The kth upper generalized exponent set for primitive matrices*

Yan-Ling Shao

Department of Mathematics
North China Institute of Technology
Taiyuan, Shanxi 030051, P. R. China

Yu-Bin Gao

Department of Mathematics
University of Science and Technology of China
Hefei, Anhui 230026, P. R. China
ybgao@mail.ustc.edu.cn

Abstract

Let $P_{n,d}$ be the set of $n \times n$ non–symmetric primitive matrices with exactly d nonzero diagonal entries. For each positive integer $2 \leq k \leq n-1$, we determine the kth upper generalized exponent set for $P_{n,d}$ and characterize the extremal matrices by using a graph theoretical method.

1 Introduction

An $n \times n$ nonnegative matrix A is called *primitive* if there exists some positive integer t such that $A^t > 0$. The least such positive integer t is called the *exponent* of A, denoted by $\gamma(A)$.

In [1], Brualdi and Liu defined the kth upper generalized exponent $F(A, k)$ as follows.

Definition 1.1 [1] *Let A be a primitive matrix of order n and $1 \leq k \leq n-1$. Set*

$$F(A,k) = \min\{p \mid \text{ no set of } k \text{ rows of } A^p \text{ has a column of all zeros }\}.$$

$F(A, k)$ is called the kth upper generalized exponent of A.

*Research supported by Shanxi Youth Science Foundation 981005.

The kth upper generalized exponent is a generalization of the traditional concept of the exponent. Background can be found in [1].

It is well-known that for each nonnegative matrix A there exists an associated digraph $D(A)$ whose adjacency matrix has the same zero entries as A. A digraph D is primitive iff D is strongly connected and $\text{g.c.d}(r_1, r_2, \cdots, r_\lambda) = 1$, where $\{r_1, r_2, \cdots, r_\lambda\}$ is the set of distinct lengths of the directed cycles of D. A is primitive iff $D(A)$ is primitive.

Definition 1.2 [1] *Let X be the vertex subset of a primitive digraph D. The exponent $\exp_D(X)$ is the smallest positive integer p such that for each vertex y of D, there exists a walk of length p from at least one vertex in X to y.*

Definition 1.3 [1] *Let D be a primitive digraph of order n and $1 \leq k \leq n-1$. Set*

$$F(D,k) = \max\{\exp_D(X) \mid X \subseteq V(D), |X| = k\}. \tag{1.1}$$

$F(D,k)$ is called the kth upper generalized exponent of D.

It is obvious that

$$F(A,k) = F(D(A),k). \tag{1.2}$$

Let $P_{n,d}$ be the set of $n \times n$ non-symmetric primitive matrices with exactly d nonzero diagonal entries, $E_{nd}(k)$ the set of kth upper generalized exponents of the matrices in $P_{n,d}$. In this paper, we determine the exponent set $E_{nd}(k)$ and characterize the extremal matrices.

Notice that if $k = 1$, then $F(A,k) = \gamma(A)$. In this case, the exponent set $E_{nd}(1)$ has already been determined in [3]. So we will only consider the cases $2 \leq k \leq n-1$.

We will make use of the following notation. Let D be a primitive digraph with $D = (V(D), E(D))$. We denote the distance from vertex x to vertex y of D by $d(x,y)$. If $i, j \in V(D)$, then (i,j) denotes an arc from vertex i to vertex j and $[i,j]$ denotes an edge between two vertices i and j, i.e. a 2-cycle.

2 The generalized exponent set $E_{nd}(k)$

Theorem 2.1 *Let n, d, k be positive integers with $2 \leq k \leq n-1$ and $A \in P_{n,d}$. Then*

$$F(D(A),k) \leq 2n - k - d. \tag{2.1}$$

Proof. Let X be any k-vertex subset of $D(A)$ and let W be the set of loop vertices of $D(A)$.

(1) $k \leq n - d$.

