On the Ramsey number $R(C_n \text{ or } K_{n-1}, K_m)$ $(m = 3, 4)$

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Abstract

The Ramsey number $R(C_n \text{ or } K_{n-1}, K_m)$ is the smallest integer p such that every graph G on p vertices contains either a cycle C_n with length n or a K_{n-1}, or an independent set of order m. In this paper we prove that $R(C_n \text{ or } K_{n-1}, K_3) = 2(n-2)+1$ $(n \geq 5)$, $R(C_n \text{ or } K_{n-1}, K_4) = 3(n-2)+1$ $(n \geq 7)$. In particular, we prove that $R(C_4 \text{ or } K_3, K_3) = 6$, $R(C_4 \text{ or } K_3, K_4) = 8$, $R(C_5 \text{ or } K_4, K_4) = 11$ and $R(C_6 \text{ or } K_5, K_4) = 14$.

1. Introduction.

We shall consider only graphs without multiple edges or loops.

The Ramsey number $R(C_n \text{ or } K_{n-1}, K_m)$ is the smallest integer p such that every graph G on p vertices contains either a cycle C_n with length n or a complete graph K_{n-1} on $n-1$ vertices, or an independent set of order m.

In 1976, R.H. Schelp and R.J. Faudree in [2] stated the following problem:

Problem 1.1 ([2]). *Is it true that $R(C_n \text{ or } K_{n-1}, K_m) = (n-2)(m-1)+1$ $(n \geq m)$?*

With this problem, the aim of Schelp and Faudree was to solve the following problem:

Problem 1.2 ([2]). *Find the range of integers n and m such that $R(C_n, K_m) = (n-1)(m-1)+1$. In particular, show that the equality holds for $n \geq m$.*

However, we think that Problem 1.1 is more difficult than Problem 1.2. And in fact, the statement is false for $m \leq n \leq 2(m-1)$. (See Lemma 2.3 below.)

In [3], we proved that $R(C_n, K_4) = 3(n-1)+1$ $(n \geq 4)$.

In this paper, we prove that $R(C_n \text{ or } K_{n-1}, K_3) = 2(n-2)+1$ $(n \geq 5)$ and $R(C_n \text{ or } K_{n-1}, K_4) = 3(n-2)+1$ $(n \geq 7)$. In particular, we prove that $R(C_4$ or

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$K_3, K_3) = 6$, $R(C_4$ or $K_3, K_4) = 8$, $R(C_5$ or $K_4, K_4) = 11$ and $R(C_6$ or $K_5, K_4) = 14$.

The following notation will be used in this paper. If G is a graph, the vertex set (resp. edge set) of G is denoted by $V(G)$ (resp. $E(G)$). For $x \in V(G)$, $N(x) = \{v \in V(G) \mid xv \in E(G)\}$ and $N[x] = N(x) \cup \{x\}$. If $X \subset V(G)$, then $\langle X \rangle$ is the subgraph induced by X. We denote by $\alpha(G)$ the independence number of G, and by $g(G)$ the girth of G.

2. Lemmas.

For convenience, in Lemma 1 to Lemma 3 below, we assume G is a graph that contains the cycle $(v_1, v_2, \cdots, v_n)$ of length n but no cycle of length $n+1$.

Lemma 2.1 ([3]). *Let $X \subseteq V(G) \setminus \{v_1, v_2, \cdots, v_n\}$. Then*
(a) *No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.*
(b) *If $x \in X$ is adjacent to v_i and v_j, then $v_{i+1}v_{j+1} \notin E(G)$.*
(c) *If $x \in X$ is adjacent to v_i and v_j, then no vertex $x' \in X$ is adjacent to both v_{i+1} and v_{j+2}.*

Lemma 2.2. *Let I_{m-1} be an independent set of order $m-1$ with $I_{m-1} \subseteq V(G) \setminus \{v_1, v_2, \cdots, v_n\}$. If $n \geq 2m-3$ and $|N(x) \cap \{v_1, v_2, \cdots, v_n\}| = k$, where $x \in I_{m-1}$, then $k \leq m-3$.*

