent quasigroup of order $2s$ which can be obtained from the TD$(3,2s)$, described in Lemma 3.8, by permuting the elements of the third group as follows. For $0 \le i,j \le 2s-1$, replace the triple $(i,j,i+j)$ by $(i,j,i*j)$, where

$$i*j = \begin{cases} (i+j)/2 & \text{if } i+j \text{ is even,} \\ (i+j-1)/2+s & \text{if } i+j \text{ is odd,} \end{cases}$$

(arithmetic in Z). The resulting TD$(3,2s)$ is isomorphic to the original, and the operation $*$ defines the quasigroup. The quasigroup is commutative $((i*j) = (j*i))$ and half-idempotent (the corresponding Latin square has diagonal entries $0,1,2,\ldots,s-1,0,1,2,\ldots,s-1$ in that order). Being isomorphic to the original TD, the new TD may have its triples partitioned into at most $8\lfloor\sqrt{2s}\rfloor - 8$ triangle-free classes.

The point set of the STS$(6s+1)$ is $\{0,1,\ldots,2s-1\} \times \{a,b,c\} \cup \{\infty\}$. The triples of the STS$(6s+1)$ are as follows:

$$\{(x,a),(x,b),(x,c)\} : x \in \{0,1,\ldots,s-1\}$$
$$\{(x,a),(x-s,b),\infty\} : x \in \{s,s+1,\ldots,2s-1\}$$

$$\{(x,b),(x-s,c),\infty\} : x \in \{s,s+1,\ldots,2s-1\}$$
$$\{(x,c),(x-s,a),\infty\} : x \in \{s,s+1,\ldots,2s-1\}$$
$$\{(x,a),(y,a),(x*y,b)\} : x,y \in \{0,1,\ldots,2s\}, x \neq y$$
$$\{(x,b),(y,b),(x*y,c)\} : x,y \in \{0,1,\ldots,2s\}, x \neq y$$
$$\{(x,c),(y,c),(x*y,a)\} : x,y \in \{0,1,\ldots,2s\}, x \neq y.$$

If A is a triangle-free class in the new TD then the set $\{\{(x,a),(y,a),(x*y,b)\} : (x,y,x*y) \in A\}$ will be a triangle-free set of triples of the STS$(6s+1)$. The same argument applies to the sets of triples $\{\{(x,b),(y,b),(x*y,c)\} : (x,y,x*y) \in A\}$ and $\{\{(x,c),(y,c),(x*y,a)\} : (x,y,x*y) \in A\}$. Finally, the two sets of triples $\{\{(x,a),(x,b),(x,c)\} : x \in \{0,1,\ldots,s-1\}\}$ and $\{\{(x,a),(x-s,b),\infty\},\{(x,b),(x-s,c),\infty\},\{(x,c),(x-s,a),\infty\} : x \in \{s,s+1,\ldots,2s-1\}\}$ are also triangle-free. Thus the triples of the STS$(6s+1)$ are partitioned into at most $24\lfloor\sqrt{2s}\rfloor - 22$ triangle-free classes. With $v = 6s+1$, the result follows. □

Corollary 3.2 *For all admissible v, $\chi(B_5, v) \leq Av^{1/2}$ for some absolute constant A.*
Proof. The result follows immediately from Theorems 3.3 and 3.4. In fact, $A \leq 8\sqrt{3}$. □

4 Concluding Remarks

The final result of Section 3 is unlikely to be best possible. It relies on using specific TDs whose triples can be partitioned into triangle-free classes in a particular way, i.e. by combining parallel and partial parallel classes. The estimation of the partition size for this particular approach is itself dependent on a sufficient, but by no means necessary, partition of the points of "neighbouring" projective planes. The degree of closeness inherent in the word "neighbouring" is dependent on knowledge of the distribution of primes via Bertrand's Postulate. Moreover, Theorem 3.1 suggests that the exponent 1/2 in Corollary 3.2 may be significantly reduced. It would be interesting to determine a more precise bound valid for all admissible v.

Of independent interest is the question, partially addressed in Section 2, of determining the minimum number of caps required to partition PG$(n,3)$. The best result we have here is Theorem 2.4. Again, the possibility of improvement remains open. One might also reasonably ask the corresponding question regarding PG(n,q).

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