Case 1: $X \cap W \neq \emptyset$. Then $\exp_{D(A)}(X) \leq \max\limits_{y \in V(D(A))} d(X \cap W, y) \leq n-1 < 2n-k-d$.

Case 2: $X \cap W = \emptyset$. Let $l_y = d(W,y) = d(w_y, y)$ $(w_y \in W)$ and $h_y = d(X, w_y)$ for any $y \in V(D)$. Then $l_y \leq n-d$, $h_y \leq n-k$ and $\exp_{D(A)}(X) \leq \max\limits_{y \in V(D(A))} (h_y + l_y) \leq 2n - k - d$.

(2) $k \geq n-d+1$.

X must include at least one loop vertex. $|X \cap W| \geq k-(n-d) = k+d-n$. Notice that $\max_{y \in V(D(A))} d(X \cap W, y) \leq n-(k+d-n) = 2n-k-d$. We have $\exp_{D(A)}(X) \leq 2n-k-d$.

The proof of the theorem is completed. ∎

Theorem 2.2 *Let n, d, k be positive integers with $2 \leq k \leq n-1, d=1$. Then*

$$\{k+1, k+2, \cdots, 2n-k-1\} \subseteq E_{n1}(k). \tag{2.2}$$

Proof. Firstly, suppose $k \leq m \leq n-1$. We consider $D_1 = D(A)$ with vertex set $V(D_1) = \{1, 2, \cdots, n\}$ and arc set $E(D_1) = \{(1,1), (1,2), (2,3), \cdots, (m-1,m), (m,m+1), (m,m+2), \cdots, (m,n), (m+1,1), (m+2,1), \cdots, (n,1)\}$.

It is obvious that $A \in P_{n,1}$. Take $X_0 = \{2, 3, \cdots, k+1\}$. It is not difficult to verify that there is no walk of length $2m-k$ from any vertex of X_0 to the vertex $m+1$. So we have

$$F(D_1, k) \geq \exp_{D_1}(X_0) \geq 2m-k+1. \tag{2.3}$$

On the other hand, let X be any k–vertex subset of D_1. If $1 \in X$, then

$$\exp_{D_1}(X) \leq m < 2m-k+1. \tag{2.4}$$

If $1 \notin X$, letting i be the vertex of X which is closest to 1, then $d(i,1) \leq m+1-k-1+1 = m-k+1$ and

$$\exp_{D_1}(X) \leq m-k+1+m = 2m-k+1. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) we have

$$F(D_1, k) = 2m-k+1. \tag{2.6}$$

Next, suppose $k+1 \leq m \leq n-1$.We consider $D_2 = D(A)$ with vertex set $V(D_2) = \{1, 2, \cdots, n\}$ and arc set $E(D_2) = \{(1,1), [1,2], (2,3), (3,4), \cdots, (m-1,m), (m,m+1), (m,m+2), \cdots, (m,n), (m+1,1), (m+2,1), \cdots, (n,1)\}$.

It is obvious that $A \in P_{n,1}$. Take $X_0 = \{3, 4, \cdots, k+2\}$. It is not difficult to verify that there is no walk of length $2m-k-1$ from any vertex of X_0 to the vertex $m+1$. Then $F(D_2, k) \geq \exp_{D_2}(X_0) \geq 2m-k$.

On the other hand, let X be any k–vertex subset of D_2. If $\{1,2\} \cap X \neq \emptyset$, then $\exp_{D_2}(X) \leq m+1 \leq 2m-k$. If $\{1,2\} \cap X = \emptyset$, letting j be the vertex of X which is closest to 1, then $d(j,1) \leq m+1-k-2+1 = m-k$ and $\exp_{D_2}(X) \leq m-k+m = 2m-k$.

So we have

$$F(D_2, k) = 2m-k. \tag{2.7}$$

Notice that $k \leq m \leq n-1$ for D_1 and $k+1 \leq m \leq n-1$ for D_2. Combining (2.6) and (2.7) we obtain (2.2). ∎

Theorem 2.3 *Let n, d, k be positive integers with $2 \le k \le n-1, d=1$. Then*

$$\{2,3,\cdots,k\} \subseteq E_{n1}(k). \tag{2.8}$$

Proof. (1) $2 \le k \le n-2$.

