On the Plotkin arrays

W. H. Holzmann and H. Kharaghani*

Department of Mathematics & Computer Science
University of Lethbridge
Lethbridge, Alberta, T1K 3M4
Canada

Abstract

Using a new method we construct a class of orthogonal designs,

$$OD(4(1+p)n; n, n, n, n, pn, pn, pn, pn)$$

for $p = 1$, p a prime power $\equiv 3 \pmod 4$, and $n = 3, 5, 7$. This class includes new Plotkin arrays of order 24, and for the first time, of orders 40 and 56.

1 Preliminaries

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_k)$ denoted $OD(n; s_1, s_2, \ldots, s_k)$ in variables $x_1, x_2, \ldots, x_k$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$AA^t = \sum_{i=1}^{k} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^t = \sum_{i=1}^{k} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^tR & -B_3^tR \\ -B_3R & -B_4^tR & B_1 & B_2^tR \\ -B_4R & B_3^tR & -B_2^tR & B_1 \end{pmatrix}$$

*The research is supported by an NSERC operating grant.
email: hadi@cs.uleth.ca

where R is the back–diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_k)$. See page 107 of [2] for details.

Plotkin [6] showed that if there is an Hadamard matrix of order $2t$, then there is an $OD(8t; t, t, t, t, t, t, t, t)$. In the same paper he also constructed an $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$. Although this OD has appeared in [7], [1] and in [2] it is conjectured that there is an $OD(8n; n, n, n, n, n, n, n, n)$ for each odd integer n, none except $n = 3$ is found yet. In this paper using a new method we construct many new Plotkin ODs of order 24 and two new Plotkin ODs of orders 40 and 56. Actually our construction provides many new orthogonal designs in 8 variables which includes the Plotkin ODs of order 40 and 56.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^t - BA^t = 0$ ($AB^t + BA^t = 0$). Following [3] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n}(A_{\sigma(2i-1)}A^t_{\sigma(2i)} - A_{\sigma(2i)}A^t_{\sigma(2i-1)}) = 0$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ is said to be of *type* $(s_1, s_2, \ldots, s_k)$ and in variables $x_1, x_2, \ldots, x_k$ if it satisfy an additive property

$$\sum_{i=1}^{n} B_i B_i^t = \sum_{i=1}^{k}(s_i x_i^2) I_m.$$

2 Some new Arrays

First we need the following array from [3].

Theorem 1 *Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices of type $(s_1, s_2, \ldots, s_k)$. Then the array*

$$H = \begin{pmatrix} A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\ -A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\ -A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^tR_n & A_7^tR_n & A_6^tR_n & -A_5^tR_n \\ -A_3R_n & A_4R_n & -A_2 & A_1 & A_7^tR_n & A_8^tR_n & -A_5^tR_n & -A_6^tR_n \\ -A_6R_n & -A_5R_n & A_8^tR_n & -A_7^tR_n & A_1 & A_2 & -A_4^tR_n & A_3^tR_n \\ -A_5R_n & A_6R_n & -A_7^tR_n & -A_8^tR_n & -A_2 & A_1 & A_3^tR_n & A_4^tR_n \\ -A_8R_n & -A_7R_n & -A_6^tR_n & A_5^tR_n & A_4^tR_n & -A_3^tR_n & A_1 & A_2 \\ -A_7R_n & A_8R_n & A_5^tR_n & A_6^tR_n & -A_3^tR_n & -A_4^tR_n & -A_2 & A_1 \end{pmatrix},$$

is an $OD(8m; s_1, s_2, \ldots, s_k)$.

We also need the following result from [3].

Theorem 2 *For each prime (power) $p \equiv 3 \pmod 4$ there is an array of order $4(p+1)$ suitable for any amicable set of eight circulant matrices $\{A_i\}_{i=1}^{8}$ for which $\sum_{i=1}^{4}(A_{2i-1}A_{2i-1}^t + pA_{2i}A_{2i}^t)$ is a multiple of the identity matrix.*

A more general version of the following array appeared first in [4].

Theorem 3 *Let $X = x_1P_1 + x_2P_2$ and $Y = y_1Q_1 + y_2Q_2$ be a pair of amicable orthogonal designs of order n and type $((u_1, u_2); (s_1, s_2))$, and in variables x_i and y_i, $i = 1, 2$, respectively, where $Q_i = P_iR_n$, $i = 1, 2$ Let $\{A_i\}_{i=1}^{i=8}$ be an amicable set of circulant $\{0, \pm 1\}$–matrices of order m such that*

$$u_1A_1A_1^t + u_2A_2A_2^t + s_1A_3A_3^t + s_2A_4A_4^t + s_1A_5A_5^t + s_2A_6A_6^t + s_1A_7A_7^t + s_2A_8A_8^t$$

is a multiple of the identity matrix I_m. Let $B_i = A_iR$, $C_i = A_i^tR$, and $\overline{Q_i} = -Q_i$ then

