On a conjecture involving cycle-complete graph Ramsey numbers

Béla Bollobás

Department of Mathematical Sciences
Campus Box 526429, University of Memphis
Memphis, TN 38152-6429 USA

Chula Jayawardene*

Department of Mathematics
University of Colombo
Columbo, Sri Lanka

Jiansheng Yang, Huang Yi Ru

Department of Mathematics
Shanghai University
Shanghai 201800, P. R. China

Cecil Rousseau

Department of Mathematical Sciences
Campus Box 526429, University of Memphis
Memphis, TN 38152-6429 USA

Zhang Ke Min

Department of Mathematics
Nanjing University
Nanjing 210093, P. R. China

Abstract

It has been conjectured that $r(C_n, K_m) = (m-1)(n-1)+1$ for all $(n,m) \neq (3,3)$ satisfying $n \geq m$. We prove this for the case $m = 5$.

*This author is currently pursuing post-doctoral studies under Prof. Bollobás

1 Introduction

The *independence number* $\alpha(G)$ of a graph G is the cardinality of its largest independent set. Given a graph H without isolated vertices, the *Ramsey number* $r(H, K_m)$ is the smallest integer N such that every graph G of order N either contains H as a subgraph or satisfies $\alpha(G) \geq m$. In one of the earliest contributions to graphical Ramsey theory [1], Bondy and Erdős proved the following result for the case where $H \cong C_n$, a cycle of length n.

Theorem (Bondy, Erdős). *For all* $n \geq m^2 - 2$,

$$r(C_n, K_m) = (m-1)(n-1) + 1.$$

The condition $n \geq m^2 - 2$ is required because of the proof technique, and it has been thought from the beginning that the conclusion is likely to hold under a rather less restrictive hypothesis. The problem of determining for each m the smallest n for which $r(C_n, K_m) = (m-1)(n-1) + 1$ is among those given in [3], and it is conjectured in [8] and elsewhere that $r(C_n, K_m) = (m-1)(n-1) + 1$ for all $(n, m) \neq (3, 3)$ satisfying $n \geq m$. This is trivial for $m = 2$. It was confirmed for $m = 3$ in early work on graphical Ramsey theory [4], and recently it was proved for $m = 4$ [9]. In this paper, we shall prove that the conjecture is true for $m = 5$.

Theorem 1. *For all* $n \geq 5$, $r(C_n, K_5) = 4n - 3$.

Note. The condition $n \geq 5$ is best possible. From early work of Clancy [2], it is known that $r(C_4, K_5) = 14$. There is a unique graph G of order 13 such that $C_4 \not\subset G$ and $\alpha(G) \leq 4$. This graph is exhibited in [6] and elsewhere.

To reach our goal, it is only necessary to prove that for $n \geq 5$ every C_n-free graph G of order $4n - 3$ satisfies $\alpha(G) \geq 5$. The fact that $r(C_n, K_5) \geq 4n - 3$ follows from the simple example of $G \cong 4K_{n-1}$, which contains no C_n and has independence number $\alpha(G) = 4$.

2 Proofs

The proof of Theorem 1 will be given through a sequence of lemmas. As usual, $\delta(G)$ denotes the minimum degree, that is $\delta(G) = \min_{v \in V(G)} \deg v$.

Lemma 1. *Suppose that for some* $n \geq 4$ *there exists a graph* G *of order* $4(n-1)+1$ *such that* $C_n \not\subset G$ *and* $\alpha(G) \leq 4$. *Then* $\delta(G) \geq n - 1$.

Proof. Suppose to the contrary that some vertex $v \in V(G)$ satisfies $\deg v \leq n - 2$. Deleting v and its neighborhood, we obtain a graph H of order at least $3(n-1)+1$. By the result in [9] either $C_n \subset H$ or $\alpha(H) \geq 4$. Since $C_n \not\subset G$, we must assume that latter. But then v together with the appropriate four vertices from $V(H)$ yields a five-element independent set in G, a contradiction. □

The following lemma is proved in [7].