Suppose $2 \le m \le k$. We consider $D_3 = D(A)$ with vertex set $V(D_3) = \{1,2,\cdots,n\}$ and arc set $E(D_3) = \{(1,1),[1,2],(2,3),(3,4),\cdots,(m-1,m),(m,m+1),(m,m+2),\cdots,(m,n),(m+1,1),(m+2,1),\cdots,(n,1),(m+1,2),(m+2,2),\cdots,(n,2)\}$.

It is obvious that $A \in P_{n,1}$. Take $X_0 = \{n, n-1, \cdots, n-k+1\}$. Then $|X_0| = k$. Since $n-k+1 \ge 3$, it is not difficult to verify that there is no walk of length $m-1$ from any vertex of X_0 to the vertex $m+1$. Then $F(D_3,k) \ge \exp_{D_3}(X_0) \ge m$.

On the other hand, let X be any k–vertex subset of D_3. If $1 \in X$, then $\exp_{D_3}(X) \le m$. If $1 \notin X$, then $X \cap \{m+1, m+2, \cdots, n\} \ne \emptyset$ and $\exp_{D_3}(X) \le m$.

So we have $F(D_3,k) = m$. Noticing that $2 \le m \le k$, we obtain (2.8).

(2) $k = n-1$.

Suppose $1 \le m \le n-2$. We consider $D_1 = D(A)$ in Theorem 2.2.

Take $X_0 = \{2,3,\cdots,n\}$. Then $|X_0| = n-1$. It is not difficult to verify that there is no walk of length m from any vertex of X_0 to the vertex $m+1$. Then $F(D_1,k) \ge \exp_{D_1}(X_0) \ge m+1$.

On the other hand, let X be any k–vertex subset of D_1. Since $X \cap \{m+1, m+2, \cdots, n\} \ne \emptyset$, $\exp_{D_1}(X) \le m+1$.

So we have $F(D_1,k) = m+1$. Noticing that $1 \le m \le n-2$, we obtain (2.8). ∎

Theorem 2.4 *Let n, d, k be positive integers with $2 \le k \le n-d, d \ge 2$. Then*

$$\{d+k-1, d+k, \cdots, 2n-k-d\} \subseteq E_{nd}(k). \tag{2.9}$$

Proof. Suppose $d+k-1 \le m \le n-1$. Firstly, we consider $D_4 = D(A)$ with vertex set $V(D_4) = \{1,2,\cdots,n\}$ and arc set $E(D_4) = \{(1,1),(2,2),\cdots,(d,d),(1,2),(2,3),\cdots,(m-1,m),(m,m+1),(m,m+2),\cdots,(m,n),(m+1,1),(m+2,1),\cdots,(n,1)\}$.

It is obvious that $A \in P_{n,d}$. Take $X_0 = \{d+1, d+2, \cdots, d+k\}$. It is not difficult to verify that there is no walk of length $2m-d-k+1$ from any vertex of X_0 to the vertex $m+1$. So we have

$$F(D_4,k) \ge \exp_{D_4}(X_0) \ge 2m-d-k+2. \tag{2.10}$$

On the other hand, let X be any k–vertex subset of D_4. If $\{1,2,\cdots,d\} \cap X \ne \emptyset$, then

$$\exp_{D_4}(X) \le m < 2m-d-k+2. \tag{2.11}$$

If $\{1,2,\cdots,d\} \cap X = \emptyset$, letting i be the vertex of X which is closest to 1, then $d(i,1) \le m+1-d-k+1 = m-k-d+2$ and

$$\exp_{D_4}(X) \le m-d-k+2+m = 2m-d-k+2. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12) we have

$$F(D_4,k) = 2m-d-k+2. \tag{2.13}$$

Next, we consider $D_5 = D(A)$ with vertex set $V(D_5) = \{1, 2, \cdots, n\}$ and arc set $E(D_5) = \{(1,1), (2,2), \cdots, (d,d), [1,2], (2,3), (3,4), \cdots, (m-1, m), (m, m+1), (m, m+2), \cdots, (m, n), (m+1, 1), (m+2, 1), \cdots, (n, 1), (m+1, 2), (m+2, 2), \cdots, (n, 2)\}$.