$$\begin{pmatrix} P_1\otimes A_1 + P_2\otimes A_2 & Q_1\otimes B_3 + Q_2\otimes B_4 & Q_1\otimes B_5 + Q_2\otimes B_6 & Q_1\otimes B_7 + Q_2\otimes B_8 \\ \overline{Q_1}\otimes B_3 + \overline{Q_2}\otimes B_4 & P_1\otimes A_1 + P_2\otimes A_2 & Q_1\otimes C_7 + \overline{Q_2}\otimes C_8 & \overline{Q_1}\otimes C_5 + Q_2\otimes C_6 \\ \overline{Q_1}\otimes B_5 + \overline{Q_2}\otimes B_6 & \overline{Q_1}\otimes C_7 + Q_2\otimes C_8 & P_1\otimes A_1 + P_2\otimes A_2 & Q_1\otimes C_3 + \overline{Q_2}\otimes C_4 \\ \overline{Q_1}\otimes B_7 + \overline{Q_2}\otimes B_8 & Q_1\otimes C_5 + \overline{Q_2}\otimes C_6 & \overline{Q_1}\otimes C_3 + Q_2\otimes C_4 & P_1\otimes A_1 + P_2\otimes A_2 \end{pmatrix}$$

is a (Goethals–Seidel–type) array.

Our first result is an extension of the result of Plotkin that if there is an Hadamard matrix of order $2t$, then there is an $OD(8t; t, t, t, t, t, t, t, t)$.

Lemma 4 *Let $2t$ be the order of an Hadamard matrix H, then there are amicable orthogonal designs X, Y of type $((t, t); (t, t))$ in order $2t$, satisfying the conditions of theorem 3.*

PROOF. Let $K_1 = I_2 \otimes I_t$, $K_2 = P \otimes I_t$ and $R = R_2 \otimes R_t$, where R_k is the back diagonal identity matrix of order k and

$$P = \begin{pmatrix} 0 & 1 \\ - & 0 \end{pmatrix}.$$

It is straight forward to show that $\{P_1 = \frac{1}{2}H(K_1 + K_2), P_2 = \frac{1}{2}H(K_1 - K_2)\}$ and $\{Q_1 = \frac{1}{2}H(K_1R + K_2R), Q_2 = \frac{1}{2}H(K_1R - K_2R)\}$ are anti-amicable sets of matrices and $P_iQ_j^t = Q_jP_i^t$, $i, j = 1, 2$. Let $X = x_1P_1 + x_2P_2$ and $Y = y_1Q_1 + y_2Q_2$. Then X, Y are the required orthogonal designs. Note that $Q_i = P_iR$, $i = 1, 2$.

Theorem 5 *(The First Multiplication Theorem)*
If there is an amicable set of circulant matrices $\{A_i\}_{i=1}^{8}$ of order m and type $(s_1, s_2, \ldots, s_k)$ and an Hadamard matrix of order $2t$, then there is an $OD(mt; ts_1, ts_2, \ldots, ts_k)$.

PROOF.
Let H be an Hadamard matrix of order $2t$. Let P_1, P_2, Q_1 and Q_2 be matrices of Lemma 4. It follows from Lemma 3 that the circulant matrices $\{A_i\}_{i=1}^{8}$ satisfy the conditions of the Theorem 3. The result now follows from Theorem 3.