Lemma 2. *Suppose $\delta(G) \geq 4$ and $C_5 \not\subset G$. Then $\alpha(G) \geq \Delta(G)$.*

The following result is given in [5]. In the interest of completeness, it is included here with proof.

Lemma 3. $r(C_5, K_5) = 17$.

Proof. Suppose there exists a graph G of order 17 such that $C_5 \not\subset G$ and $\alpha(G) \leq 4$. By Lemma 1 we know that $\delta(G) \geq 4$. Let $u \in V(G)$ be a vertex of degree $\delta(G)$, let Γ denote the neighborhood of u, and let W denote the set of vertices that remain after u and its neighborhood have been deleted. There are two cases.

Case (i): $\delta(G) = 4$. In this case $\langle W \rangle$ is a C_5-free graph of order 12 with no four-element independent set. All such graphs are found in [7], and they are listed in the Appendix (§3) of this paper for the reader's convenience. Inspection shows that each one contains a K_4 with at least two vertices of degree three. In particular, for each possibility there is a cycle $(w_1, w_2, w_3, w_4, w_1)$ in which w_1 and w_2 have degree three in $\langle W \rangle$. Since $\delta(G) = 4$, w_1 is adjacent to some vertex in Γ and so is w_2. If w_1 and w_2 are each adjacent to $v \in \Gamma$ then $(v, w_1, w_4, w_3, w_2, v)$ is a C_5 in G. If w_1 and w_2 are adjacent to v_1 and v_2, respectively, where $v_1 \neq v_2$, then $(u, v_1, w_1, w_2, v_2, u)$ is a C_5 in G. In either case, we have obtained the desired contradiction.

Case (ii): $\delta(G) \geq 5$. In this case $\alpha(G) \geq \Delta(G) \geq 5$ by Lemma 2, a contradiction. □

Lemma 4. $r(C_6, K_5) = 21$.

Proof. Suppose there exists a graph G of order 21 such that $C_6 \not\subset G$ and $\alpha(G) \leq 4$. Let $V(G) = \{v_1, v_2, \ldots, v_{21}\}$. By Lemma 1, $\delta(G) \geq 5$. Also, $r(K_1 + P_4, K_5) = 19$ [5] and $r(C_6, K_4) = 16$, so we may assume that v_1 is adjacent to each vertex of the path (v_2, v_3, v_4, v_5), and $I \stackrel{\text{def}}{=} \{v_6, v_7, v_8, v_9\}$ is an independent set. It is easy to check that since $C_6 \not\subset G$, no vertex in $\{v_6, v_7, \ldots, v_{21}\}$ is adjacent to two or more vertices of $\{v_2, v_3, v_4, v_5\}$. [If w is adjacent to v_2 and v_3 then $(w, v_2, v_1, v_5, v_4, v_3, w)$ is a C_6 in G, if w is adjacent to v_2 and v_4 then $(w, v_2, v_3, v_1, v_5, v_4, w)$ is a C_6 in G, and so on.] Since $\alpha(G) \leq 4$ each vertex of $V(G) \setminus I$ is adjacent to at least one vertex of I. In view of these two facts, we may assume $\{v_2v_6, v_3v_7, v_4v_8, v_5v_9\} \subset E(G)$. No vertex in $\{v_{10}, \ldots, v_{21}\}$ is adjacent to two or more vertices of I; otherwise, G contains a C_6. Consider v_6. Note that $v_1v_6 \notin E(G)$; otherwise $(v_1, v_5, v_4, v_3, v_2, v_6, v_1)$ is a C_6 in G. Since $\delta(G) \geq 5$ we may assume that $v_6v_j \in E(G)$ for $10 \leq j \leq 13$. Note that $\{v_6, v_{10}, v_{11}, v_{12}, v_{13}\}$ spans a complete subgraph; if $v_iv_j \notin E(G)$ for some $\{i, j\} \subset \{10, 11, 12, 13\}$, then $\{v_7, v_8, v_9, v_i, v_j\}$ is a five-element independent set in G. Now the argument can be repeated, except instead of simply containing $K_1 + P_4$, we may assume that the subgraph induced by $\{v_1, v_2, \ldots, v_5\}$ is complete. Then either some $i \leq 5$ makes $\{v_i, v_6, v_7, v_8, v_9\}$ a five-element independent set in G or else some $v_j \in I$ is adjacent to two or more vertices of $\{v_1, v_2, \ldots, v_5\}$ yielding a C_6 in G, a contradiction. □