It is obvious that $A \in P_{n,d}$. Take $X_0 = \{d+1, d+2, \cdots, d+k\}$. It is not difficult to verify that there is no walk of length $2m - d - k$ from any vertex of X_0 to the vertex $m+1$. Then $F(D_5, k) \geq \exp_{D_5}(X_0) \geq 2m - d - k + 1$.

On the other hand, let X be any k–vertex subset of D_5. If $\{1, 2, \cdots, d\} \cap X \neq \emptyset$, then $\exp_{D_5}(X) \leq m \leq 2m - d - k + 1$. If $\{1, 2, \cdots, d\} \cap X = \emptyset$, letting j be the vertex of X which is closest to 2, then $d(j, 2) \leq m + 1 - d - k + 1 = m - k - d + 2$ and $\exp_{D_5}(X) \leq m - k - d + 2 + m - 1 = 2m - k - d + 1$.

So we have

$$F(D_5, k) = 2m - k - d + 1. \tag{2.14}$$

Notice that $d + k - 1 \leq m \leq n - 1$. Combining (2.13) and (2.14) we obtain (2.9). ∎

Theorem 2.5 *Let n, d, k be positive integers with $2 \leq k \leq n - d, d \geq 2$. Then*

$$\{2, 3, \cdots, d + k - 2\} \subseteq E_{nd}(k). \tag{2.15}$$

Proof. Suppose $2 \leq m \leq d + k - 2$. We consider $D_5 = D(A)$ in Theorem 2.4.

Take $X_0 = \{n, n-1, \cdots, n-k+1\}$. Then $|X_0| = k$. Since $n - k + 1 \geq d + 1$, it is not difficult to verify that there is no walk of length $m - 1$ from any vertex of X_0 to the vertex $m+1$. Then $F(D_5, k) \geq \exp_{D_5}(X_0) \geq m$.

On the other hand, let X be any k–vertex subset of D_5. If $\{1, 2, \cdots, d\} \cap X \neq \emptyset$, then $\exp_{D_5}(X) \leq m$. If $\{1, 2, \cdots, d\} \cap X = \emptyset$, then $X \cap \{m+1, m+2, \cdots, n\} \neq \emptyset$ and $\exp_{D_5}(X) \leq m$.

So we have $F(D_5, k) = m$. Noticing that $2 \leq m \leq d + k - 2$, we obtain (2.15). ∎

Theorem 2.6 *Let n, d, k be positive integers with $n - d + 1 \leq k \leq n - 1, d \geq 2$. Then*

$$\{1, 2, \cdots, 2n - k - d\} \subseteq E_{nd}(k). \tag{2.16}$$

Proof. Suppose $k + d - n \leq m \leq n - 1$. We consider $D_4 = D(A)$ in Theorem 2.4.

Take $X_0 = \{1, 2, \cdots, k+d-n, d+1, d+2, \cdots, n\}$. Then $|X_0| = k$. It is not difficult to verify that there is no walk of length $m+1-(k+d-n)-1 = n+m-k-d$ from any vertex of X_0 to the vertex $m+1$. Then $F(D_4, k) \geq \exp_{D_4}(X_0) \geq n + m - k - d + 1$.

On the other hand, let X be any k–vertex subset of D_4 and $W = \{1, 2, \cdots, d\}$. Since $|X \cap W| \geq k + d - n > 0$, $\max\limits_{y \in V(D_4)} d(X \cap W, y) \leq m + 1 - (k + d - n) = n + m + 1 - k - d$, then $\exp_{D_4}(X) \leq n + m - k - d + 1$.