$$
\begin{bmatrix}
5&6&7&1&2&3&4&1&\mathbf{2}&5&\mathbf{8}&\mathbf{7}&1&\mathbf{4}&\mathbf{3}&8&5&\mathbf{6}&\mathbf{6}&7&\mathbf{8}&\mathbf{2}&3&\mathbf{4}\\
7&5&6&3&1&2&1&\mathbf{2}&4&\mathbf{8}&\mathbf{7}&5&\mathbf{4}&\mathbf{3}&1&5&\mathbf{6}&8&7&\mathbf{8}&\mathbf{6}&3&\mathbf{4}&\mathbf{2}\\
6&7&5&2&3&1&\mathbf{2}&4&1&\mathbf{7}&5&\mathbf{8}&\mathbf{3}&1&\mathbf{4}&\mathbf{6}&8&5&\mathbf{8}&\mathbf{6}&7&\mathbf{4}&\mathbf{2}&3\\
1&2&3&\mathbf{5}&\mathbf{6}&\mathbf{7}&\mathbf{5}&8&7&4&1&\mathbf{2}&8&\mathbf{5}&6&1&\mathbf{4}&\mathbf{3}&2&\mathbf{3}&4&\mathbf{6}&7&\mathbf{8}\\
3&1&2&\mathbf{7}&\mathbf{5}&\mathbf{6}&8&7&\mathbf{5}&1&\mathbf{2}&4&\mathbf{5}&6&\mathbf{8}&\mathbf{4}&\mathbf{3}&1&\mathbf{3}&4&2&7&\mathbf{8}&\mathbf{6}\\
2&3&1&\mathbf{6}&\mathbf{7}&\mathbf{5}&7&\mathbf{5}&8&\mathbf{2}&4&1&6&\mathbf{8}&\mathbf{5}&\mathbf{3}&1&\mathbf{4}&4&2&\mathbf{3}&\mathbf{8}&\mathbf{6}&7\\
\mathbf{4}&\mathbf{1}&2&\mathbf{5}&8&7&5&6&7&1&2&3&7&\mathbf{6}&\mathbf{8}&\mathbf{3}&2&4&4&\mathbf{1}&3&5&8&\mathbf{6}\\
\mathbf{1}&2&\mathbf{4}&8&7&\mathbf{5}&7&5&6&3&1&2&\mathbf{6}&\mathbf{8}&7&2&4&\mathbf{3}&\mathbf{1}&3&4&8&\mathbf{6}&5\\
2&\mathbf{4}&\mathbf{1}&7&\mathbf{5}&8&6&7&5&2&3&1&\mathbf{8}&7&\mathbf{6}&4&\mathbf{3}&2&3&4&\mathbf{1}&\mathbf{6}&5&8\\
5&\mathbf{8}&\mathbf{7}&\mathbf{4}&\mathbf{1}&2&1&2&3&\mathbf{5}&\mathbf{6}&\mathbf{7}&3&\mathbf{2}&\mathbf{4}&7&\mathbf{6}&\mathbf{8}&\mathbf{5}&\mathbf{8}&6&4&\mathbf{1}&3\\
\mathbf{8}&\mathbf{7}&5&\mathbf{1}&2&\mathbf{4}&3&1&2&\mathbf{7}&\mathbf{5}&\mathbf{6}&\mathbf{2}&\mathbf{4}&3&\mathbf{6}&\mathbf{8}&7&\mathbf{8}&6&\mathbf{5}&\mathbf{1}&3&4\\
\mathbf{7}&5&\mathbf{8}&2&\mathbf{4}&\mathbf{1}&2&3&1&\mathbf{6}&\mathbf{7}&\mathbf{5}&\mathbf{4}&3&\mathbf{2}&\mathbf{8}&7&\mathbf{6}&6&\mathbf{5}&\mathbf{8}&3&4&\mathbf{1}\\
\mathbf{1}&4&3&\mathbf{8}&\mathbf{5}&6&\mathbf{7}&6&8&3&\mathbf{2}&\mathbf{4}&5&6&7&1&2&3&1&4&\mathbf{2}&8&\mathbf{5}&7\\
4&3&\mathbf{1}&\mathbf{5}&6&\mathbf{8}&6&8&\mathbf{7}&\mathbf{2}&\mathbf{4}&3&7&5&6&3&1&2&4&\mathbf{2}&1&\mathbf{5}&7&8\\
3&\mathbf{1}&4&6&\mathbf{8}&\mathbf{5}&8&\mathbf{7}&6&\mathbf{4}&3&\mathbf{2}&6&7&5&2&3&1&\mathbf{2}&1&4&7&8&\mathbf{5}\\
8&5&\mathbf{6}&\mathbf{1}&4&3&\mathbf{3}&2&4&\mathbf{7}&6&8&1&2&3&\mathbf{5}&\mathbf{6}&\mathbf{7}&\mathbf{8}&5&\mathbf{7}&1&4&\mathbf{2}\\
5&\mathbf{6}&8&4&3&\mathbf{1}&2&4&\mathbf{3}&6&8&\mathbf{7}&3&1&2&\mathbf{7}&\mathbf{5}&\mathbf{6}&5&\mathbf{7}&\mathbf{8}&4&\mathbf{2}&1\\
\mathbf{6}&8&5&3&\mathbf{1}&4&4&\mathbf{3}&2&8&\mathbf{7}&6&2&3&1&\mathbf{6}&\mathbf{7}&\mathbf{5}&\mathbf{7}&\mathbf{8}&5&\mathbf{2}&1&4\\
6&\mathbf{7}&8&2&\mathbf{3}&4&\mathbf{4}&1&\mathbf{3}&\mathbf{5}&\mathbf{8}&6&\mathbf{1}&\mathbf{4}&2&\mathbf{8}&5&\mathbf{7}&5&6&7&1&2&3\\
\mathbf{7}&8&6&\mathbf{3}&4&2&1&\mathbf{3}&\mathbf{4}&\mathbf{8}&6&\mathbf{5}&\mathbf{4}&2&\mathbf{1}&5&\mathbf{7}&\mathbf{8}&7&5&6&3&1&2\\
8&6&\mathbf{7}&4&2&\mathbf{3}&\mathbf{3}&\mathbf{4}&1&6&\mathbf{5}&\mathbf{8}&2&\mathbf{1}&\mathbf{4}&\mathbf{7}&\mathbf{8}&5&6&7&5&2&3&1\\
\mathbf{2}&3&\mathbf{4}&6&\mathbf{7}&8&5&8&\mathbf{6}&\mathbf{4}&1&\mathbf{3}&8&\mathbf{5}&7&\mathbf{1}&\mathbf{4}&2&1&2&3&\mathbf{5}&\mathbf{6}&\mathbf{7}\\
3&\mathbf{4}&\mathbf{2}&\mathbf{7}&8&6&8&\mathbf{6}&5&1&\mathbf{3}&\mathbf{4}&\mathbf{5}&7&8&\mathbf{4}&2&\mathbf{1}&3&1&2&\mathbf{7}&\mathbf{5}&\mathbf{6}\\
\mathbf{4}&\mathbf{2}&3&8&6&\mathbf{7}&\mathbf{6}&5&8&\mathbf{3}&\mathbf{4}&1&7&8&\mathbf{5}&2&\mathbf{1}&\mathbf{4}&2&3&1&\mathbf{6}&\mathbf{7}&\mathbf{5}
\end{bmatrix}
$$