The following lemma provides tools which will be used repeatedly in the remaining proofs. Parts (a) and (b) were used in [1] and parts (c) and (d) appear in [9].

Lemma 5. *Suppose G contains the cycle $(u_1, u_2, \dots, u_{n-1}, u_1)$ of length $n-1$ but no cycle of length n. Let $X \subseteq V(G) \setminus \{u_1, u_2, \dots, u_{n-1}\}$. Then*

(a) No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.

(b) If $x \in X$ is adjacent to u_i and u_j then $u_{i+1}u_{j+1} \notin E(G)$.

(c) If $x \in X$ is adjacent to u_i and u_j then no vertex $x' \in X$ is adjacent to both u_{i+1} and u_{j+2}.

(d) Suppose $\alpha(G) = m-1$ where $m \leq (n+3)/2$, and $\{x_1, x_2, \dots, x_{m-1}\} \subset X$ is an $(m-1)$-element independent set. Then no member of this set is adjacent to $m-2$ or more vertices on the cycle.

Proof. (a) Obvious.

(b) If $x \in X$ is adjacent to u_i and u_j where $u_{i+1}u_{j+1} \in E(G)$ then

$$(u_i, x, u_j, u_{j-1}, \dots, u_{i+1}, u_{j+1}, \dots, u_{i-1}, u_i)$$

is a C_n in G, a contradiction.

(c) If x is adjacent to u_i and u_j and x' is adjacent to u_{i+1} and u_{j+2} then

$$(u_i, x, u_j, u_{j-1}, \dots, u_{i+1}, x', u_{j+2}, \dots, u_{i-1}, u_i)$$

is a C_n in G, a contradiction.

(d) First notice as did Bondy and Erdős that no $x \in X$ can be adjacent to $m-1$ or more vertices of the cycle. For, if $1 \leq j_1 < j_2 < \dots < j_{m-1} \leq n-2$ and $x \in X$ satisfies $xu_j \in E(G)$ for all $j \in J = \{j_1, j_2, \dots, j_{m-1}\}$, then in view of (a) and (b) we see that $\{x\} \cup \{u_{j+1} | \ j \in J\}$ is an m-element independent set. Now suppose that $1 \leq k_1 < k_2 < \dots < k_{m-2} \leq n-3$ and $x \in \{x_1, x_2, \dots, x_{m-1}\}$ satisfies $xu_k \in E(G)$ for all $k \in K = \{k_1, k_2, \dots, k_{m-2}\}$. [The condition $n \geq 2m-3$ ensures that there is such an indexing of the vertices on the cycle.] By what was just proved, x is not adjacent to any more vertices on the cycle, in particular x is not adjacent to v_s where $s = k_{m-2}+2$. But v_s is adjacent to some $x' \in \{x_1, x_2, \dots, x_{m-1}\}$ since otherwise there would be an m-element independent set. By (b) we know that $\{u_{k+1} | \ k \in K\}$ is an independent set, and by (c) no member of this set is adjacent to x'. It follows that $\{x, x'\} \cup \{u_{k+1} | \ k \in K\}$ is an m-element independent set, a contradiction. □