So we have $F(D_4, k) = n + m - k - d + 1$. Noticing that $k + d - n \leq m \leq n - 1$, we obtain (2.16). ∎

Theorem 2.7 *Let n, d, k be positive integers with $2 \leq k \leq n - 1$. Then*

$$E_{nd}(k) = \{1, 2, 3, \cdots, 2n - k - d\}. \tag{2.17}$$

Proof. We consider $D = D(A)$ with vertex set $V(D) = \{1, 2, \cdots, n\}$ and arc set $E(D) = \{(i, j) \mid i, j = 1, 2, \cdots, n\} \setminus \{(2, 1), (d+1, d+1), (d+2, d+2), \cdots, (n, n)\}$.

It is obvious that $A \in P_{n,d}$ and $F(D, k) = 1$. So $1 \in E_{nd}(k)$.

Combining (2.1), (2.2), (2.8), (2.9), (2.15) and (2.16) we obtain (2.17). ∎

3 The extremal matrices

In this section,we characterize the extremal matrices of kth upper generalized exponent for $P_{n,d}$.

Theorem 3.1 *Let n, d, k be positive integers with $k \leq n - d$, $A \in P_{n,d}$, $D = D(A)$. Then $F(D, k) = 2n - k - d$ iff D is isomorphic to one of the digraphs D_1^*, where D_1^* are strongly connected digraphs with vertex set $V(D_1^*) = \{1, 2, \cdots, n\}$ and arc set $E(D_1^*) = \{(1, 1), (2, 2), \cdots, (d, d), (1, 2), (2, 3), \cdots, (n-1, n)\} \cup \Phi$.*

(1) If $k = n - d$, then Φ is a subset of $\{(d+i, 1) \mid 1 \leq i \leq n-d\} \cup \{(i, j) \mid 1 \leq j < i \leq d\}$.

(2) If $k < n - d$, then Φ is a subset of $\{(i, j) \mid 1 \leq j < i \leq d\} \cup \{(d+i, d+j) \mid 1 \leq j < i \leq n-d\} \cup \{(d+i, 1) \mid 1 \leq i \leq n-d\}$ such that D_1^* satisfies the conditions:

(i) There exists $d < x_0 \leq n$ such that $d(x_0, 1) = n - k - d + 1$;

(ii) Let P be a shortest path from x_0 to 1. Then the vertex set of P has the form

$$V(P) = \{x_0, 1, d+m, d+m+1, \cdots, d+s, d+l, d+l+1, \cdots, n-k+l+m-s-2\}$$

where $m \geq 1$, $n - k + l + m - s - 2 \leq n$ and $l \geq s + 1$;

(iii) For any $j > d$ and $j \notin V(P) \setminus \{x_0\}$, there is no walk of length $2n - k - d - 1$ from j to n.

Proof. Take $X_0 = \{d+1, d+2, \cdots, n\}$ (when $k = n-d$) or $X_0 = \{j \in V(D_1^*) \mid j > d \text{ and } j \notin V(P) \setminus \{x_0\}\}$ (when $k < n - d$). It is not difficult to verify that $|X_0| = k$ and there is no walk of length $2n - k - d - 1$ from any vertex of X_0 to the vertex n in D_1^*, so we have

$$F(D_1^*, k) \geq \exp_{D_1^*}(X_0) \geq 2n - k - d.$$

Combining with Theorem 2.1, we obtain

$$F(D_1^*, k) = 2n - k - d.$$

On the other hand, let $k \leq n - d$, $A \in P_{n,d}$, $D = D(A)$, $F(D, k) = 2n - k - d$, X be a k-vertex subset of D with $\exp_D(X) = 2n - k - d$, and let W be the set of loop vertices of D.

Case 1: $W \cap X \neq \emptyset$. Then $\exp_D(X) \leq \max_{y \in V(D)} d(X \cap W, y) \leq n - 1 < 2n - k - d$. It is a contradiction.