Table 1: $OD(24;3,3,3,3,3,3,3,3)$

7885516633221168855477445112236633244778
5788531663211628554874457122316332647784
5578833166116225548844577223113326677844
8557863316162215488545774231123266378447
8855766331622114885557744311222663384477
1663378855885542211611223774454477866332
3166357885855482116212231744574778463326
3316655788554881162222311445777784433266
6331685578548851622123112457747844732663
6633188557488556221131122577448447726633
2211688554788551663333662774484477522113
2116285548578853166336623744874775421132
1162255488557883316666233448777754411322
1622154885855786331662336487747544713221
6221148855885576633123366877445447732211
8855422116166337885577448336622211344775
8554821162316635788574487366232113247754
5548811622331665578844877662331132277544
5488516221633168557848774623361322175447
4885562211663318855787744233663221154477
7744511223336627744878855166331122655884
7445712231366237448757885316631226158845
4457722311662334487755788331662261188455
4577423112623364877485578633162611284558
5774431122233668774488557663316112245588
1122377445774483366216633788555588411226
1223174457744873662331663578855884512261
2231144577448776623333166557888845522611
2311245774487746233663316855788455826112
3112257744877442336666331885574558861122
6633244778447752211311226558847885516633
6332647784477542113212261588455788531663
3326677844775441132222611884555578833166
3266378447754471322126112845588557863316
2663384477544773221161122455888855766331
4477866332221134477555884112261663378855
4778463326211324775458845122613166357885
7784433266113227754488455226113316655788
7844732663132217544784558261126331685578
8447726633322115447745588611226633188557

Table 2: $OD(40; 5, 5, 5, 5, 5, 5, 5, 5)$

66686241117153537177724868887153555684244486242221735333
46668623111715371777548688821535557842444662422287353331
24666865311171717775386888245355571424446824222863533317
62466681531117177753768882483555715244468442228625333173
86246667153111777537188824865557153444684222286243331735
68624661715311775371788248685571535446842422862423317353
66862461171531753717782486885715355468424428624223173533
11171536668624248688853717776842444715355517353338624222
31117154666862486888237177758424446153555773533316242228
53111712466686868882471777534244468535557135333172422286
15311176246668688824817775372444684355571553331734222862
71531118624666888248677753714446842555715333317352228624
17153116862466882486877537174468424557153533173532286242
11715316686246824868875371774684244571535531735332862422
53717772486888666862411171532242682335371355351754424864
37177754868882466686231117152426822353713353517554248644
71777538688824246668653111714268222537133335175552486444
17775376888248624666815311172682224371333551755534864442
77753718882486862466671531116822242713335317555358644424
77537178824868686246617153118222426133353775553516444248
75371778248688668624611715312224268333537155535174442486
24868885371777111715366686243353713224268244248645535175
48688823717775311171546668623537133242682242486445351755
86888247177753531117124666865371333426822224864443517555
68882481777537153111762466683713335268222448644425175553
88824867775371715311186246667133353682224286444241755535
88248687753717171531168624661333537822242664442487555351
82486887537177117153166862463335371222426844424865553517
71535556842444224268233537136668624111715377173578868428
15355578424446242682235371334666862311171571735778684288
53555714244468426822253713332466686531117117357776842888
35557152444684268222437133356246668153111773577718428886
55571534446842682224271333538624666715311135777174288868
55715354468424822242613335376862466171531157771732888684
57153554684244222426833353716686246117153177717358886842
68424447153555335371322426821117153666862488684287717357
84244461535557353713324268223111715466686286842887173577
42444685355571537133342682225311171246668668428881735777
24446843555715371333526822241531117624666884288867357771
44468425557153713335368222427153111862466642888683577717
44684245571535133353782224261715311686246628886845777173
46842445715355333537122242681171531668624688868427771735
86242221735333553517544248647717357886842866686241117153
62422287353331535175542486447173577868428846668623111715
24222863533317351755524864441735777684288824666865311171
42228625333173517555348644427357771842888662466681531117
22286243331735175553586444243577717428886886246667153111
22862423317353755535164442485777173288868468624661715311
28624223173533555351744424867771735888684266862461171531
17353338624222442486455351758868428771735711171536668624
73533316242228424864453517558684288717357731117154666862
35333172422286248644435175556842888173577753111712466686
53331734222862486444251755538428886735777115311176246668
33317352228624864442417555354288868357771771531118624666
33173532286242644424875553512888684577717317153116862466
31735332862422444248655535178886842777173511715316686246

Table 3: $OD(56; 7,7,7,7,7,7,7,7)$

type	A_1 A_2	A_3 A_4	A_5 A_6	A_7 A_8
$(3,3,3,3,3,3,3,3)$	(a,b,c) (e,f,g)	$(-b,a,d)$ $(-g,-h,e)$	$(-c,-d,a)$ $(-f,e,h)$	$(-h,g,-f)$ $(-d,c,-b)$
$(2,2,2,2,2,2,2,2)$	$(a,b,0)$ $(c,d,0)$	$(a,-b,0)$ $(c,-d,0)$	$(e,f,0)$ $(g,h,0)$	$(e,-f,0)$ $(g,-h,0)$

