On maximal (k, b)-linear-free sets of integers and its spectrum*

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Abstract

Let k and b be integers with $k > 1$. A set A of integers is called (k, b)-linear-free if $x \in A$ implies $kx + b \notin A$. Such a set A is maximal in $[1, n] = \{1, 2, ..., n\}$ if $A \cup \{t\}$ is not (k, b)-linear-free for any t in $[1, n] \setminus A$. Let $M = M(n, k, b)$ be the set of all maximal (k, b)-linear-free subsets of $[1, n]$ and define $f(n, k, b) = max\{|A| : A \in M\}$ and $g(n, k, b) = min\{|A| : A \in M\}$. In this paper a new method for constructing maximal (k, b)-linear-free subsets of [1,n] is given and formulae for $f(n, k, b)$ and $g(n, k, b)$ are obtained. Also, we investigate the spectrum of maximal (k, b)-linear-free subsets of $[1, n]$, and prove that there is a maximal (k, b)-linear-free subset of $[1, n]$ with x elements for any integer x between the minimum and maximum possible orders.

1 Introduction

Throughout the paper n, k and b are fixed integers, $k > 1$. For integers c and d, let $[c, d]=\{x : x \text{ is an integer and } c \leq x \leq d\}$. We denote $(k^i - 1)/(k - 1)$ by $\langle k^i \rangle$.

A set A of integers is called k-multiple-free if $x \in A$ implies $kx \notin A$. Such a set A is maximal in $[1, n]$ if $A \cup \{t\}$ is not k-multiple-free for any t in $[1, n] \setminus A$. Let $f(n, k)=\max\{|A| : A \subseteq [1, n] \text{ is } k\text{-multiple-free}\}$. A subset A of $[1, n]$ with $|A| = f(n, k)$ is called a maximal k-multiple-free subset of $[1, n]$.

In [1], E.T.H. Wang investigated 2-multiple-free subsets of $[1, n]$ (these are called double-free subsets) and gave a recurrence relation and a formula for $f(n, 2)$. In [3] Leung and Wei obtained a recurrence and a formula for $f(n, k)$.

Naturally the concept of multiple-free can be generalized to multiple and translation-free, or linear-free. A set A of integers is called (k, b)-linear-free if $x \in A$ implies $kx + b \notin A$. Clearly, if $b = 0$, A is k-multiple-free; if $b = 0, k = 2$, A is double-free.

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Such a set A is maximal in $[1,n]$ if $A \cup \{t\}$ is not (k,b)-linear-free for any t in $[1,n]\backslash A$. We write $M = M(n,k,b)$ for the set of all maximal (k,b)-linear-free subsets of $[1,n]$ and define $f(n,k,b) = max\{|A| : A \in M\}$, $g(n,k,b) = min\{|A| : A \in M\}$.

In this paper we focus on three problems concerning $f(n,k,b)$ and $g(n,k,b)$: (1) constructing maximal (k,b)-linear-free subsets of $[1,n]$ and obtaining formulae for $f(n,k,b)$ and $g(n,k,b)$; (2) determining the spectrum $\{|A| : A \in M\}$; (3) giving several formulae in some special cases. As it turns out, we deal with the same topic as the work of Liu and Zhou ([5]), but our approach and results are different.

2 Main results

First we introduce some preliminary results.

A subset A of $[1,n]$ is adjacency-free if A never contains both i and $i+1$ for any i, and such an A is maximal adjacency-free if $A \cup \{t\}$ is not adjacency-free for any t in $[1,n] \setminus A$.

Lemma 1 [4] There is a maximal adjacency-free subset A of $[1,n]$ if and only if $\lceil \frac{n}{3} \rceil \leq |A| \leq \lceil \frac{n}{2} \rceil$.

Put P=$\{p : p \in [1,n]$ and $p \neq km+b$ for any $m \in N\}$, and define $n(p) = \lfloor \log_k \frac{n+b/(k-1)}{p+b/(k-1)} \rfloor$, $Q_p = \{p, pk+b, pk^2+b\langle k^2\rangle, \cdots, pk^{n(p)}+b\langle k^{n(p)}\rangle\}$, for any $p \in P$.

Lemma 2 $[1,n] = \cup_{p \in P} Q_p$

Proof. For any $s \in [1,n]$, if $s \neq km+b$ then $s \in Q_s$, otherwise $s = km+b$ for some $m \in [1,n]$. In this case, if $m \neq kq+b$ then $s, m \in Q_m$, otherwise $m = kq+b, s = qk^2+b\langle k^2\rangle$ for some $q \in N$. By repeating the above procedure, we will eventually obtain $s = rk^j + b\langle k^j\rangle \in Q_r$ for some $j \in N$, $r \in [1,n]$, and $r \neq kt+b$ for any $t \in N$. So $[1,n] \subseteq \cup_{p \in P} Q_p$.