The Standard Configuration. To prove that $r(C_n, K_5) = 4(n-1)+1$ for $n \geq 7$, we shall in each case assume to the contrary that there exists a graph G of order $4(n-1)+1$ such that $C_n \not\subset G$ and $\alpha(G) \leq 4$. By Lemma 1, $\delta(G) \geq n-1$. By induction, $r(C_{n-1}, K_5) = 4(n-2)+1$. Hence we may assume that $(u_1, u_2, \dots, u_{n-1}, u_1)$ is a cycle of length $n-1$ in G and, disjoint from this cycle, there is a four-element independent set $I = \{v_1, v_2, v_3, v_4\}$. Let $C = V(C_{n-1}) = \{u_1, u_2, \dots, u_{n-1}\}$ denote the set of vertices on the cycle, and let $W = V(G) \setminus (C \cup I) = \{w_1, w_2, \dots, w_{3n-6}\}$ denote the set of vertices disjoint from $C \cup I$. Since $\alpha(G) \leq 4$ each vertex in C is adjacent to at least one vertex in I. In view of part (d) of Lemma 5 (with $m = 5$), no member of I is adjacent to 3 or more vertices on the cycle. Thus the set of edges $E(C, I) = \{uv | \ u \in C, v \in I\}$ satisfies $|C| \leq |E(C, I)| \leq 8$. If $v \in I$ is adjacent to u_i and u_j and these two vertices have no other neighbors in I then $u_i u_j \in E(G)$; otherwise, u_i, u_j and the three members of $I \setminus \{v\}$ yield a five-element independent set. Note that each vertex in I is adjacent to at least $n - 3$ vertices in W. Since $4(n-3) > 3n - 6$, we may assume (if needed) that there are two vertices in I that are commonly adjacent to some vertex $w \in W$. The structure just described will be called the *standard configuration.*

Lemma 6. $r(C_7, K_5) = 25$.

Proof. Assume the standard configuration. Then $6 \leq |E(C, I)| \leq 8$. The proof is divided into two parts. The first part deals with the possibility $7 \leq |E(C, I)| \leq 8$ and the second part with $|E(C, I)| = 6$.

Part I: $7 \leq |E(C, I)| \leq 8$. Note that each vertex in I is adjacent to at least one vertex in C. If not, then some other vertex in I is adjacent to at least $\lceil 7/3 \rceil = 3$ vertices in C, contradicting part (d) of Lemma 5 (with $m = 5$). In case (i) below, we use the prerogative of assuming that v_1 and v_2 are commonly adjacent to some $w \in W$. We may assume that v_1 is adjacent to two vertices in C. There are two cases.

Case (i): v_1 is adjacent to u_1 and u_3. Note that $u_2 u_4 \notin E(G)$ and $u_2 u_6 \notin E(G)$, both by part (b) of Lemma 5. Also $u_4 v_2 \notin E(G)$; otherwise $(w, v_1, u_1, u_2, u_3, u_4, v_2, w)$ is a C_7 in G. In the same way, $u_6 v_2 \notin E(G)$. We now make two claims.

Claim 1: $u_5 v_2 \notin E(G)$. Suppose $u_5 v_2 \in E(G)$. Then $u_2 v_2 \notin E(G)$ by part (c) of Lemma 5 and $u_4 u_6 \notin E(G)$ as well; otherwise $(w, v_1, u_1, u_6, u_4, u_5, v_2, w)$ is a C_7 in G. In this case, $\{u_2, u_4, u_6, v_1, v_2\}$ is a five-element independent set in G, a contradiction.

Claim 2: $u_2 v_2 \in E(G)$. Suppose $u_2 v_2 \notin E(G)$. Then $u_4 u_6 \in E(G)$ since otherwise $\{u_2, u_4, u_6, v_1, v_2\}$ is a five-element independent set in G. Then $u_1 v_2 \notin E(G)$; otherwise $(w, v_1, u_3, u_4, u_6, u_1, v_2, w)$ is a C_7 in G. In the same way $u_3 v_2 \notin E(G)$. Then $u v_2 \notin E(G)$ for all $u \in C$, a contradiction.