Table 4: Amicable sets for order 24 ODs in 8 variables each repeated an equal number of times

type	A_1 A_3 A_5 A_7	A_2 A_4 A_6 A_8
$(5,5,5,5,5,5,5,5)$	$(a,f,f,c,-c)$ $(-f,a,a,b,-b)$ $(e,d,d,-g,g)$ $(-b,c,c,-f,f)$	$(g,h,h,e,-e)$ $(-d,e,e,-h,h)$ $(c,b,b,-a,a)$ $(-h,g,g,d,-d)$

Table 5: Amicable sets for a Plotkin OD of order 40

Let A,B,C,D be a set of circulant matrices of type (s_1,s_2,s_3,s_4) and in variables a,b,c,d. Let E,F,G,H be circulant matrices obtained from A,B,C,D by switching a to e, b to f, c to g and d to h. If there is a matching between the sets $\{A,B,C,D\}$ and $\{E,F,G,H\}$ in such a way that the set $\{A,B,C,D,E,F,G,H\}$ is amicable, we call the set $\{A,B,C,D,E,F,G,H\}$ to be a *special amicable* set of circulant matrices of type $(s_1,s_1,s_2,s_2,s_3,s_3,s_4,s_4)$ and *initial circulant matrices* A,B,C,D.

Theorem 6 *(The Second Multiplication Theorem)*
If there is a special amicable set of circulant matrices of order n and type $(s_1,s_1,s_2,s_2,s_3,s_3,s_4,s_4)$, then there is an

$$OD(4n(1+p); s_1, s_2, s_3, s_4, ps_1, ps_2, ps_3, ps_4),$$

for each prime (power) $p \equiv 3 \pmod 4$, or $p=1$.

PROOF.
Let A,B,C,D be the initial circulant matrices of order n for the special amicable set of matrices. Then $AA^t+BB^t+CC^t+DD^t=(s_1a^2+s_2b^2+s_3c^2+s_4d^2)I_n$, and $EE^t+FF^t+GG^t+HH^t=(s_1e^2+s_2f^2+s_3g^2+s_4h^2)I_n$. Without loss of generality, we can assume that the matching in the amicability condition is A with E, B with F, C with G and D with H. It follows that, $AA^t+BB^t+CC^t+DD^t+p(EE^t+FF^t+GG^t+HH^t)=(s_1a^2+s_2b^2+s_3c^2+s_4d^2+p(s_1e^2+s_2f^2+s_3g^2+s_4h^2))I_n$. Therefore the result, for the case that p is a prime (power) $p \equiv 3 \pmod 4$ follows

type	A_1 A_3 A_5 A_7	A_2 A_4 A_6 A_8
$(1,1,1,1,1,1,1,1)$	$(a,0,0,0,0,0,0)$ $(c,0,0,0,0,0,0)$ $(e,0,0,0,0,0,0)$ $(g,0,0,0,0,0,0)$	$(b,0,0,0,0,0,0)$ $(d,0,0,0,0,0,0)$ $(f,0,0,0,0,0,0)$ $(h,0,0,0,0,0,0)$
$(2,2,2,2,2,2,2,2)$	$(a,g,0,0,0,0,0)$ $(a,-g,0,0,0,0,0)$ $(e,c,0,0,0,0,0)$ $(e,-c,0,0,0,0,0)$	$(h,f,0,0,0,0,0)$ $(h,-f,0,0,0,0,0)$ $(d,b,0,0,0,0,0)$ $(d,-b,0,0,0,0,0)$
$(3,3,3,3,3,3,3,3)$	$(a,g,-e,0,0,0,0)$ $(a,-g,0,c,0,0,0)$ $(a,0,e,-c,0,0,0)$ $(g,e,c,0,0,0,0)$	$(h,f,-d,0,0,0,0)$ $(h,-f,0,b,0,0,0)$ $(h,0,d,-b,0,0,0)$ $(f,d,b,0,0,0,0)$
$(4,4,4,4,4,4,4,4)$	$(a,g,e,c,0,0,0)$ $(a,-g,e,-c,0,0,0)$ $(a,g,-e,-c,0,0,0)$ $(a,-g,-e,c,0,0,0)$	$(h,f,d,b,0,0,0)$ $(h,-f,d,-b,0,0,0)$ $(h,f,-d,-b,0,0,0)$ $(h,-f,-d,b,0,0,0)$
$(5,5,5,5,5,5,5,5)$	$(a,a,-a,g,0,g,0)$ $(-g,-g,g,a,0,a,0)$ $(-e,-e,e,-c,0,-c,0)$ $(-c,-c,c,e,0,e,0)$	$(-f,-f,f,h,0,h,0)$ $(h,h,-h,f,0,f,0)$ $(-b,-b,b,d,0,d,0)$ $(-d,-d,d,-b,0,-b,0)$
$(7,7,7,7,7,7,7,7)$	$(-a,a,a,g,a,e,c)$ $(-g,g,g,-a,g,c,-e)$ $(-e,e,e,-c,e,-a,g)$ $(-b,b,b,d,b,-f,-h)$	$(-f,f,f,-h,f,b,-d)$ $(-h,h,h,f,h,d,b)$ $(-d,d,d,-b,d,-h,f)$ $(-c,c,c,e,c,-g,-a)$

