The metamorphosis of lambda-fold 4-wheel systems into lambda-fold bowtie systems

Elizabeth J. Billington*

Centre for Discrete Mathematics and Computing,
Department of Mathematics,
The University of Queensland, Queensland 4072, Australia

Abstract

A 4-wheel is a simple graph on 5 vertices with 8 edges, consisting of a 4-cycle, with a fifth vertex joined to each vertex in the 4-cycle. A λ-fold 4-wheel system of order n is an edge-disjoint decomposition of λK_n into 4-wheels. If two non-adjacent edges of the 4-cycle are removed, the result is a bowtie (that is, two triangles with a common vertex). In this paper necessary and sufficient conditions are given for the metamorphosis of a λ-fold 4-wheel system of order n into a λ-fold bowtie system of order n, by retaining the bowtie subgraph from each 4-wheel, and rearranging the disjoint pairs of removed edges from each 4-wheel into further bowties. (There remain three isolated unresolved values of n when $\lambda = 2$, namely: 24, 72, 88. Currently no 2-fold 4-wheel systems of these orders are known.)

1 Introduction and necessary conditions

Let G and H be simple graphs, and let λH denote the graph H with each of its edges replicated λ times. A *λ-fold G-system* of λH is a pair (X, K) where X is the vertex set of H and K is a collection of isomorphic copies of the graph G whose edges partition the edges of λH. If H is a complete graph K_n, we refer to such a λ-fold G-system as one of *order* n. Also if $\lambda = 1$, we drop the term "1-fold".

A 4-wheel G is a simple graph with 5 vertices $\{c, a_1, a_2, a_3, a_4\}$ and 8 edges $\{\{c, a_i\} \mid 1 \leq i \leq 4\} \cup \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\}$; it will be denoted by c–(a_1, a_2, a_3, a_4) (or possibly c–$(a_i, a_{i+1}, a_{i+2}, a_{i+3})$, or c–$(a_i, a_{i-1}, a_{i-2}, a_{i-3})$, for $i = 1, 2, 3$ or 4 (subscript addition modulo 4).

A bowtie G' is a simple graph with 5 vertices and 6 edges, consisting of two triangles sharing one common vertex. If the two triangles have vertices $\{a, b, c\}$ and $\{a, d, e\}$, we shall denote the bowtie by $\{a, b, c; a, d, e\}$.

*Research supported by the Australian Research Council grant A69701550.

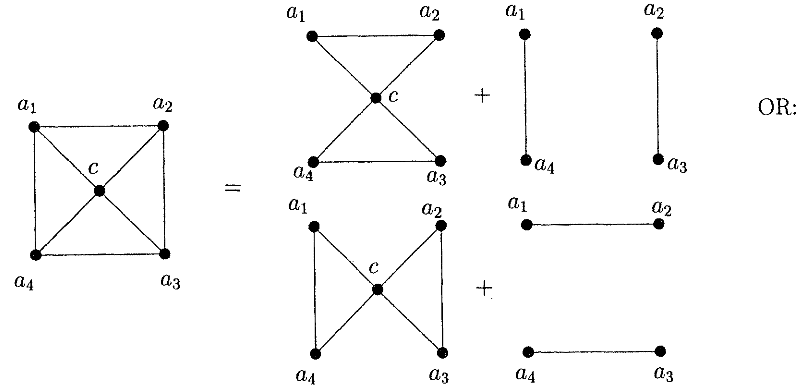

Figure 1. Bowtie from 4-wheel

Clearly a 4-wheel $G = c\text{–}(a_1, a_2, a_3, a_4)$ contains a bowtie G' as a proper subgraph: either $\{c, a_1, a_2; c, a_3, a_4\}$ or else $\{c, a_1, a_4; c, a_2, a_3\}$. The former bowtie excludes edges $\{a_1, a_4\}$ and $\{a_2, a_3\}$ from the 4-wheel, while the latter bowtie excludes edges $\{a_1, a_2\}$ and $\{a_3, a_4\}$ from the 4-wheel.

Suppose there exists a λ-fold G-system (X, K) of order n, and let G' be a proper subgraph of G. (The reader may consider the case at hand where G is a 4-wheel and G' is a bowtie.) Let G'' denote the complement of G' in G, so that $G = G' \cup G''$. (In our case, G'' is a pair of disjoint edges.) For each copy of G in K, we retain a subgraph of G isomorphic to G' (placing each one in a collection K') and take all the remaining edges in the subgraphs G''; these edges in the subgraphs G'' are rearranged (if possible) into further copies of G', which are also placed in K'. The result is an edge-disjoint decomposition of λK_n into copies of G', which is a *metamorphosis* of the G-system (X, K) into the G'-system (X, K').

Such metamorphoses from G-systems into G'-systems have been considered previously; see for example [5], [6] and [7]. In particular the first of these deals with the metamorphosis of λ-fold 4-wheel systems into λ-fold 4-cycle systems, taking $G = c\text{–}(a_1, a_2, a_3, a_4)$ with subgraph the 4-cycle $G' = (a_1, a_2, a_3, a_4)$. Here we deal with the problem of finding a metamorphosis of a λ-fold 4-wheel system into a λ-fold bowtie system. Bowtie systems have been considered previously; see for example [9], where any Steiner triple system with an even number of triples is shown able to be arranged into bowties. Further work on bowties appears in [3] and [4].

Metamorphosis problems are of particular interest in that they provide a link between a G-system and a G'-system of the same order. Since G' is a subgraph of G, they may be regarded as giving rise to a type of subdesign where (some of the) new blocks are subsets of the original blocks.

Let us start by considering the necessary conditions for existence of λ-fold 4-wheel systems and λ-fold bowtie systems; these are easily calculated (see Table 1.1 below). The intersection of these conditions is needed to obtain the admissible orders of a λ-fold 4-wheel system with *potential* for metamorphosis into a λ-fold bowtie system. We tabulate these necessary conditions in Table 1.2.