In view of part (a) of Lemma 5 and previously established facts, this means that v_2 is adjacent to precisely one vertex in C. Hence if $|E(C, I)| = 8$, we have already reached a contradiction. Now we may assume that u_4 and u_6 are both adjacent to v_3 and u_5 is adjacent to v_4. Then $u_4 u_6 \in E(G)$; otherwise $\{u_2, u_4, u_6, v_1, v_4\}$ is a five-element independent set in G. Note that $u_2 v_4 \notin E(G)$ by part (c) of Lemma 5. Also, $u_1 v_4 \notin E(G)$ and $u_3 v_4 \notin E(G)$, both by part (b) of Lemma 5. It follows that v_4 is adjacent to precisely one vertex on the cycle, so $|E(C, I)| = 2 + 1 + 2 + 1 = 6$, a contradiction. This completes the proof in case (i).

Case (ii): v_1 is adjacent to u_1 and u_4. In this case, we do not make use of the assumption that v_1 and v_2 are commonly adjacent to $w \in E(G)$. This means that the three vertices v_2, v_3, v_4 are on an equal footing. A simple argument, sketched below, shows that a second vertex, which we may take to be v_2, is adjacent to u_2 and u_5 or to u_3 and u_6. [If we deny this conclusion and use part (c) of Lemma 5, we find that if $v \in \{v_2, v_3, v_4\}$ is adjacent to two vertices in C, then one of those vertices must be u_1 or u_4. For each such v there is an extra edge in $E(C, I)$ over the six that are required by the fact that each vertex in C is adjacent to at least one vertex in I. Suppose there are k such vertices. By the observation just made, $|E(C, I)| \geq 6 + k$. On the other hand, the appropriate degree sum for vertices in I yields $|E(C, I)| = 2(k + 1) + (3 - k) = 5 + k$.] Hence there are two subcases.

Subcase (a): v_2 is adjacent to u_2 and u_5. Then $u_3u_6 \notin E(G)$ by part (b) of Lemma 5. For $v \in \{v_3, v_4\}$, either $u_3v \in E(G)$ or $u_6v \in E(G)$; otherwise $\{u_3, u_6, v_1, v_2, v\}$ is a five-element independent set in G. If $u_3v \in E(G)$ then $u_1v \notin E(G)$ and $u_4v \notin E(G)$, by parts (c) and (a), respectively, of Lemma 5. If $u_6v \in E(G)$ then $u_1v \notin E(G)$ and $u_4v \notin E(G)$ by parts (a) and (c), respectively, of Lemma 5. In view of this, $\{u_1, u_4, v_2, v_3, v_4\}$ is a five-element independent set in G, a contradiction.

Subcase (b): v_2 is adjacent to u_3 and u_6. The proof is similar to that of subcase (a). First $u_2u_5 \notin E(G)$ by part (b) of Lemma 5. Then for $v \in \{v_3, v_4\}$ either $u_2v \in E(G)$ or $u_5v \in E(G)$. Finally, for $u_1v \notin E(G)$ and $u_4v \notin E(G)$ for $v \in \{v_3, v_4\}$, so $\{u_1, u_4, v_1, v_3, v_4\}$ is a five-element independent set in G, a contradiction. This completes the proof in Part I.

Part II: $|E(C, I)| = 6$. In this part, each vertex in C is adjacent to precisely one vertex in I, so if $v \in I$ is adjacent to u_i and u_j then $u_iu_j \in E(G)$. Do not assume that v_1 and v_2 are both adjacent to $w \in W$, only that some pair $v_i, v_j \in I$ have this property. Without loss of generality, v_1 is adjacent to two vertices in C. There are two cases.