Case 2: $W \cap X = \emptyset$. Let $l_y = d(W, y) = d(w_y, y)$ ($w_y \in W$) and $h_y = d(X, w_y)$ for any $y \in V(D)$. If $l_y < n - d$ or $h_y < n - k$ for any $y \in V(D)$, then $\exp_D(X) \leq \max_{y \in V(D)} (h_y + l_y) < 2n - k - d$. It is also a contradiction. If $l_y = n - d$ and $h_y = n - k$

for some $y \in V(D)$, then there exists a Hamilton path in D, and d loop vertices of D are consecutive at the beginning of this Hamilton path. Now assume that $\{(1,1),(2,2),\cdots,(d,d),(1,2),(2,3),\cdots,(n-1,n)\} \subseteq E(D)$. In this assumption, we have $y=n$, $w_y=d$ and $d(1,n)=n-1$. Further, $(i,j) \notin E(D)$ for $1 \le i \le n-2$ and $i+1<j\le n$.

Subcase 2.1: $k=n-d$. It is clear that $X=\{d+1,d+2,\cdots,n\}$. Since that there is no walk of length $2n-k-d-1$ from i to n for any $i \in X$, we have that $(i,j) \notin E(D)$ for $d+1 \le i \le n$ and $2 \le j < i$. Thus, D is isomorphic to one of the D_1^*.

Subcase 2.2: $k<n-d$. Then D satisfies the conditions:

(1) For $i>d$ and $j \le d$, if $(i,j) \in E(D)$, then $i \notin X$, $j=1$, and i is unique. Else $h_y<n-k$. It is a contradiction.

(2) $d(X,1)=n-k-d+1$. Further, letting $d(X,1)=d(x_0,1)$ $(x_0 \in X)$ and P be a shortest path from x_0 to 1 in D, the vertex set of P has the form

$$V(P)=\{x_0,1,d+m,d+m+1,\cdots,d+s,d+l,d+l+1,\cdots,n-k+l+m-s-2\}$$

where $m \ge 1$, $n-k+l+m-s-2 \le n$ and $l \ge s+1$.

Notice that $\exp_D(X)=2n-k-d$. Combining (1) and (2), D must be isomorphic to one of the D_1^*. ∎

Theorem 3.2 *Let n,d,k be positive integers with $k \ge n-d+1$, $A \in P_{n,d}$, $D=D(A)$. Then $F(D,k)=2n-k-d$ iff D is isomorphic to one of the digraphs D_2^*, where D_2^* are strongly connected digraphs with vertex set $V(D_2^*)=\{1,2,\cdots,n\}$ and arc set $E(D_2^*)=\{(1,1),(2,2),\cdots,(d,d),(k+d-n,k+d-n+1),(k+d-n+1,k+d-n+2),\cdots,(n-1,n)\} \cup \Phi$, where Φ is a subset of $\{(i,j) \mid 1 \le i \le k+d-n; 1 \le j \le k+d-n)\} \cup \{(i,k+d-n+1) \mid 1 \le i \le k+d-n-1)\} \cup \{(i,j) \mid k+d-n+1 \le i \le d; 1 \le j \le i-1\} \cup \{(i,j) \mid d+1 \le i \le n; 1 \le j \le k+d-n+1\} \cup \{(i,j) \mid d+2 \le i \le n; d+1 \le j \le i-1\}$. If $\{(d+i_1,d+j_1),(d+i_2,d+j_2),\cdots,(d+i_t,d+j_t)\} \subseteq \Phi$ $(j_1 \le j_2 \le \cdots \le j_t)$, then there are no nonnegative integers $k_1,k_2,\cdots,k_t$ such that $k_1(i_1-j_1+1)+k_2(i_2-j_2+1)+\cdots+k_t(i_t-j_t+1)=n-k+m-1$ for $1 \le m \le j_1$.*