4-wheel system

λ (mod 8)	order
1, 3, 5, 7	0, 1 (mod 16)
2, 6	0, 1 (mod 8)
4	0, 1 (mod 4)
8	any $n \geq 5$

bowtie system

λ (mod 6)	order
1, 5	1, 9 (mod 12)
2, 4	0, 1 (mod 3)
3	1 (mod 4)
6	any $n \geq 5$

Table 1.1

4-wheel system with potential for metamorphosis into bowtie system

λ (mod 24)	order
1, 5,7,11,13,17,19,23	1, 33 (mod 48)
2, 10,14,22	0,1,9,16 (mod 24)
3, 9,15,21	1 (mod 16)
4, 20	0,1,4,9 (mod 12)
6, 18	0, 1 (mod 8)
8, 16	0, 1 (mod 3)
12	0, 1 (mod 4)
24	any $n \geq 5$

Table 1.2

Henceforth, any λ-fold 4-wheel system which has a metamorphosis into a λ-fold bowtie system we shall call a *λ-fold B-wheel system* for short. We shall also sometimes drop the prefix 4, so that "wheel system" will always mean 4-wheel system here.

In this paper we solve the problem of constructing a λ-fold B-wheel system of all admissible orders given in the above Table 1.2, apart from three isolated cases when $\lambda = 2$ (namely, 24, 72, 88). (To date, there are no 2-fold 4-wheel systems of these orders known, let alone any 2-fold B-wheel systems of these orders.)

In Section 2 we deal with the construction of λ-fold B-wheel systems when λ is 1 or 3, while Section 3 deals with this construction when λ is 2, 4, 6, 8, 12 or 24. Then Section 4 applies the results to *any* value of λ.

Our main construction is the following (see [5]; we include it here for completeness). For definition of a group divisible design (GDD), and notation used here, see for instance [8]. Many of the small 4-wheel systems used here were found with *autogen* (Adams [1]); the metamorphoses into bowtie systems were found by hand.

THE 3-GDD CONSTRUCTION

Let the vertex set of a complete graph of order $s\ell+h$ be $\{\infty_i \mid 1 \le i \le h\} \cup \{(i,j) \mid 1 \le i \le s,\ 1 \le j \le \ell\}$. (If $h = 0$ then none of the elements ∞_i will occur.)

Suppose that there exists a 3-GDD of type p^1q^r where $p + rq = s$, and that a λ-fold B-wheel system exists for order $p\ell + h$. Suppose further that, for the graph $K_{q\ell+h} \setminus K_h$ (a complete graph of order $q\ell + h$ set of h vertices removed from $K_{q\ell+h}$), there exists a λ-fold B-wheel system. Finally, suppose that there is a λ-fold B-wheel system of $K_{\ell,\ell,\ell}$.

Then the 4-wheels in our B-wheel system of order $s\ell + h$ are as follows:

1. If the 3-GDD group of size p is $\{a_1, a_2, \ldots, a_p\}$, place a λ-fold B-wheel system of order $p\ell+h$ on the vertex set $\{\infty_i \mid 1 \le i \le h\} \cup \{(a_i, j) \mid 1 \le i \le p,\ 1 \le j \le \ell\}$. (Possibly $p = q$ here.)

2. For each 3-GDD group of size q, say $\{b_1, \ldots, b_q\}$, place a λ-fold B-wheel system of $K_{q\ell+h} \setminus K_h$ on the vertex set $\{\infty_i \mid 1 \le i \le h\} \cup \{(b_i, j) \mid 1 \le i \le q,\ 1 \le j \le \ell\}$, where the h "hole" elements are $\{\infty_i \mid 1 \le i \le h\}$.

3. For each block $\{x, y, z\}$ of the 3-GDD, on the vertex set $\{(x,j) \mid 1 \le j \le \ell\}$, $\{(y,j) \mid 1 \le j \le \ell\}$, $\{(z,j) \mid 1 \le j \le \ell\}$, place a λ-fold B-wheel system of $K_{\ell,\ell,\ell}$.

2 The cases $\lambda = 1$ or 3

2.1 $\boxed{\lambda = 1}$

We start with some crucial building blocks.

EXAMPLE 2.1 There is a B-wheel system of $K_{8,8,8}$.

A 4-wheel system of $K_{8,8,8}$ is given in [2] by $(\mathbb{Z}_{24}, W)$ where $W = \{i\text{–}(1+i, 5+i, 22+i, 14+i) \mid 0 \le i \le 23\}$, and where the vertex partition is

$$\{3i \mid 0 \le i \le 7\}, \{3i+1 \mid 0 \le i \le 7\}, \{3i+2 \mid 0 \le i \le 7\}.$$

Remove the edges at difference 4 and difference 8 from each 4-wheel, leaving the bowties $\{(i, i+1, i+14; i, i+5, i+22) \mid 0 \le i \le 23\}$. We may then use the removed edges to make a further eight bowties, thus yielding a bowtie system with 32 bowties altogether:

$$\{(i, i+4, i+8; i, i+16, i+20), (i+12, i+8, i+16; i+12, i+4, i+20) \mid 0 \le i \le 3\}.$$

□

EXAMPLE 2.2 A B-wheel system of K_{33} is given by (V, W) with metamorphosis into the bowtie system (V, B), where $V = \{i_j \mid 0 \le i \le 10,\ 1 \le j \le 3\}$ and

$$\begin{aligned} W = \{ & i_3\text{–}((i+1)_1, (i+3)_3, (i+7)_2, (i+2)_3),\ i_3\text{–}(i_1, (i+3)_2, (i+2)_1, (i+6)_3), \\ & i_2\text{–}((i+2)_1, (i+10)_3, (i+9)_3, (i+5)_3),\ i_2\text{–}((i+5)_1, (i+4)_2, (i+9)_2, (i+1)_2), \\ & i_1\text{–}((i+3)_1, (i+4)_1, (i+9)_1, i_2),\ i_1\text{–}((i+4)_2, (i+5)_3, (i+5)_2, (i+7)_3 \mid \\ & 0 \le i \le 10\}, \end{aligned}$$