Proof. For $x \in I_{m-1}$ suppose the neighbors of x on the cycle are $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. By parts (a) and (b) of Lemma 2.1 we know that that $\{x, v_{i_1+1}, \ldots, v_{i_k+1}\}$ is an independent set; hence $k+1 \leq m-1$. To prove that $k \leq m-3$, suppose to the contrary that $k = m-2$. Then $2k = 2m-4$ so since $n \geq 2m-3$ we may put $z = v_{i_k+2}$, where $i_k + 2 \not\equiv i_1 \pmod{n}$. Then $xz \notin E(G)$. If $x'z \in E(G)$ for some $x' \in I_{m-1}$ then by part (c) of Lemma 2.1 $\{x, x', v_{i_1}, \ldots, v_{i_k}\}$ is an m-element independent set; otherwise $I_{m-1} \cup \{z\}$ is an m-element independent set. Hence $k \leq m-3$. $\square$

Corollary. *If $n > (m-1)(m-3)+1$ and G contains a C_{n-1} and a vertex disjoint independent set I_{m-1} with size $m-1$, then G either contains a C_n or an independent set of m vertices.*

Proof. If there is no independent set of m vertices then each vertex on the C_{n-1} is adjacent to at least one vertex in I_{m-1}. But then some vertex in I_{m-1} is adjacent to at least $\lceil (n-1)/(m-1) \rceil \geq m-2$ vertices on the cycle, contradicting Lemma 2.2. $\square$

Lemma 2.3.
(1) $R(C_n$ *or* $K_{n-1}, K_m) \geq (n-2)(m-1)+1$ $(n \geq m)$.
(2) $R(C_n$ *or* $K_{n-1}, K_m) \geq (n-2)(m-1)+2$ $(m \leq n \leq 2(m-1))$.

Proof.
(1) This is trivial.
(2) Starting with the cycle $(x_1, y_2, \ldots, x_{m-1}, y_{m-1}, x_m)$, let G be the graph obtained by replacing each y_i by a K_{n-3}. (Thus each vertex in the K_{n-3} that

replaces y_i is adjacent to x_i and x_{i+1}.) It is easy to see that G contains no K_{n-1} and $\alpha(G) \leq m-1$. If the edge $x_m x_1$ is removed, then each block of the resulting graph has $n-1$ vertices; hence there is no C_n. Any other cycle in G must use the edge $x_m x_1$, and then it must have length at least $2(m-1)+1 \geq n+1$. □

Lemma 2.4 [1]. *Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then G either is pancyclic or else $G = K_{n/2,n/2}$.*

3. The Ramsey number $R(C_n$ or $K_{n-1}, K_m)$ for $m = 3, 4$.

Theorem 3.1. $R(C_n$ *or* $K_{n-1}, K_3) = 2(n-2)+1$ $(n \geq 5)$.

Proof.

Let G be a graph with order $2(n-2)+1$. Suppose $\alpha(G) \leq 2$ and suppose G contains neither a C_n nor a K_{n-1}.

Let $x \in V(G)$ and $V_x = V(G) \setminus N(x)$. Then $\langle V_x \rangle$ is a clique of G. Since G does not contain a K_{n-1} then $|V_x| \leq n-2$. Thus $d(x) \geq n-2$.

If $d(x) \geq n-1$ for every $x \in V(G)$ then by Lemma 2.4 G is pancyclic, a contradiction.

Thus there is a vertex $x \in V(G)$ such that $d(x) \leq n-2$. (Note $n \geq 5$). Hence we have $d(x) = n-2$ and $\langle V_x \rangle \cong K_{n-2}$. It is clear that there are two nonadjacent vertices in $N(x)$, say y_1, y_2. Since $\alpha(\overline{G}) \leq n-2$, there is a vertex z_1 in V_x such that $z_1 \notin N(y_1)$. Thus $z_1 \in N(y_2)$ since $\alpha(G) \leq 2$. Similarly, there is a vertex in V_x, say z_2, such that $z_2 \notin N(y_2)$ and $z_2 \in N(y_1)$.

Thus $(x, y_1, v_1, v_2, \cdots, v_{n-4}, v_{n-3}, y_2)$ is a cycle of G, where $v_1 = z_1$, $v_{n-3} = z_2$ and $\{v_2, v_3, \cdots, v_{n-4}\} \subset V_x \setminus \{z_1, z_2\}$, a contradiction.

Therefore $R(C_n$ or $K_{n-1}, K_3) = 2(n-2)+1$ $(n \geq 5)$. □

Theorem 3.2.