Clearly $\cup_{p \in P} Q_p \subseteq [1,n]$. We have $\cup_{p \in P} Q_p = [1,n]$.□

It is evident that $Q_p \cap Q_r = \emptyset$ if p and r are distinct elements of P. So we have

Lemma 3 Let S be a subset of [1,n], $S_p = S \cap Q_p$ for any $p \in P$. Then $S = \cup_{p \in P} S_p$, and S is a maximal (k,b)-linear-free subset of $[1,n]$ if and only if S_p is a maximal (k,b)-linear-free subset in Q_p.

Now we define a one-to-one correspondence φ from Q_p to $[1, n(p)+1]$ by $\varphi(pk^i + b\langle k^i\rangle) = i+1$. Then we have

Lemma 4 S_p is a maximal (k,b)-linear-free subset in Q_p if and only if $\varphi(S_p)$ is maximal adjacency-free in $[1, n(p)+1]$.

Let $N_{n(p)}$=$\{Q_i : i \in P$, and $|Q_i| = n(p)+1\}$ for any $p \in P$. Clearly, $Q_p \in N_{n(p)}$, so $N_{n(p)} \neq \emptyset$.

In the following Lemma, if $a < b$, we define $\lfloor \frac{a-b}{c} \rfloor = 0$.

Lemma 5

$$|N_{n(p)}| = \begin{cases} \lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \rfloor - 2\lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)+1}} \rfloor + \lfloor \frac{n-b\langle k^{n(p)+2}\rangle}{k^{n(p)+2}} \rfloor & \text{for } n(p) < n(1) \\ \lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \rfloor & \text{for } n(p) = n(1). \end{cases}$$

Proof. Case 1. If $n(p) < n(1)$, for any $i \in [1,n]$ such that $|Q_i| = n(p)+1$, we have $ik^{n(p)} + b\langle k^{n(p)}\rangle \le n$ and $ik^{n(p)+1} + b\langle k^{n(p)+1}\rangle > n$, then $i \in [\lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)+1}} \rfloor + 1, \lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \rfloor]$.

If $i = km + b$ for some $m \in N$, then $km + b \in [\lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)+1}} \rfloor + 1, \lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \rfloor]$, so $m \in [\lfloor \frac{n-b\langle k^{n(p)+2}\rangle}{k^{n(p)+2}} \rfloor + 1, \lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)+1}} \rfloor]$.

Clearly,

$$\begin{aligned}
|N_{n(p)}| &= |\{i : i \in P \text{ and } |Q_i| = n(p)+1\}| \\
&= |\{i : i \in [1,n] \text{ and } |Q_i| = n(p)+1\}| \\
&\qquad - |\{i : i \in [1,n],\ |Q_i| = n(p)+1 \text{ and } i \notin P\}| \\
&= \left(\left\lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \right\rfloor - \left\lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)=1}} \right\rfloor\right) \\
&\qquad - \left(\left\lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)=1}} \right\rfloor - \left\lfloor \frac{n-b\langle k^{n(p)+2}\rangle}{k^{n(p)+2}} \right\rfloor\right) \\
&= \left\lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \right\rfloor - 2\left\lfloor \frac{n-b\langle k^{n(p)+1}\rangle}{k^{n(p)+1}} \right\rfloor + \left\lfloor \frac{n-b\langle k^{n(p)+2}\rangle}{k^{n(p)+2}} \right\rfloor.
\end{aligned}$$

Case 2. If $n(p) = n(1)$, then $k^{n(1)} + b\langle k^{n(1)}\rangle \le n$ and $(k+b)k^{n(1)} + b\langle k^{n(1)}\rangle = k^{n(1)+1} + b\langle k^{n(1)+1}\rangle > n$. Hence $1 \le i < k+b$ for any $i \in [1,n]$ such that $|Q_i| = n(1)+1$, so $i \ne km+b$ for any $m \in N$. We obtain $|N_{n(1)}| = \lfloor \frac{n-b\langle k^{n(p)}\rangle}{k^{n(p)}} \rfloor$.□

Theorem 1

(i) $f(n,k,b) = \sum_{p\in P} \lceil \frac{n(p)+1}{2} \rceil = \sum_{i=1}^{n(1)} |N_i| \lceil \frac{i+1}{2} \rceil$;

(ii) $g(n,k,b) = \sum_{p\in P} \lceil \frac{n(p)+1}{3} \rceil = \sum_{i=1}^{n(1)} |N_i| \lceil \frac{i+1}{3} \rceil$.

Proof. (i) Let S be a (k,b)-linear-free subset of $[1,n]$. By Lemma 1 and Lemma 4, for each $p \in P$, $\{|S_p| : S_p$ is a maximal (k,b)-linear-free subset in $Q_p\} = \{|\varphi(S_p)| : \varphi(S_p)$ is a maximal adjacency-free subset in $[1, n(p)+1]\} = [\lceil \frac{n(p)+1}{3} \rceil, \lceil \frac{n(p)+1}{2} \rceil]$.