addition modulo 11, with subscripts fixed, and

$$B = \{(i_3, (i+1)_1, (i+3)_3; i_3, (i+2)_3, (i+7)_2),$$
$$(i_3, i_2, (i+3)_2; i_3, (i+2)_1, (i+6)_3),$$
$$(i_2, (i+2)_1, (i+5)_3; i_2, (i+9)_3, (i+19)_3),$$
$$(i_2, (i+1)_2, (i+5)_1; i_2, (i+4)_2, (i+9)_2),$$
$$(i_1, (i+3)_1, i_2; i_1, (i+4)_1, (i+9)_1),$$
$$(i_1, (i+4)_2, (i+5)_3; i_1, (i+5)_2, (i+7)_3) \mid 0 \le i \le 10\} \cup$$
$$\{(i_1, (i+1)_1, (i+2)_2; i_1, (i+6)_3, (i+10)_2),$$
$$(i_3, (i+7)_3, (i+10)_1; i_3, i_2, (i+8)_2) \mid 0 \le i \le 10\}.$$

□

EXAMPLE 2.3 A B-wheel system of K_{49} is given by $(\mathbb{Z}_{49}, W)$ with metamorphosis into the bowtie system $(\mathbb{Z}_{49}, B)$, where

$$W = \{i\text{–}(i+7, i+32, i+34, i+38),\ i\text{–}(i+5, i+6, i+26, i+35),$$
$$i\text{–}(i+10, i+13, i+21, i+37) \mid 0 \le i \le 48\},$$

and

$$B = \{(i, i+7, i+32; i, i+34, i+38),\ (i, i+5, i+6; i, i+26, i+35),$$
$$(i, i+10, i+37; i, i+13, i+21) \mid 0 \le i \le 48\} \cup$$
$$\{(i, i+2, i+20; i, i+3, i+19) \mid 0 \le i \le 48\}.$$

□

We now use the 3-GDD construction. For order $48t+1$, take $\ell = 8$, $h = 1$, $s = 6t$, a 3-GDD of type 6^t for $t \ge 3$ ([8]), and B-wheel systems of K_{49} and $K_{8,8,8}$. (When $t = 2$, see the Appendix for the isolated case K_{97}.)

For order $48t + 33$, take $\ell = 8$, $h = 1$, $s = 6t + 4$, a 3-GDD of type $6^t 4^1$ for $t \ge 3$, and B-wheel systems of K_{33}, K_{49} and $K_{8,8,8}$. (When $t = 1$ or 2, see the Appendix for the isolated cases of orders 81 and 129.)

Thus we have

THEOREM 2.4 *There exist 4-wheel systems of orders $48t+1$ and $48t+33$, which each have a metamorphosis into a bowtie system of the same order, for all $t \ge 0$.*

2.2 $\boxed{\lambda = 3}$

We need one further example in this case.

EXAMPLE 2.5 A B-wheel system of $3K_{17}$ is given by $(\mathbb{Z}_{17}, W)$ with metamorphosis into the bowtie system $(\mathbb{Z}_{17}, B)$, where

$$W = \{i\text{–}(1+i, 11+i, 2+i, 5+i),\ i\text{–}(1+i, 5+i, 9+i, 3+i),$$
$$i\text{–}(7+i, 14+i, 6+i, 5+i) \mid 0 \le i \le 16\}$$

and

$$B = \{(i, 1+i, 5+i; i, 2+i, 11+i),\ (i, 1+i, 3+i; 1, 5+i, 9+i),$$
$$(i, 5+i, 7+i; i, 6+i, 14+i) \mid 0 \le i \le 16\} \ \cup$$
$$\{(i, 3+i, 7+i; i, 10+i, 11+i) \mid 0 \le i \le 16\}.$$

□

THEOREM 2.6 *There exists a 3-fold 4-wheel system of order $16t+1$ which has a metamorphosis into a 3-fold bowtie system of that order, for all $t \ge 1$.*

Proof We use the 3-GDD construction with $h = 1$, $\ell = 8$, $s = 2t$, 3-fold B-wheel systems of order 33 (three copies of Example 2.2) and order 17 (Example 2.5), and three copies of a B-wheel system of $K_{8,8,8}$, together with a 3-GDD of type 2^t, $t \ge 3$, if $t \equiv 0$ or 1 (mod 3), or type $4^1 2^{t-2}$, $t \ge 5$, if $t \equiv 2$ (mod 3). This deals with all orders $16t+1$. □

3 The cases $\lambda = 2$, 4, 6, 8, 12 and 24

3.1 $\boxed{\lambda = 2}$

We start with some necessary examples.

EXAMPLE 3.1 A B-wheel system of $2K_9$ is given by $(\mathbb{Z}_9, W)$ where $W = \{i\text{–}(1+i, 3+i, 2+i, 6+i) \mid 0 \le i \le 8\}$. This has a metamorphosis into the 2-fold bowtie system $(\mathbb{Z}_9, B)$ as follows. (Here we take edges with differences 2, 4 from six of the nine 4-wheels, and edges with differences 1, 4 from three of the nine 4-wheels.)

$$B = \{(0,1,6;0,2,3),\ (1,2,4;1,3,7),\ (2,3,8;2,4,5),$$
$$(3,4,0;3,5,6),\ (4,5,7;4,6,1),\ (5,2,6;5,7,8),$$
$$(6,3,7;6,0,8),\ (7,1,8;7,0,4),\ (8,0,5;8,1,2)\} \ \cup$$
$$\{(0,1,5;0,2,7),\ (3,4,8;3,1,5),\ (6,7,2;6,4,8)\}.$$ □

EXAMPLE 3.2 A B-wheel system of $2K_{16}$ is given by (X, W) where the vertex set X is $\{\infty\} \cup \{(i,j) \mid 0 \le i \le 4,\ 1 \le j \le 3\}$, and the wheels W are:

$\{(i,1)\text{–}((i,3),(4+i,3),(3+i,2),(2+i,2)),\ (i,2)\text{–}((i,3),(2+i,3),(3+i,2),(4+i,1)),$
$(i,1)\text{–}((i,3),(3+i,1),(2+i,1),(3+i,3)),\ (i,2)\text{–}((i,1),(3+i,2),(i,3),(4+i,3)),$
$(i,1)\text{–}((4+i,2),(i,2),(3+i,3),\infty),\ (i,2)\text{–}((4+i,1),(1+i,3),\infty,(3+i,1)) \mid$
$0 \le i \le 4\};$

(here ∞ is fixed, the second entries of pairs are fixed, and addition is modulo 5 in the first entry of each pair).