(1) $R(C_4$ *or* $K_3, K_3) = 6$.
(2) $R(C_4$ *or* $K_3, K_4) = 8$.
(3) $R(C_5$ *or* $K_4, K_4) = 11$.
(4) $R(C_6$ *or* $K_5, K_4) = 14$.

Proof.

(1) It is clear that $R(C_4$ or $K_3, K_3) = 6$.

(2) Suppose G is of order eight and girth at least five. We shall prove that $\alpha(G) \geq 4$. If G is bipartite, this conclusion is immediate, so we assume that G contains an odd cycle. If $\langle X \rangle \cong C_7$ is the shortest odd cycle in G, then the remaining vertex u is adjacent to at most one vertex in X. But any five-element subset of X contains a three-element independent set; hence $\{x_i, x_j, x_k, u\}$ is an independent set for appropriate i, j, k. If $\langle X \rangle \cong C_5$ is the shortest odd cycle in G, then since $\{u, v, w\} = V(G) \setminus X$ does not span K_3 we may assume that u and v are nonadjacent. Since $g(G) \geq 5$ neither u nor v is adjacent to more than one vertex in X. Hence there are three vertices in X, none of which is adjacent to either u or v. Since G contains no K_3, we thus find that $\{x_i, x_j, u, v\}$ is an independent set for appropriate i, j.

(3) Suppose G is a graph of order eleven that contains neither C_5 nor K_4, and $\alpha(G) \leq 3$. In view of the result $R(C_5 \text{ or } K_4, K_3) = 7$ obtained earlier, we have $\delta(G) \geq 4$. Using $R(C_4, K_4) = 10$ as well, we may assume that

$$V(G) = X \cup Y \cup Z = \{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3\} \cup \{z_1, z_2, z_3, z_4\},$$

where (x_1, x_2, x_3, x_4) is a C_4 in G and Y is an independent set. Since each vertex in X is adjacent to at least one vertex in Y and G contains no C_5, there is no loss of generality in assuming $x_1y_1, x_3y_1, x_2y_2 \in E(G)$. Then $x_2x_4 \notin E(G)$; otherwise, $(x_1, y_1, x_3, x_2, x_4)$ is a C_5 in G. Since there is no C_5 in G, it is apparent that $x_2y_1 \notin E(G)$ and $x_4y_1 \notin E(G)$. In the same way $x_1y_2 \notin E(G)$ and $x_3y_2 \notin E(G)$. Since $\delta(G) \geq 4$, we have $y_1z \in E(G)$ for some $z \in Z$. Note that $x_1z \notin E(G)$, $y_2z \notin E(G)$, $zx_3 \notin E(G)$; otherwise G contains (z, x_1, x_2, x_3, y_1), (z, y_2, x_2, x_3, y_1), (z, x_3, x_2, x_1, y_1), respectively. Now $x_1x_3 \in E(G)$; otherwise $\{x_1, x_3, y_2, z\}$ is an independent set. Then $x_4y_2 \notin E(G)$; otherwise G contains the cycle $(x_4, y_2, x_2, x_1, x_3)$. Since $x_4y_1 \notin E(G)$ and $x_4y_2 \notin E(G)$, we have $x_4y_3 \in E(G)$ and thus $x_3y_3 \notin E(G)$. Finally, if $zy_3 \in E(G)$ then G contains the cycle (z, y_3, x_4, x_1, y_1) and if $zy_3 \notin E(G)$ then $\{x_3, y_2, y_3, z\}$ is an independent set.

(4) Suppose G is a graph of order fourteen that contains neither C_6 nor K_5, and $\alpha(G) \leq 3$. In view of the results $R(C_5, K_4) = 12$ and $R(C_6 \text{ or } K_5, K_3) = 9$, we may assume that $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3\}$ are disjoint subsets of $V(G)$ such that $(x_1, \ldots, x_5)$ is a C_5 in G and Y is an independent set. Since $6 > (4-1)(4-3)+1$, the desired result follows from the corollary to Lemma 2.2. □

Lemma. *If G is a graph of order $2m$ having independence number $\alpha(G) < 3$ and containing neither K_{m+1} nor C_{m+2} then $G \supseteq 2K_m$.*