Case (i): v_1 is adjacent to u_1 and u_4. Then we may assume that u_2 is adjacent to v_2. In view of parts (a), (b), and (c) of Lemma 5 and the fact that each vertex in C is adjacent to precisely one vertex in I, it is clear that $u_iv_2 \notin E(G)$ for $i \neq 2$. In the same way, we may assume that u_3 is adjacent to v_3 and then find that $u_iv_3 \notin E$ for $i \neq 3$. Then we may assume that u_5 is adjacent to v_4. Finally, however, v_6 cannot be adjacent to any vertex in I, a contradiction.

Case (ii): v_1 is adjacent to u_1 and u_3. Then $u_1u_3 \in E(G)$. We may assume that u_2 is adjacent to v_2. As before, we then find that $u_iv_2 \notin E(G)$ for $i \neq 2$. Then, in the only acceptable configuration, u_4 and u_6 are both adjacent to v_3, $u_4u_6 \in E(G)$, $u_5v_4 \in E(G)$ and $u_iv_4 \notin E(G)$ for $i \neq 5$. Now we use the fact that there are two vertices $v_i, v_j \in I$ that are both adjacent to $w \in W$. If v_1 and v_3 are both adjacent to w then $(w, v_1, u_1, u_2, u_3, u_4, v_3, w)$ is a C_7 in G. If v_1 and v_4 are both adjacent to w then $(w, v_1, u_1, u_3, u_4, u_5, v_4, w)$ is a C_7 in G. If v_2 and v_4 are adjacent to w then $(w, v_2, u_2, u_3, u_4, u_5, v_4, w)$ is a C_7 in G. Hence, by symmetry, we may assume that v_1 and v_2 are both adjacent to $w \in W$. Let $Z = \{u_1, \ldots, u_6, v_1, \ldots, v_4, w\}$ and $Z' = V(G) \setminus Z$.

As one may readily verify, for each vertex $z \in Z \setminus \{v_1, v_2, w\}$ there is a path of length six from w to z. Also for each $z \in Z \setminus \{u_4, u_6, v_4\}$ there is a path of length six from u_5 to z. Since $C_7 \not\subset G$, the degrees of u_5, v_2, v_3, w in $\langle Z \rangle_G$ are $3, 2, 2, 2$, respectively. Since $\delta(G) \geq 6$ there are at least $3 + 4 + 4 + 4 = 15$ edges joining $S \overset{\text{def}}{=} \{u_5, v_2, v_3, w\}$ and Z'. Since $|Z'| = 14$, there must be two vertices in S that are adjacent to the same $w' \in Z'$. Finally, the following path system shows that any two vertices in S are joined by a path of length five in $\langle Z\rangle_G$:

$$(u_5, u_4, u_3, u_1, u_2, v_2), \qquad (u_5, u_4, u_3, u_1, u_6, v_3),$$
$$(u_5, u_4, u_3, u_2, v_2, w), \qquad (v_2, u_2, u_1, u_3, u_4, v_3),$$
$$(v_2, u_2, u_1, u_3, v_1, w), \qquad (v_3, u_4, u_3, u_2, v_2, w).$$

Since there are two vertices in S that are both adjacent to $w' \in Z'$, this gives a C_7 in G, a contradiction. □

Lemma 7. $r(C_8, K_5) = 29$.

Proof. Assume the standard configuration. The edge count $7 = |C| \leq |E(C, I)| \leq 8$ gives two cases for consideration.

Case (i): $|E(C, I)| = 7$. In this case, each vertex in C is adjacent to exactly one vertex in I, one (exceptional) vertex in I is adjacent to only one vertex in C, and the other three are each adjacent to two vertices on the cycle. We may assume that v_1 is not the exceptional vertex. Let N denote the neighbors of v_1 in C. By symmetry, there are two subcases.