Table 6: Amicable sets for order 56 ODs in 8 variables each repeated an equal number of times

type	A_1 A_3 A_5 A_7	A_2 A_4 A_6 A_8
$(1,1,1,1,9,9,9,9)$	$(a,e,f,-f,-e)$ $(-h,h,h,h,h)$ $(-e,e,e,e,e)$ $(d,h,-g,g,-h)$	$(-g,g,g,g,g)$ $(b,f,-e,e,-f)$ $(c,g,h,-h,-g)$ $(-f,f,f,f,f)$
$(2,2,2,2,8,8,8,8)$	$(a,f,e,f,-e)$ $(a,-f,-e,-f,e)$ $(d,g,-h,g,h)$ $(d,-g,h,-g,-h)$	$(c,h,g,h,-g)$ $(c,-h,-g,-h,g)$ $(b,e,-f,e,f)$ $(b,-e,f,-e,-f)$
$(2,2,5,5,5,5,8,8)$	$(c,e,g,-g,e)$ $(a,c,g,g,-c)$ $(-a,e,g,g,-e)$ $(-e,c,g,-g,c)$	$(b,d,h,h,-d)$ $(d,f,h,-h,f)$ $(-f,d,h,-h,d)$ $(-b,f,h,h,-f)$

Table 7: Amicable sets for full non-Plotkin ODs of order 40

from Theorem 2 by taking $A_1 = A$, $A_2 = E$, $A_3 = B$, $A_4 = F$, $A_5 = C$, $A_6 = G$, $A_7 = D$ and $A_8 = H$. The case $p = 1$ is a consequence of Theorem 1.
Applying Theorem 1, to the first set of special amicable matrices of Table 4 gives a Plotkin $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$ presented in Table 2. In this and the following tables, the variables are $x_1, \ldots, x_8$ but only the subscripts are shown. Further, if the entry is $-x_i$ then i is shown in boldface.
Applying Theorem 1, to the special amicable matrices of Table 5 gives a Plotkin $OD(40; 5,5, 5, 5, 5, 5, 5, 5)$ presented in Table 2.
Applying Theorem 5, to the circulant matrices $A_1 = \text{circ}(a, g, -g)$, $A_2 = \text{circ}(f, h, h)$, $A_3 = \text{circ}(c, g, -g)$, $A_4 = \text{circ}(-f, h, h)$, $A_5 = \text{circ}(e, g, g)$, $A_6 = \text{circ}(b, h, -h)$, $A_7 = \text{circ}(-e, g, g)$, and $A_8 = \text{circ}(d, h, -h)$ together with a Paley Hadamard matrix of order 12 gives an $OD(144; 6, 6, 6, 6, 12, 12, 48, 48)$. See
http://www.cs.uleth.ca/~holzmann/research/plotkin/ for images and copies of this orthogonal design.
Applying Theorem 1, to the amicable matrices of Table 6 gives the Plotkin $OD(56; 7, 7, 7, 7, 7, 7, 7, 7)$ presented in Table 3.

3 The Search

Although arrays of Theorems 1 and 5 requires only an amicable set of eight circulant matrices, we concentrated our search for special amicable set of matrices. By feeding an already known set of four circulant matrices A, B, C, and D of type (s_1, s_2, s_3, s_4) and of order 3, 5, 7, see the tables in [2], we constructed the sets E, F, G and H as described earlier. We then did an extensive computer search for the special amicable sets of matrices. The search turned up amicable matrices as listed in Table 4 for