Proof. Take $X_0=\{1,2,\cdots,k+d-n,d+1,d+2,\cdots,n\} \subseteq V(D_2^*)$. It is not difficult to verify that $|X_0|=k$ and there is no walk of length $2n-k-d-1$ from any vertex of X_0 to the vertex n, so we have

$$F(D_2^*,k) \ge \exp_{D_2^*}(X_0) \ge 2n-k-d.$$

Combining with Theorem 2.1, we obtain

$$F(D_2^*,k)=2n-k-d.$$

On the other hand, let $k \ge n-d+1$, $A \in P_{n,d}$, $D=D(A)$, $F(D,k)=2n-k-d$, let X be a k-vertex subset of D with $\exp_D(X)=2n-k-d$, and let W be the set of loop vertices of D.

If $|X \cap W| > k+d-n$ or $\max_{v \in V(D)} d(X \cap W, v) < n-|X \cap W|$, we have $\exp_D(X) < 2n-k-d$. It is a contradiction.

If $|X \cap W| = k + d - n$ and $\max_{v \in V(D)} d(X \cap W, v) = d(X \cap W, y) = n - |X \cap W| = 2n - k - d$, noticing $\exp_D(X) = d(X \cap W, y) = 2n - k - d$, there exists a directed path of length $2n - k - d$ in D with $n - k + 1$ loop vertices. The loop vertices are consecutive at the beginning of this directed path. Now assume that $\{(1,1), (2,2), \cdots, (d,d), (k+d-n, k+d-n+1), (k+d-n+1, k+d-n+2), \cdots, (n-1, n)\} \subseteq E(D)$. In this assumption, we have $y = n, X = \{1, 2, \cdots, k+d-n, d+1, d+2, \cdots, n\}$. D also satisfies the conditions:

(1) $(i,j) \notin E(D)$ for $k + d - n \leq i \leq n - 2$ and $i + 2 \leq j \leq n$. Otherwise, $d(X \cap W, y) < 2n - k - d$, which gives a contradiction.

(2) $(i,j) \notin E(D)$ for $1 \leq i \leq k + d - n$ and $k + d - n + 2 \leq j \leq n$. Otherwise, $d(X \cap W, y) < 2n - k - d$, which gives a contradiction.

(3) $(i,j) \notin E(D)$ for $d + 1 \leq i \leq n$ and $k + d - n + 2 \leq j \leq d$. Otherwise, $\exp_D(X) \leq 2n - k - d - 1$, which gives a contradiction.

(4) For $d + 1 \leq i \leq n$, there is no walk of length $2n - k - d - 1$ from i to n. This implies that if $\{(d+i_1, d+j_1), (d+i_2, d+j_2), \cdots, (d+i_t, d+j_t)\} \subseteq E(D)$, where $j_1 \leq j_2 \leq \cdots \leq j_t$ and $i_s > j_s \geq 1$ for $1 \leq s \leq t$. Then there are no nonnegative integers $k_1, k_2, \cdots, k_t$ such that $k_1(i_1 - j_1 + 1) + k_2(i_2 - j_2 + 1) + \cdots + k_t(i_t - j_t + 1) = n - k + m - 1$ for $1 \leq m \leq j_1$.

Noticing that $\exp_D(X) = 2n - k - d$ and D is a primitive digraph, D must be isomorphic to one of the D_2^*. ∎

Acknowledgements

The authors would like to thank the referee for his many helpful suggestions and comments on an earlier version of this paper.

References

[1] Brualdi R.A. and Liu Bolian, Generalized exponents of primitive directed graphs, *J. Graph Theory,* **14** (1990), 483–499.

[2] Gao Yubin and Shao Yanling, The kth upper generalized exponent set for the class of non-symmetric primitive matrices, *Australas. J. Combin.,* **19** (1999), 191–201.

[3] Liu Bolian, About the distribution of exponent set for the primitive matrices, *Acta Math. Sinica,* **6** (1989), 803–809.

(Received 31/8/99; revised 20/12/99)