Subcase (a): $N = \{u_1, u_3\}$. Then $u_1u_3 \in E(G)$, and we may assume that u_2 is adjacent to v_2. It is easily checked that there is a path of order eight joining v_2 and u_i for $i = 4, 5, 6, 7$. Since there would be a C_8 otherwise, we may assume that $u_iv_2 \notin E(G)$ for $i = 1, 3, 4, 5, 6, 7$, so v_2 must be the exceptional vertex. Then we may assume that v_3 is adjacent to u_4 and u_6, and that v_4 is adjacent to u_5 and u_7, so $u_5u_7 \in E(G)$. But this violates part (b) of Lemma 5.

Subcase (b): $N = \{u_1, u_4\}$. Then $u_1u_4 \in E(G)$, and we may assume that that u_2 is adjacent to v_2 and u_3 is adjacent to v_3. Note that there is a path of order eight joining v_i and u_j for $i = 2, 3$ and $j = 5, 6, 7$. But v_2 and v_3 are not both exceptional, so we have a contradiction.

Case (ii): $|E(C, I)| = 8$. In this case, one (exceptional) vertex in C is adjacent to two vertices in I, and each vertex in I is adjacent to two vertices in C. As noted earlier, we may assume that there is a vertex $w \in W$ that is adjacent to both v_1 and v_2. Again let N denote the neighbors of v_1 in C.

Subcase (a): $N = \{u_1, u_3\}$. Note that there is a path of order eight joining v_2 and u_i for $i = 4, 5, 6, 7$, so v_2 cannot be adjacent to u_4, u_5, u_6 or u_7. Also v_2 cannot be adjacent to u_1 and u_2 or to u_2 and u_3 by part (a) of Lemma 5. Finally, v_2 cannot be adjacent to both u_1 and u_3 since there is only one exceptional vertex in C. Hence there do not exist two vertices on the cycle that can serve as neighbors of v_2, a contradiction.

Subcase (b): $N = \{u_1, u_4\}$. Note that there is a path of order eight joining v_2 and u_i for $i = 1, 4, 5, 7$. Hence we may assume that v_2 is adjacent to u_2 and u_6. However, this violates part (c) of Lemma 5.

Since a contradiction arises in each subcase, the lemma is proved. □

Lemma 8. $r(C_9, K_5) = 33$.

Proof. Assume the standard configuration. The edge count $8 = |C| \leq |E(C, I)| \leq 8$ requires each vertex in C to be adjacent to exactly one vertex of I and each vertex in

I to be adjacent to exactly two vertices in C. We may assume that there is a vertex $w \in W$ that is adjacent to both v_1 and v_2. Let $N = \{u_i, u_j\}$ denote the neighbors of v_1 on the cycle. Since there is no five-element independent set, $u_i u_j \in E(G)$. By symmetry, there are three cases.

Case (i): $N = \{u_1, u_3\}$. It is easily checked that for $4 \leq i \leq 8$ there is a path of order seven joining v_1 and u_i. The paths $(v_1, u_1, u_8, u_7, u_6, u_5, u_4)$ and $(v_1, u_3, u_1, u_8, u_7, u_6, u_5)$ serve for $i = 4$ and $i = 5$, respectively, and their counterparts by symmetry take care of $i = 8$ and $i = 7$. The required path for $i = 6$ may be taken to be $(v_1, u_1, u_2, u_3, u_4, u_5, u_6)$. Hence there do not exist two vertices on the cycle that can serve as neighbors of v_2.

Case (ii): $N = \{u_1, u_4\}$. In this case for $5 \leq i \leq 8$ there is a path of order seven joining v_1 and u_i. The paths $(v_1, u_4, u_1, u_8, u_7, u_6, u_5)$ and $(v_1, u_1, u_2, u_3, u_4, u_5, u_6)$ serve for $i = 5$ and $i = 6$, respectively, and symmetric counterparts take care of $i = 8$ and $i = 7$. Therefore v_2 cannot be adjacent to any of the vertices u_5, u_6, u_7, u_8. By part (a) of Lemma 5, v_2 cannot be adjacent to u_2 and u_3. Hence there do not exist two vertices on the cycle that can serve as the neighbors of v_2.