By Lemma 3 , S is a maximal (k,b)-linear-free subset of $[1,n]$ if and only if S_p is a maximal adjacency-free subset in $[1, n(p)+1]$. If $|S| = f(n,k,b)$, we can choose $|S_p| = \lceil \frac{n(p)+1}{2} \rceil$ for each $p \in P$. So $f(n,k,b) = \sum_{p\in P} \lceil \frac{n(p)+1}{2} \rceil = \sum_{i=1}^{n(1)} |N_i| \lceil \frac{i+1}{2} \rceil$ by the definition of $|N_{n(p)}|$.

The proof of (ii) is similar.□

Example 1. Let $n = 63$, $k = 2$ and $b = 1$. Then $n(1) = 5$, $|N_{n(1)}| = \lfloor \frac{63-\langle 2^5\rangle}{2^5} \rfloor = 1$, and $|N_i| = \lfloor \frac{63-\langle 2^i\rangle}{2^i} \rfloor - 2\lfloor \frac{63-\langle 2^{i+1}\rangle}{2^{i+1}} \rfloor + \lfloor \frac{63-\langle 2^{i+2}\rangle}{2^{i+2}} \rfloor = 2^{4-i}$, for $0 \le i \le 4$.

$f(63,2,1) = \sum_{i=0}^{n(1)} |N_i| \lceil \frac{i+1}{2} \rceil = 2^4\times1 + 2^3\times1 + 2^2\times2 + 2^1\times2 + 2^0\times3 + 1\times3 = 42.$

$g(63,2,1) = \sum_{i=0}^{n(1)} |N_i| \lceil \frac{i+1}{3} \rceil = 2^4\times1 + 2^3\times1 + 2^2\times1 + 2^1\times2 + 2^0\times2 + 1\times2 = 36.$

Now we consider the spectrum $\{|A| : A \in M\}$. We have

Theorem 2 For any value x in $[g(n,k,b), f(n,k,b)]$, there is a maximal (k,b)-linear-free subset of $[1,n]$ with x elements.

Proof. Suppose $S \in M$. By Lemma 1 and Theorem 1, we can choose S_p to have any value in the range $[\lceil\frac{n(p)+1}{3}\rceil, \lceil\frac{n(p)+1}{2}\rceil]$ for each $p \in P$. So we can obtain a maximal (k,b)-linear-free subset of $[1,n]$ and $|S| = x \in [g(n,k,b), f(n,k,b)]$. Also, when x is in $(g(n,k,b), f(n,k,b))$ there is more than one subset S which satisfies $|S| = x$.□

Theorem 3 If $n = k^m + b\langle k^m\rangle$ for some $m \in N$, then

(i) $f(n,k,b) = \lceil\frac{m+1}{2}\rceil + (k+b-2)\lceil\frac{m}{2}\rceil + \sum_{i=1}^{m-1} k^{i-1}(k+b-1)(k-1)\lceil\frac{m-i}{2}\rceil$;

(ii) $g(n,k,b) = \lceil\frac{m+1}{3}\rceil + (k+b-2)\lceil\frac{m}{3}\rceil + \sum_{i=1}^{m-1} k^{i-1}(k+b-1)(k-1)\lceil\frac{m-i}{3}\rceil$.

Proof. Case 1. If $n = k^m + b\langle k^m\rangle$, then $n(1) = m$ and $|N_{n(1)}| = 1$.

Case 2. Suppose $n(p) = m-1$ for some $p \in P$. Then $pk^{m-1} + b\langle k^{m-1}\rangle \le n = k^m + b\langle k^m\rangle = (k+b)k^{m-1} + b\langle k^{m-1}\rangle$, so $2 \le p \le k+b-1$, and $|N_{m-1}| = k+b-2$.

Case 3. Suppose $n(p) = m-i-1$, $i \in [1, m-1]$ for some $p \in P$. By Lemma 5, $|N_{n(p)}| = \lfloor\frac{n-b\langle k^{m-i-1}\rangle}{k^{m-i-1}}\rfloor - 2\lfloor\frac{n-b\langle k^{m-i}\rangle}{k^{m-i}}\rfloor + \lfloor\frac{n-b\langle k^{m-i+1}\rangle}{k^{m-i+1}}\rfloor = k^{i-1}(k+b-1)(k-1)$.

By Theorem 1, (i) and (ii) are obtained.□

As we expected, if $b = 0$, formula (ii) of Theorem 3 is exactly the same as Theorem 5 of Lai ([2]).

Theorem 4 Suppose $k + b > 2$. Then $f(n,k,b)$=$g(n,k,b)$ if and only if $n < k^2 + kb + b$.