Then a metamorphosis into a bowtie system is given by the following bowties:

$$\{((i,1),(i,3),(4+i,3);(i,1),(2+i,2),(3+i,2)),$$
$$((i,2),(2+i,3),(3+i,2);(i,2),(i,3),(4+i,1)),$$
$$((i,1),(i,3),(3+i,3);(i,1),(2+i,1),(3+i,1)),$$
$$((i,2),(i,1),(4+i,3);(i,2),(i,3),(3+i,2)),$$
$$((i,1),(i,2),(3+i,3);(i,1),(4+i,2),\infty),$$
$$((i,2),(3+i,1),(4+i,1);(i,2),(1+i,3),\infty)\} \cup$$
$$\{((i,1),(1+i,3),(2+i,3);(i,1),(3+i,2),(4+i,2)),$$
$$((i,3),(2+i,3),(4+i,2);(i,3),(3+i,1),\infty)\}.$$

□

EXAMPLE 3.3 A 4-wheel system of $2K_{4,4,4}$ with vertex set partitioned $\{\{A,B,C,D\},\{1,2,3,4\},\{5,6,7,8\}\}$ is given by

$$\begin{array}{lll} A\text{–}(1,5,2,6), & 1\text{–}(C,6,D,8), & 5\text{–}(B,1,D,4), \\ B\text{–}(1,7,2,8), & 2\text{–}(C,5,D,7), & 6\text{–}(B,2,D,3), \\ C\text{–}(3,5,4,6), & 3\text{–}(A,5,B,8), & 7\text{–}(A,1,C,3), \\ D\text{–}(3,7,4,8), & 4\text{–}(A,6,B,7), & 8\text{–}(A,2,C,4). \end{array}$$

This has a metamorphosis into bowties as follows:

$$\begin{array}{llll} (A,1,6;A,2,5), & (B,1,8;B,2,7), & (C,3,6;C,4,5), & (D,3,8;D,4,7), \\ (1,8,C;1,6,D), & (2,7,C;2,5,D), & (3,8,A;3,5,B), & (4,7,A;4,6,B), \\ (5,4,B;5,1,D), & (6,3,B;6,2,D), & (7,3,A;7,1,C), & (8,4,A;8,2,C), \\ (5,1,A;5,3,C), & (6,2,A;6,4,C), & (D,4,8;D,3,7), & (B,2,8;B,1,7). \end{array}$$

□

EXAMPLE 3.4 A B-wheel system of $2K_{40}$ is as follows. Take the vertex set $\{\infty\} \cup \{i_j \mid 0 \le i \le 12,\ 1 \le j \le 3\}$. Then 4-wheels may be taken as follows, where the subscripts are fixed. and the addition is modulo 13:

$$\{\ i_1\text{–}(i_2,(i+4)_3,(i+2)_3,\infty),$$
$$i_1\text{–}((i+4)_2,(i+7)_2,(i+11)_3,\infty),$$
$$(i+2)_3\text{–}((i+2)_1,(i+9)_3,(i+12)_1,(i+12)_3),$$
$$(i+2)_1\text{–}(i+7)_2,(i+7)_3,(i+1)_3,(i+10)_3),$$
$$(i+4)_3\text{–}((i+8)_2,(i_5)_3,(i+12)_2,(i+9)_3),$$
$$(i+2)_1\text{–}((i+9)_2;(i+5)_3,(i+3)_3,(i+7)_3),$$
$$(i+9)_1\text{–}((i+1)_2,(i+5)_2,(i+7)_2,(i+6)_2),$$
$$(i+8)_1\text{–}((i+3)_1,(i+9)_2,(i+3)_2,(i+11)_2),$$
$$(i+2)_1\text{–}((i+3)_2,(i+5)_2,(i+3)_3,(i+8)_3),$$
$$(i+6)_3\text{–}((i+4)_1,i_2,(i+3)_3,(i+4)_2),$$
$$i_1\text{–}((i+1)_1,(i+2)_1,(i+4)_1,(i+8)_1),$$
$$i_1\text{–}((i+3)_1,(i+10)_1,(i+8)_3,(i+9)_3),$$
$$i_1\text{–}((i+2)_2,(i+6)_2,(i+7)_3,(i+12)_2),$$
$$i_1\text{–}((i+2)_2,(i+4)_3,(i+10)_2,(i+9)_3),$$
$$i_1\text{–}((i+4)_2,(i+11)_2,(i+12)_2,(i+12)_3)\ \mid\ 0 \le i \le 12\}$$

These 4-wheels have a metamorphosis into the following bowties:

$$\begin{array}{l}
\{(i_1, i_2, \infty; i_1, (i+2)_3, (i+4)_3),\\
(i_1, (i+4)_2, \infty; i_1, (i+7)_2, (i+11)_3),\\
((i+2)_3, (i+2)_1, (i+9)_3; (i+2)_3, (i+12)_1, (i+12)_3),\\
((i+2)_1, (i+7)_2, (i+10)_3; (i+2)_1, (i+7)_3, (i+1)_3),\\
((i+4)_3, (i+8)_2, (i+9)_3; (i+4)_3, (i+5)_3, (i+12)_2),\\
((i+2)_1, (i+9)_2, (i+5)_3; (i+2)_1, (i+3)_3, (i+7)_3),\\
((i+9)_1, (i+1)_2, (i+5)_2; (i+9)_1, (i+6)_2, (i+7)_2),\\
((i+8)_1, (i+3)_1, (i+9)_2; (i+8)_1, (i+3)_2, (i+11)_2),\\
((i+2)_1, (i+3)_2, (i+8)_3; (i+2)_1, (i+5)_2, (i+3)_3),\\
((i+6)_3, (i+4)_1, i_2; (i+6)_3, (i+3)_3, (i+4)_2),\\
(i_1, (i+1)_1, (i+8)_1; i_1, (i+2)_1, (i+4)_1),\\
(i_1, (i+3)_1, (i+10)_1; i_1, (i+8)_3, (i+9)_3),\\
(i_1, (i+2)_2, (i+12)_2; i_1, (i+6)_2, (i+7)_3),\\
(i_1, (i+2)_2, (i+9)_3; i_1, (i+4)_3, (i+10)_2),\\
(i_1, (i+4)_2, (i+12)_3; i_1, (i+11)_2, (i+12)_2)\}\\
\quad \cup \quad \{(i_1, (i+1)_1, (i+11)_3; i_1, (i+4)_1, (i+10)_3),\\
\qquad (i_2, i_1, (i+8)_2; i_2, (i+10)_2, (i+10)_3),\\
\qquad (i_2, (i+2)_2, (i+6)_2; i_2, (i+3)_3, (i+11)_3),\\
\qquad (i_2, i_3, (i+4)_3; i_2, (i+2)_2, (i+12)_3),\\
\qquad (i_3, (i+5)_2, (i+11)_2; i_3, (i+11)_3, \infty)\}.
\end{array}$$

□

We now have the ingredients to prove the main existence result for $\lambda = 2$.

THEOREM 3.5 *There exist 2-fold 4-wheel systems for all orders congruent to 0, 1, 9 or 16 (mod 24) which have a metamorphosis into a 2-fold bowtie system of the same order, except possibly for orders 24, 72, 88.*

Proof We deal with orders 1 or 9 (mod 24) first. For order $24t+1$ we use the 3-GDD construction with $h=1$, $\ell=4$, $s=6t$; then B-wheel systems of $2K_{4,4,4}$ (Example 3.3), $2K_9$ (Example 3.1), and a 3-GDD of type 2^{3t} (which exists for all $t \geq 1$) complete the construction.

For order $24t+9$ we use the 3-GDD construction with $h=1$, $\ell=4$, $s=6t+2$; then B-wheel systems of $2K_{4,4,4}$ (Example 3.3), $2K_9$ (Example 3.1), and a 3-GDD of type 2^{3t+1} (which exists for all $t \geq 1$) complete the construction.

Cases of orders $24t$ and $24t+16$ remain. We write these as $48t$, $48t+24$, $48t+16$ and $48t+40$.

For order $48t$, we use the 3-GDD construction with $h=0$, $\ell=4$ and $s=12t$. We have B-wheel systems of $2K_{4,4,4}$ (Example 3.3), $2K_{16}$ (Example 3.2), and a 3-GDD of type 4^{3t} for all $t \geq 1$.

For order $48t+16$, the 3-GDD construction with $h=0$, $\ell=4$ and $s=12t+4$ together with B-wheel systems of $2K_{4,4,4}$ (Example 3.3), $2K_{16}$ (Example 3.2) and a 3-GDD of type 4^{3t+1} (which exists for all $t \geq 1$) suffice.

For order $48t+24$, we use the 3-GDD construction with $h=0$, $\ell=4$, $s=12t+6$, a 3-GDD of type $10^1 4^{3t-1}$ (which exists for all $t \geq 2$), and 2-fold B-wheel systems of orders 40 (Example 3.4) and 16 (Example 3.2). Existence of B-wheel systems of orders 24 and 72 (when $t=0$ and 1 respectively) remains open at this stage.

For order $48t+40$, we repeat the above for order $48t+24$, but with $s=12t+10$ and a 3-GDD of type $10^1 4^{3t}$ (which exists for all $t \geq 2$). In the case $t=0$, a B-wheel system of $2K_{40}$ exists (Example 3.4), while in the case $t=1$, existence of a B-wheel system of $2K_{88}$ remains open at this stage.

This completes the theorem. □

3.2 $\boxed{\lambda = 4}$

Once again we start with some necessary examples.

EXAMPLE 3.6 There is a B-wheel system (V, W) of $4K_{2,2,2}$ with a metamorphosis into a 4-fold bowtie system, (V, B). Let $V = \{\{1,2\},\{3,4\},\{5,6\}\}$, and take

$$W = \{1\text{–}(3,5,4,6),\ 2\text{–}(3,5,4,6),\ 3\text{–}(1,5,2,6),\ 4\text{–}(1,5,2,6),\ 5\text{–}(1,3,2,4),\ 6\text{–}(1,3,2,4)\}.$$

Then

$$B = \{(1,3,6;1,4,5),(2,3,5;2,4,6),(3,1,6;3,2,5),(4,1,5;4,2,6),\\ (5,1,4;5,2,3),(6,1,4;6,2,3)\} \cup \{(5,1,3;5,2,4),(6,1,3;6,2,4)\}.$$

□

EXAMPLE 3.7 There is a B-wheel system (V, W) of $4K_{12}$ with a metamorphosis into a 4-fold bowtie system, (V, B). Let $V = \{\infty\} \cup \mathbb{Z}_{11}$, and

$$W = \{\infty\text{–}(4+i,9+i,i,5+i),\ i\text{–}(1+i,4+i,6+i,8+i),\ i\text{–}(1+i,2+i,7+i,4+i) \mid 0 \leq i \leq 10\}$$

where addition is modulo 11. Then a metamorphosis into (V, B) is given by

$$B = \{(\infty,4+i,5+i;\infty,i,9+i),\ (i,1+i,4+i;i,6+i,8+i),\ (i,1+i,4+i;i,2+i,7+i) \mid \\ 0 \leq i \leq 10\} \cup \{(i,3+i,9+i;i,6+i,10+i) \mid 0 \leq i \leq 10\}.$$

□

EXAMPLE 3.8 There is a B-wheel system (V, W) of $4K_{13}$ with a metamorphosis into a 4-fold bowtie system, (V, B), where $V = \mathbb{Z}_{13}$,

$$W = \{i\text{–}(i+1,i+2,i+3,i+4), i\text{–}(i+2,i+5,i+9,i+7), i\text{–}(i+2,i+8,i+3,i+9) \mid 0 \leq i \leq 12\},$$

and $B = \{(i,i+1,i+4;i,i+2,i+3),\ (i,i+2,i+7;i,i+5,i+9),\ (i,i+2,i+9;i,i+8,i+3) \mid 0 \leq i \leq 12\} \cup \{(i,i+2,i+3;i,i+1,i+7) \mid 0 \leq i \leq 12\}$.