Case (iii): $N = \{u_1, u_5\}$. In this case, there is a path of order seven joining v_1 to u_i for $i = 2, 4, 6, 8$, so the neighbors of v_2 on the cycle must be u_3 and u_7. Without loss of generality, u_2 is adjacent to v_3, and by symmetry the neighbors of v_3 on the cycle are either u_2 and u_4 or u_2 and u_6. In the first instance, $u_2 u_4 \in E(G)$ and $(v_1, u_1, u_8, u_7, v_2, u_3, u_2, u_4, u_5, v_1)$ is a C_9 in G. In the second, $u_2 u_6 \in E(G)$ and $(v_1, u_1, u_8, u_7, u_6, u_2, u_3, u_4, u_5, v_1)$ is a C_9 in G.

Since a contradiction arises in each case, the proof is complete. □

Completion of the proof of Theorem 1. For $n \geq 10$, the edge count $n - 1 = |C| \leq |E(C, I)| \leq 8$ gives an immediate contradiction. □

3 Appendix - Possible Induced Subgraphs $\langle W \rangle$ for Case (i) in Lemma 3

Here we give the promised collection of graphs of order 12 that contain no C_5 and have independence number 3.

Proposition. *If G is a graph of order twelve such that $C_5 \not\subset G$ and $\alpha(G) = 3$ then G is isomorphic to $3K_4$ or to one of the five graphs shown below, obtained by adding appropriate edges to $3K_4$.*

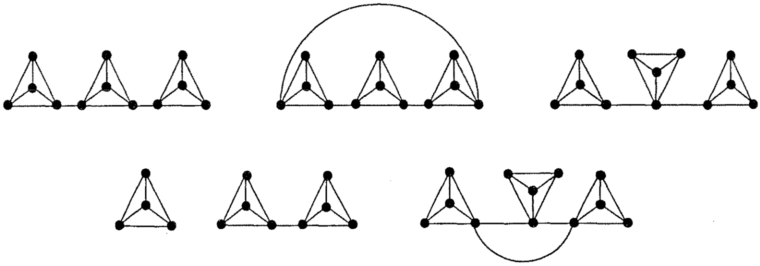

FIGURE 1. Graphs of order twelve with $C_5 \not\subseteq G$ and $\alpha(G) = 3$.

References

[1] J. A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, *J. Combin. Theory Ser. B*, **14** (1973), 46–54.

[2] M. Clancy, Some small Ramsey numbers, *J. Graph Theory*, **1** (1977), 89–91.

[3] R. J. Faudree, C. C. Rousseau, and R. H. Schelp, Problems in graph theory from Memphis, in: *The Mathematics of Paul Erdős, II*, R. L. Graham and J. Nešetřil, eds., Springer-Verlag, Berlin, 1997, 7–26.

[4] R. J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Math.*, **8** (1974), 313–329.

[5] G. R. T. Hendry, Ramsey numbers for graphs with five vertices, *J. Graph Theory*, **13** (1989), 245-248.

[6] C. J. Jayawardene and C. C. Rousseau, An upper bound for the Ramsey number of a quadrilateral versus a complete graph on seven vertices, *Cong. Numer.*, **130** (1998), 175–188.

[7] C. J. Jayawardene and C. C. Rousseau, The Ramsey number for a cycle of length five vs. a complete graph of order six, submitted.

[8] R. H. Schelp and R. J. Faudree, Some problems in Ramsey theory, in: *Theory and Applications of Graphs*, Y. Alavi and D. R. Lick, eds., *Lecture Notes in Math.* **642**, Springer-Verlag, Berlin, 1978, 500–515.

[9] Yang Jian Sheng, Huang Yi Ru and Zhang Ke Min, The value of the Ramsay number $R(C_n, K_4)$ is $3(n-1)+1$ $(n \geq 4)$, *Australas. J. Combin.*, **20** (1999), 205–206.

(Received 1/7/99)