Proof. In view of Bondy's theorem, we may assume that $\delta(G) \leq m-1$. Let $x \in V(G)$ be a vertex of degree $\delta(G)$, and set $A = N[x]$ and $B = V(G) \setminus A$. Then $|B| \geq m$ and $\langle B \rangle$ is complete since $\alpha(G) < 3$. Since $K_{m+1} \not\subseteq G$, we have $\delta(G) = m-1$. If $\langle A \rangle$ is complete then $G \supseteq 2K_m$, so let us assume $u, v \in A$ and $uv \notin E(G)$. Since $K_m \not\subseteq G$ and $\alpha(G) < 3$ there are distinct vertices $w, z \in B$ such that $uw \notin E(G)$ and $vz \notin E(G)$. Then the path w, v, x, u, z together with the appropriate path of length $m-2$ joining w and z in $\langle B \rangle$ yields a $C_{m+2} \subset G$ and thus a contradiction. □

Theorem 3.3. $R(C_n \text{ or } K_{n-1}, K_4) = 3(n-2)+1$ $(n \geq 7)$.

Proof. Suppose $n \geq 7$ and G is a graph of order $3(n-2)+1$ that contains neither C_n nor K_{n-1} and satisfies $\alpha(G) \leq 3$. Since $R(C_{n-1}, K_4) = 3(n-2)+1$ for $n \geq 5$, we may assume that $(x_1, x_2, \ldots, x_{n-1})$ is a cycle in G. With $X = \{x_1, x_2, \ldots, x_{n-1}\}$ consider the subgraph of G spanned by $V(G) \setminus X$. If this graph has independence number 3 then we have $C_n \subset G$ or $\alpha(G) \geq 4$ by the corollary to Lemma 2.2. Hence the subgraph of G spanned by $V(G) \setminus X$ has $2(n-2)$ vertices and its independence number is 2. By the preceding Lemma, we thus find a partition $V(G) \setminus X = (Y, Z)$

such that $\langle Y \rangle \cong \langle Z \rangle \cong K_{n-2}$. Since $\langle X \rangle$ is not complete, we may assume that $x_1x_k \notin E(G)$ where $k \leq \lfloor (n+1)/2 \rfloor$. If $x_1v \notin E(G)$ and $x_kv \notin E(G)$ for every $v \in Y \cup Z$ then $\{x_1, x_k, y, z\}$ is a 4-element independent set for arbitrary $y \in Y$ and $z \in Z$ such that $yz \notin E(G)$. (There must be such a z since G contains no K_{n-1}.) Hence by symmetry we may assume that $x_1y_1 \in E(G)$ and (since G contains no K_{n-1}) $x_1y_2 \notin E(G)$. Note that $x_ky_i \notin E(G)$ for all $i \neq 1$; otherwise (since $(n+1)/2 + 1 \leq n$) there is a cycle $(x_1, \ldots, x_k, y_i, \ldots, y_1)$ in G of length n. In particular, $x_ky_2 \notin E(G)$. We now consider two cases.

Case (i). $x_kz \notin E(G)$ for all $z \in Z$. If $x_1z_i \in E(G)$ for some $z_i \in Z$ then $y_2z \notin E(G)$ for all $z \in Z$; otherwise there is a cycle $(x_1, y_1, \ldots, y_2, z, z_i)$ of length n in G. Then since there is some $z_j \in Z$ such that $x_1z_j \notin E(G)$ we find that $\{x_1, x_k, y_2, z_j\}$ is an independent set. If $x_1z \notin E(G)$ for all $z \in Z$ then we can pick a vertex $z_j \in Z$ such that $y_2z_j \notin E(G)$ and then $\{x_1, x_k, y_2, z_j\}$ is an independent set.

Case (ii). $x_kz_1 \in E(G)$ and $x_ky_2 \notin E(G)$. By repeating an earlier argument, we have $x_1z_2 \notin E(G)$. If $y_2z_2 \notin E(G)$ then $\{x_1, x_k, y_2, z_2\}$ is an independent set. Otherwise, $(x_1, \ldots, x_k, z_1, \ldots, z_2, y_2, \ldots, y_1)$ is a cycle in G of length $\geq k+4$ and G contains a C_n provided $n \geq \lfloor (n+1)/2 \rfloor + 4$. This completes the proof in case $n \geq 8$. In case $n = 7$, we are left to consider the case $k = 4$. In particular, we may assume $x_1x_3 \in E(G)$ and then the argument proceeds as before except that $(x_1, x_3, x_4, z_1, z_2, y_2, y_1)$ provides the C_7. $\square$

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