□

We now have the ingredients to construct 4-fold B-wheel systems in all cases.

THEOREM 3.9 *There exist 4-fold 4-wheel systems of all orders congruent to 0, 1, 4 or 9 (modulo 12), which have a metamorphosis into a 4-fold bowtie system of the same order.*

Proof For orders 0 or 1 (mod 12), let the order be $12t$ or $12t+1$. We use the 3-GDD construction with $h = 0$ or 1 (respectively), $\ell = 2$ and $s = 6t$. This uses a 3-GDD of type 6^t (which exists for all $t \geq 3$), a B-wheel system of $4K_{2,2,2}$ (Example 3.6), and 4-fold B-wheel systems of orders 12 or 13 (respectively). (See Examples 3.7 and 3.8.) When $t = 2$ we have the isolated cases of orders 24 or 25; 4-fold B-wheel systems of these orders are given in the Appendix.

Now let the order be $12t+4$. We use the 3-GDD construction with $h = 0$, $\ell = 2$ and $s = 6t+2$, together with a 3-GDD of type $8^1 6^{t-1}$ which exists for all $t \geq 4$. This also uses B-wheel systems of $4K_{2,2,2}$ (Example 3.6), $4K_{16}$ (two copies of Example 3.2) and $4K_{12}$ (Example 3.7). The isolated cases which remain are $4K_{28}$ (see the Appendix) and $4K_{40}$ (take two copies of Example 3.4).

Finally, consider the order $12t+9$. We use the 3-GDD construction with $h = 1$, $\ell = 2$, $s = 6t+4$, a 3-GDD of type $4^1 6^t$ (which exists for all $t \geq 3$), and B-wheel systems given in Example 3.1 (take two copies to obtain $\lambda = 4$), and in Examples 3.6 and 3.8. In the case $t = 1$, see the Appendix for $4K_{21}$; when $t = 2$, for $4K_{33}$ take four copies of Example 2.2.

This completes the case $\lambda = 4$. □

3.3 $\boxed{\lambda = 6}$

We need one extra example in this case.

EXAMPLE 3.9 There is a B-wheel system (V, W) of $6K_8$.
Let $V = \{\infty\} \cup \mathbb{Z}_7$, and take
$W = \{i\text{–}(\infty, i+1, i+3, i+2),\ i\text{–}(\infty, i+3, i+6, i+4),\ i\text{–}(i+1, i+2, i+4, i+3) \mid 0 \leq i \leq 6\}$. Then

$$\begin{aligned} B = \{&(i, \infty, i+2; i, i+1, i+3),\ (i, \infty, i+4; i, i+3, i+6),\\ &(i, i+1, i+3; i, i+2, i+4) \mid 0 \leq i \leq 6\}\\ &\cup\ \{(i, \infty, i+6; i, i+1, i+2) \mid 0 \leq i \leq 6\}. \end{aligned}$$

□

THEOREM 3.10 *There exist 6-fold 4-wheel systems for all orders 0 or 1 (mod 8) which have a metamorphosis into a 6-fold bowtie system of the same order.*

Proof We use the 3-GDD construction for order $8t + \epsilon$ where ϵ is 0 or 1. Take $h = \epsilon$, $\ell = 4$ and $s = 2t$. Then we use:
- three copies of a B-wheel system of $2K_{4,4,4}$ (see Example 3.3);
- a 3-GDD of type 2^t (if $t \equiv 0$ or 1 (mod 3), and $t \geq 3$) or of type $4^1 2^{t-2}$ (if $t \equiv 2$ (mod 3), and $t \geq 5$);

• a B-wheel system of $6K_8$ or $6K_9$ (according as $\epsilon = 0$ or 1; see Example 3.9 or three copies of Example 3.1);
• a B-wheel system of $6K_{16}$ or $6K_{17}$ (according as $\epsilon = 0$ or 1; see three copies of Example 3.2 or two copies of Example 2.5).
This completes all cases. □

3.4 $\boxed{\lambda = 8}$

Here the order is 0 or 1 (mod 3). When the order is 0, 1, 4 or 9 (mod 12), we may take two copies of a 4-fold B-wheel system, found above. So we only need consider the orders congruent to 3, 6, 7 or 10 (mod 12).

EXAMPLE 3.11 There is a B-wheel system (V, W) of $8K_6$.
Let $V = \{\infty\} \cup \mathbb{Z}_5$ and take $W =$

$\{\infty\text{–}(i, i{+}2, i{+}4, i{+}3),\ \infty\text{–}(i, i{+}1, i{+}4, i{+}3),\ i\text{–}(i{+}1, i{+}2, i{+}3, i{+}4) \mid 0 \le i \le 4\}$.