type	A_1 A_3 A_5 A_7	A_2 A_4 A_6 A_8
$(1,1,1,1,1,1,4,4)$	$(a,0,0,0,0,0,0)$ $(c,0,0,0,0,0,0)$ $(g,e,-g,0,0,0,0)$ $(g,0,g,0,0,0,0)$	$(b,0,0,0,0,0,0)$ $(d,0,0,0,0,0,0)$ $(h,0,h,0,0,0,0)$ $(h,f,-h,0,0,0,0)$
$(1,1,1,1,2,2,2,2)$	$(a,0,0,0,0,0,0)$ $(c,0,0,0,0,0,0)$ $(f,e,0,0,0,0,0)$ $(f,-e,0,0,0,0,0)$	$(b,0,0,0,0,0,0)$ $(d,0,0,0,0,0,0)$ $(h,g,0,0,0,0,0)$ $(h,-g,0,0,0,0,0)$
$(1,1,1,1,2,2,8,8)$	$(g,a,-g,0,0,0,0)$ $(g,c,-g,0,0,0,0)$ $(g,e,g,0,0,0,0)$ $(g,-e,g,0,0,0,0)$	$(h,f,h,0,0,0,0)$ $(h,-f,h,0,0,0,0)$ $(h,b,-h,0,0,0,0)$ $(h,d,-h,0,0,0,0)$
$(1,1,1,1,4,4,4,4)$	$(a,0,0,0,0,0,0)$ $(c,0,0,0,0,0,0)$ $(f,f,e,-e,0,0,0)$ $(e,e,-f,f,0,0,0)$	$(b,0,0,0,0,0,0)$ $(d,0,0,0,0,0,0)$ $(g,g,-h,h,0,0,0)$ $(h,h,g,-g,0,0,0)$
$(1,1,1,1,4,4,16,16)$	$(g,0,g,0,g,0,g)$ $(g,0,g,a,-g,0,-g)$ $(g,e,-g,0,-g,e,g)$ $(g,e,-g,c,g,-e,-g)$	$(h,f,-h,d,h,-f,-h)$ $(h,f,-h,0,-h,f,h)$ $(h,0,h,b,-h,0,-h)$ $(h,0,h,0,h,0,h)$
$(1,1,1,1,5,5,5,5)$	$(a,0,0,0,0,0,0)$ $(c,0,0,0,0,0,0)$ $(f,f,-f,e,0,e,0)$ $(e,e,-e,-f,0,-f,0)$	$(b,0,0,0,0,0,0)$ $(d,0,0,0,0,0,0)$ $(g,g,-g,-h,0,-h,0)$ $(h,h,-h,g,0,g,0)$
$(1,1,1,1,8,8,8,8)$	$(f,e,a,-e,-f,0,0)$ $(h,-g,0,-g,h,0,0)$ $(f,e,0,e,f,0,0)$ $(-h,g,d,-g,h,0,0)$	$(h,g,0,g,h,0,0)$ $(-f,e,c,-e,f,0,0)$ $(h,g,b,-g,-h,0,0)$ $(f,-e,0,-e,f,0,0)$
$(1,1,2,2,2,2,4,4)$	$(g,a,-g,0,0,0,0)$ $(g,0,g,0,0,0,0)$ $(d,c,0,0,0,0,0)$ $(d,-c,0,0,0,0,0)$	$(h,0,h,0,0,0,0)$ $(h,b,-h,0,0,0,0)$ $(f,e,0,0,0,0,0)$ $(f,-e,0,0,0,0,0)$
$(1,1,2,2,3,3,6,6)$	$(e,g,c,0,0,0,0)$ $(e,g,-c,0,0,0,0)$ $(h,b,-h,0,0,0,0)$ $(h,-f,h,0,0,0,0)$	$(f,h,d,0,0,0,0)$ $(f,h,-d,0,0,0,0)$ $(g,-e,g,0,0,0,0)$ $(g,a,-g,0,0,0,0)$

Table 8: Amicable sets for ODs of order 56 (continued)

type	A_1 A_3 A_5 A_7	A_2 A_4 A_6 A_8
$(1,1,3,3,6,6,8,8)$	$(-g,g,g,0,g,c,e)$ $(g,-g,-g,0,-g,c,e)$ $(f,b,-f,0,0,0,0)$ $(f,-d,f,0,0,0,0)$	$(-h,h,h,0,h,d,f)$ $(h,-h,-h,0,-h,d,f)$ $(e,-c,e,0,0,0,0)$ $(e,a,-e,0,0,0,0)$
$(1,1,4,4,4,4,4,4)$	$(c,a,-c,0,0,0,0)$ $(c,0,c,0,0,0,0)$ $(f,f,e,-e,0,0,0)$ $(e,e,-f,f,0,0,0)$	$(d,0,d,0,0,0,0)$ $(d,b,-d,0,0,0,0)$ $(g,g,-h,h,0,0,0)$ $(h,h,g,-g,0,0,0)$
$(1,1,4,4,5,5,5,5)$	$(c,a,-c,0,0,0,0)$ $(c,0,c,0,0,0,0)$ $(f,f,-f,e,0,e,0)$ $(e,e,-e,-f,0,-f,0)$	$(d,0,d,0,0,0,0)$ $(d,b,-d,0,0,0,0)$ $(g,g,-g,-h,0,-h,0)$ $(h,h,-h,g,0,g,0)$
$(2,2,2,2,4,4,4,4)$	$(a,c,0,0,0,0,0)$ $(a,-c,0,0,0,0,0)$ $(f,f,e,-e,0,0,0)$ $(e,e,-f,f,0,0,0)$	$(b,d,0,0,0,0,0)$ $(b,-d,0,0,0,0,0)$ $(g,g,-h,h,0,0,0)$ $(h,h,g,-g,0,0,0)$
$(2,2,2,2,5,5,5,5)$	$(a,c,0,0,0,0,0)$ $(a,-c,0,0,0,0,0)$ $(e,e,-e,-f,0,-f,0)$ $(f,f,-f,e,0,e,0)$	$(b,d,0,0,0,0,0)$ $(b,-d,0,0,0,0,0)$ $(h,h,-h,g,0,g,0)$ $(g,g,-g,-h,0,-h,0)$
$(2,2,2,2,10,10,10,10)$	$(f,f,-f,e,a,e,0)$ $(f,f,-f,e,-a,e,0)$ $(e,e,-e,-f,c,-f,0)$ $(e,e,-e,-f,-c,-f,0)$	$(g,g,-g,-h,d,-h,0)$ $(g,g,-g,-h,-d,-h,0)$ $(h,h,-h,g,b,g,0)$ $(h,h,-h,g,-b,g,0)$
$(2,2,3,3,4,4,6,6)$	$(e,g,0,-g,e,0,0)$ $(e,g,c,g,-e,0,0)$ $(b,d,-h,0,0,0,0)$ $(-b,d,-h,0,0,0,0)$	$(f,h,d,h,-f,0,0)$ $(f,h,0,-h,f,0,0)$ $(a,c,-g,0,0,0,0)$ $(-a,c,-g,0,0,0,0)$
$(3,3,3,3,3,3,12,12)$	$(-g,g,g,0,g,a,c)$ $(g,-g,-g,e,-g,a,0)$ $(g,-g,-g,-e,-g,0,c)$ $(0,0,0,-e,0,a,-c)$	$(-h,h,h,0,h,b,d)$ $(h,-h,-h,f,-h,b,0)$ $(h,-h,-h,-f,-h,0,d)$ $(0,0,0,-f,0,b,-d)$
$(3,3,3,3,6,6,6,6)$	$(e,g,a,-g,e,0,0)$ $(e,g,c,g,-e,0,0)$ $(d,-h,-f,0,b,0,0)$ $(d,-h,f,0,-b,0,0)$	$(f,h,d,h,-f,0,0)$ $(f,h,b,-h,f,0,0)$ $(c,-g,e,0,-a,0,0)$ $(c,-g,-e,0,a,0,0)$