Proof. For convenience we denote "if and only if" by "$\Longleftrightarrow$". By Theorem 1, $f(n,k,b) = g(n,k,b) \Longleftrightarrow \sum_{i=1}^{n(1)} |N_i|\lceil\frac{i+1}{2}\rceil = \sum_{i=1}^{n(1)} |N_i|\lceil\frac{i+1}{3}\rceil \Longleftrightarrow \lceil\frac{i+1}{2}\rceil = \lceil\frac{i+1}{3}\rceil$ for any $i \in [0, n(1)] \Longleftrightarrow n(1) < 2 \Longleftrightarrow n < k^2 + kb + b$.□

Now we give some recurrence relations for $f(n,k,b)$ and $g(n,k,b)$.

Theorem 5 Suppose $n = ks + b$ for some $s \in N$.

(i) If $1 \le i \le k$, then $f(n+i,k,b) = f(n,k,b) + i$.

(ii) Suppose $n + k = pk^m + b\langle k^m\rangle$, where $p \in P$. If $m \equiv 0$ (*mod* 2), then $f(n+k,k,b) = f(n,k,b) + k$, otherwise $f(n+k,k,b) = f(n,k,b) + k - 1$.

Proof (i) Suppose $n = ks + b$. Then $n + i \ne kr + b$ for any $r \in N$, $1 \le i \le k$. Thus

$$\left\lfloor \frac{n - b\langle k^{n(p)}\rangle}{k^{n(p)}} \right\rfloor = \begin{cases} \lfloor\frac{n+i-b\langle k^{n(p)}\rangle}{k^{n(p)}}\rfloor & \text{for } n(p) > 0 \\ \lfloor\frac{n+i-b\langle k^{n(p)}\rangle}{k^{n(p)}}\rfloor - i & \text{for } n(p) = 0. \end{cases}$$

By Theorem 1, we have $f(n+i,k,b) = f(n,k,b) + i$.

(ii) Supose $n + k = pk^m + b\langle k^m\rangle$, where $p \in P$ and $m \equiv 0$ (*mod* 2), so $m \ge 2$. Let S be a maximal (k,b)-linear-free subset in $[1, n+k]$ with $|S| = f(n+k,k,b)$. Consider Q_p. By Theorem 1, $|S_p| = |S \cap Q_p| = \lceil\frac{n(p)+1}{2}\rceil = \lceil\frac{m+1}{2}\rceil = \frac{m}{2} + 1$.

Let R be a maximal (k,b)-linear-free subset in $[1, n+k-1]$ with $|R| = f(n+k-1,k,b)$. Since $[1,n+k-1] = [1,n+k] - \{n+k = pk^m + b\langle k^m\rangle\}$, consider Q_p and $n(p) = m-1$. By Theorem 1, $|R_p| = |R \cap Q_p| = \lceil\frac{n(p)+1}{2}\rceil = \lceil\frac{(m-1)+1}{2}\rceil = \frac{m}{2}$.

We may choose R so that R and S have the same elements in any Q_q for all $q \in P$ except those in Q_p. Therefore $f(n+k-1,k,b) = f(n+k,k,b) - 1$. But by (i), $f(n+k-1,k,b) = f(n,k,b)+k-1$. Hence $f(n+k,k,b) = f(n,k,b)+k$.

If $m \not\equiv 0$ $(mod\ 2)$, then by employing the same S and R as above, we have $\lceil \frac{m+1}{2} \rceil = \frac{m+1}{2} = \lceil \frac{(m-1)+1}{2} \rceil$. This implies that $f(n+k,k,b) = f(n+k-1,k,b) = f(n,k,b)+k-1$.□

Example 2. Let $n = 61$, $k = 2$, and $b = 1.n+k = 61+2 = 63 = 1\times 2^5 + 1\times \langle 2^5 \rangle$, and $5 \equiv 1$ $(mod\ 2)$. By (ii), $f(63,2,1) = f(61,2,1)+2-1$, so by Example 1, we obtain $f(61,2,1) = 42-1 = 41$.

If $i = 1 < k = 2$. By (i), $f(62,2,1) = f(61,2,1)+1 = 41+1 = 42$

On the other hand, it is easy to prove $f(61,2,1) = 41$, and $f(62,2,1) = 42$ by Theorem 1.

Similarly, we have

Theorem 6 Suppose $n = ks + b$ for some $s \in N$.

(i) If $1 \leq i \leq k$, then $g(n+i,k,b) = g(n,k,b)+i$.

(ii) Suppose $n+k = pk^m + b\langle k^m \rangle$, and $p \in P$. If $m \equiv 0$ $(mod\ 3)$, then $g(n+k,k,b) = g(n,k,b)+k$, otherwise $g(n+k,k,b) = g(n,k,b)+k-1$.□

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