Then a metamorphosis into bowties is given by

$$B = \{(\infty, i, i{+}2; \infty, i{+}3, i{+}4),\ (\infty, i, i{+}3; \infty, i{+}1, i{+}4),\ (i, i{+}1, i{+}4; i, i{+}2, i{+}3) \mid 0 \le i \le 4\} \cup \{(i, i+1, i+2; i, i+3, i+4) \mid 0 \le i \le 4\}.$$

□

EXAMPLE 3.12 There is a B-wheel system $(\mathbb{Z}_7, W)$ of $8K_7$.
Take $W = \{i\text{–}(i{+}1, i{+}2, i{+}4, i{+}6),\ i\text{–}(i{+}2, i{+}4, i{+}3, i{+}6),\ i\text{–}(i{+}2, i{+}3, i{+}4, i{+}5) \mid 0 \le i \le 6\}$. Then a metamorphosis is given by

$$B = \{(i, i+2, i+4; i, i+1, i+6),\ (i, i+2, i+6; i, i+3, i+4), (i, i+2, i+3; i, i+4, i+5) \mid 0 \le i \le 6\} \cup \{(i, i+2, i+6; i, i+1, i+5) \mid 0 \le i \le 6\}.$$

□

EXAMPLE 3.13 There is a B-wheel system (V, W) of $8K_{10}$.
Take $V = \{i_1, i_2 \mid 0 \le i \le 4\}$ and take

$$W = \{i_2\text{–}(i_1, (i+1)_2, (i+3)_2, (i+2)_2),\ i_2\text{–}((i+1)_2, (i+3)_2, (i+2)_2, (i+4)_2), i_1\text{–}((i+4)_1, (i+1)_1, (i+3)_2, (i+4)_2)\ i_2\text{–}((i+4)_1, (i+4)_2, i_1, (i+1)_2), i_1\text{–}((i+1)_1, (i+2)_1, (i+3)_1, i_2),\ i_1\text{–}((i+1)_1, i_2, (i+2)_1, (i+1)_2), i_1\text{–}((i+1)_1, (i+3)_2, (i+2)_1, (i+4)_2),\ i_1\text{–}((i+1)_1, (i+3)_2, (i+2)_1, (i+4)_2), i_1\text{–}((i+2)_1, (i+1)_2, (i+3)_1, (i+3)_2) \mid 0 \le i \le 4\}.$$

Then a metamorphosis into an 8-fold bowtie system is given by (V, B) where

$$B = \{(i_2, i_1, (i+2)_2; i_2, (i+1)_2, (i+3)_2),\ (i_2, (i+1)_2, (i+4)_2; i_2, (i+3)_2, (i+2)_2), (i_1, (i+4)_1, (i+4)_2; i_1, (i+1)_1, (i+3)_2),\ (i_2, (i+4)_1, (i+1)_2; i_2, (i+4)_2, i_1), (i_1, (i+1)_1, i_2; i_1, (i+2)_1, (i+3)_1),\ (i_1, (i+1)_1, (i+1)_2; i_1, i_2, (i+2)_1), (i_1, (i+1)_1, (i+3)_2; i_1, (i+2)_1, (i+4)_2),\ (i_1, (i+1)_1, (i+4)_2; i_1, (i+3)_2, (i+2)_1), (i_1, (i+2)_1, (i+3)_2; i_1, (i+1)_2, (i+3)_1) \mid 0 \le i \le 4\}.$$

□

EXAMPLE 3.14 There is a B-wheel system $(\mathbb{Z}_{15}, W)$ of $8K_{15}$.

$$\begin{aligned} W = \{&i\text{–}(i+1, i+2, i+3, i+4),\ i\text{–}(i+1, i+2, i+3, i+4),\\ &i\text{–}(i+2, i+4, i+6, i+8),\ i\text{–}(i+2, i+4, i+10, i+7),\\ &i\text{–}(i+3, i+9, i+4, i+10),\ i\text{–}(i+3, i+10, i+4, i+11),\\ &i\text{–}(i+3, i+10, i+5, i+11) \mid 0 \le i \le 14\}. \end{aligned}$$

This has a metamorphosis into an 8-fold bowtie system as follows:

$$\begin{aligned} B = \{&(i, i+1, i+2; i, i+3, i+4),\ (i, i+1, i+2; i, i+3, i+4),\\ &(i, i+2, i+4; i, i+6, i+8),\ (i, i+2, i+4; i, i+10, i+7),\\ &(i, i+3, i+9;; i, i+4, i+10),\ (i, i+3, i+10; i, i+4, i+11),\\ &(i, i+3, i+10; i, i+5, i+11) \mid 0 \le i \le 14\}\\ &\quad \cup\{(i, i+1, i+7; i, i+3, i+6) \mid 0 \le i \le 14\}\\ &\quad \cup\{(i, i+5, i+10; i, i+1, i+7),\ (i, i+5, i+10; i, i+2, i+7) \mid 0 \le i \le 4\}\\ &\quad \cup\{(2i+6, 2i+5, 2i+12; 2i+6, 2i+8, 2i+13),\\ &\qquad (2i+7, 2i+6, 2i+13; 2i+7, 2i+5, 2i+12) \mid 0 \le i \le 4\}. \end{aligned}$$

□

THEOREM 3.15 *There exist 8-fold 4-wheel systems for all orders 0 or 1 (mod 3) which have a metamorphosis into an 8-fold bowtie system of the same order.*

Proof For orders congruent to 0, 1, 4 or 9 (mod 12) we may double a 4-fold system, which exists by Theorem 3.9 above. So we only need consider 3, 6, 7 or 10 (mod 12).

For orders 6 or 7 (mod 12):
We use the 3-GDD construction with $h = 0$ or 1, $\ell = 2$, $s = 6t+3$, and use a 3-GDD of type 3^{2t+1} which exists for all t (for instance, take a Kirkman triple system of order $6t+3$!). Then use two copies of B-wheel systems of $4K_{2,2,2}$ (Example 3.6), and $8K_6$ or $8K_7$ (Examples 3.11, 3.12).

For order 3 (mod 12), let the order be $12t + 3$ and use the 3-GDD construction with $h = 1$, $\ell = 2$ and a 3-GDD of type $7^1 3^{2t-2}$, which exists for all $t \ge 3$. With B-wheel systems of $8K_{2,2,2}$ (two copies of Example 3.6), $8K_{15}$ (Example 3.14) and $8K_7$ (Example 3.12), this completes the construction for this order except for $t = 2$; the isolated case $8K_{27}$ is given in the Appendix.

For order 10 (mod 12), let the order be $12t+10$. We use the 3-GDD construction with $h = 0$, $\ell = 2$ and a 3-GDD of type $5^1 3^{2t}$, which exists for all $t \ge 2$. Then B-wheel systems of $8K_{10}$ (Example 3.13), $8K_6$ (Example 3.11) and $8K_{2,2,2}$ (two copies of Example 3.6) are used. When $t = 1$ the isolated case $8K_{22}$ is needed; this is in the Appendix.