Table 9: Amicable sets for ODs of order 56 (continued)

type	A_1 A_3 A_5 A_7	A_2 A_4 A_6 A_8
$(3,3,4,4,6,6,8,8)$	$(-g,g,g,0,g,a,e)$ $(g,-g,-g,0,-g,a,e)$ $(-d,f,-b,f,d,0,0)$ $(d,f,0,-f,d,0,0)$	$(-h,h,h,0,h,b,f)$ $(h,-h,-h,0,-h,b,f)$ $(c,e,0,-e,c,0,0)$ $(-c,e,-a,e,c,0,0)$
$(4,4,4,4,5,5,5,5)$	$(a,a,c,-c,0,0,0)$ $(c,c,-a,a,0,0,0)$ $(f,0,f,e,e,-e,0)$ $(e,0,e,-f,-f,f,0)$	$(d,d,-b,b,0,0,0)$ $(b,b,d,-d,0,0,0)$ $(g,0,g,-h,-h,h,0)$ $(h,0,h,g,g,-g,0)$
$(4,4,4,4,8,8,8,8)$	$(e,e,g,-g,b,a,0)$ $(e,e,g,-g,-b,-a,0)$ $(e,-e,g,g,b,-a,0)$ $(e,-e,g,g,-b,a,0)$	$(f,f,h,-h,d,c,0)$ $(f,f,h,-h,-d,-c,0)$ $(f,-f,h,h,d,-c,0)$ $(f,-f,h,h,-d,c,0)$
$(4,4,4,4,10,10,10,10)$	$(c,f,a,f,e,e,-e)$ $(c,-f,a,-f,-e,-e,e)$ $(c,e,-a,e,-f,-f,f)$ $(c,-e,-a,-e,f,f,-f)$	$(d,h,b,h,g,g,-g)$ $(d,-h,b,-h,-g,-g,g)$ $(d,-g,-b,-g,h,h,-h)$ $(d,g,-b,g,-h,-h,h)$

Table 10: Amicable sets for ODs of order 56 (continued)

order 3, Table 5 for order 5 and Table 6 for order 7. The new orthogonal designs in 8 variables obtained in our search is listed as follow: For order 40 these are listed in Table 7. (Refer to [5] for a complete list of new orthogonal designs of order 40 including all the remaining unresolved full orthogonal designs in three variables). For order 56 these are listed in Tables 8 through 10.

Acknowledgment

We are grateful to NSERC for an equipment grant which made our extensive search possible.

We also thank the MACI (Multimedia Advanced Computational Infrastructure of Alberta) project for use of their fast computers.

References

[1] Jennifer Seberry and R. Craigen, Orthogonal Designs, in *CRC Handbook of Combinatorial Designs*, Editors Charles J. Colbourn & Jeffrey H. Dinitz, CRC Press, (1996), 400–406.

[2] A. V. Geramita and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York - Basel (1979).

[3] H. Kharaghani, Arrays for orthogonal designs, *JCD*, to appear.

[4] W. H. Holzmann and H. Kharaghani, On The Orthogonal Designs of Order 24, *Applied Discrete Math.* to appear (1999).

[5] W. H. Holzmann and H. Kharaghani, On The Orthogonal Designs of Order 40, preprint.

[6] M. Plotkin, Decomposition of Hadamard matrices, JCT A(13), (1972), 127–130.

[7] J. Seberry and M. Yamada, *Hadamard matrices, sequences, and block designs*, in Contemporary Design Theory: A Collection of Surveys, J. H. Dinitz and D. R. Stinson, eds., John Wiley & Sons, Inc., 1992, pp. 431–560.

(Received 15/12/99)