This completes the $\lambda = 8$ case. □

3.5 $\boxed{\lambda = 12}$

Here the expected orders are 0 or 1 (mod 4). For orders 0 or 1 (mod 8) we may simply double a 6-fold B-wheel design; see Theorem 3.10. So we concentrate on orders 4 or 5 (mod 8).

EXAMPLE 3.16 There is a B-wheel system of $12K_5$.
Take vertex set $\mathbb{Z}_5$, and the fifteen wheels got from taking three copies of each of $\{i\text{–}(i+1, i+2, i+4, i+3) \mid 0 \leq i \leq 4\}$. This has a metamorphosis into bowties as follows: Take three copies of

$$\{(i, i+1, i+3; i, i+2, i+4) \mid 0 \leq i \leq 4\}$$

together with the bowties

$$\{(i, i+1, i+2; i, i+3, i+4) \mid 0 \leq i \leq 4\}.$$

□

THEOREM 3.17 *There exist* 12*-fold* 4*-wheel systems of all orders* 0 *or* 1 *(mod* 4*), which have a metamorphosis into a* 12*-fold bowtie system of the same order.*

Proof As remarked above, we only need consider orders 4 and 5 (mod 8).

First suppose the order is $8t+5$. We use the 3-GDD construction with $h = 1$, $\ell = 2$, $s = 4t+2$, and a 3-GDD of type 2^{2t+1} or $4^1 2^{2t-1}$, according as $2t+1$ is 0 or 1 (mod 3), or is 2 (mod 3). (These exist for $t \geq 1$, $t \geq 2$ respectively.) This requires B-wheel systems of $12K_{2,2,2}$ (take three copies of Example 3.6), $12K_5$ (Example 3.16) and $12K_9$ (six copies of Example 3.1). There are no missing cases.

Next suppose the order is $8t+4$. We split this further into two cases. First suppose that $t \equiv 1$ or 2 (mod 3). Then we use the 3-GDD construction with $h = 0$, $\ell = 2$, $s = 4t+2$ and a 3-GDD of type $6^1 4^{t-1}$, which exists for $t \equiv 1$ or 2 (mod 3), $t \geq 4$. This then requires $12K_{2,2,2}$ (three copies of Example 3.6), $12K_{12}$ (three copies of Example 3.7) and $12K_8$ (two copies of Example 3.9). When $t = 2$, the isolated case $12K_{20}$ arises; see the Appendix for this.

Now consider order $8t+4$ where $t \equiv 0$ (mod 3). So let $t = 3T$ and consider the order $24T+4$. We use the 3-GDD construction with $h = 0$, $\ell = 2$, $s = 12T+2$, a 3-GDD of type $8^1 6^{2T-1}$ (which exists for all $T \geq 2$), and B-wheel systems of $12K_{2,2,2}$ (three copies of Example 3.6), $12K_{16}$ (six copies of Example 3.2), and $12K_{12}$ (three copies of Example 3.7). When $T = 1$ we have the isolated case $12K_{28}$; take three copies of $4K_{28}$, which is given in the Appendix.

This completes the case $\lambda = 12$. □

3.6 $\boxed{\lambda = 24}$

Here the order can be any value at least 5 (so that we have enough vertices to form a 4-wheel!). By doubling a 12-fold B-wheel system we only need consider orders 2 or

3 (mod 4). Moreover, by trebling 8-fold B-wheel systems, we also only need consider orders 2 (mod 3). Thus orders 2 and 11 (mod 12) are the only ones we need to concern ourselves with here.

As usual we start with some necessary small examples.

EXAMPLE 3.18 There is a B-wheel system $(\mathbb{Z}_{11}, W)$ of $24K_{11}$, where W is:

$$\begin{aligned} W = \{& i\text{–}(i+7, i+8, i+9, i+1),\ i\text{–}(i+3, i+6, i+5, i+8),\ i\text{–}(i+10, i+3, i+1, i+5), \\ & i\text{–}(i+2, i+7, i+3, i+8),\ i\text{–}(i+9, i+1, i+10, i+6),\ i\text{–}(i+1, i+2, i+3, i+4), \\ & i\text{–}(i+1, i+2, i+3, i+4),\ i\text{–}(i+1, i+2, i+3, i+4),\ i\text{–}(i+1, i+2, i+3, i+4), \\ & i\text{–}(i+2, i+4, i+6, i+8),\ i\text{–}(i+2, i+6, i+4, i+9),\ i\text{–}(i+2, i+6, i+4, i+9), \\ & i\text{–}(i+2, i+6, i+4, i+9),\ i\text{–}(i+2, i+7, i+3, i+8),\ i\text{–}(i+2, i+7, i+3, i+8) \\ & \mid 0 \le i \le 10\}. \end{aligned}$$

This has a metamorphosis into a 24-fold bowtie system $(\mathbb{Z}_{11}, B)$, where B is as follows.

$$\begin{aligned} B = \{& (i, i+7, i+1; i, i+8, i+9),\ (i, i+3, i+8; i, i+6, i+5), \\ & (i, i+10, i+5; i, i+3, i+1),\ (i, i+2, i+8; i, i+7, i+3), \\ & (i, i+9, i+6; i, i+1, i+10),\ (i, i+1, i+4; i, i+2, i+3), \\ & (i, i+1, i+4; i, i+2, i+3),\ (i, i+1, i+4; i, i+2, i+3), \\ & (i, i+1, i+4; i, i+2, i+3),\ (i, i+2, i+8; i, i+4, i+6), \\ & (i, i+2, i+9; i, i+4, i+6),\ (i, i+2, i+9; i, i+4, i+6), \\ & (i, i+2, i+9; i, i+4, i+6),\ (i, i+2, i+8; i, i+3, i+7), \\ & (i, i+2, i+8; i, i+3, i+7) \mid 0 \le i \le 10\} \\ & \cup \{(i, i+1, i+6; i, i+4, i+5),\ (i, i+1, i+6; i, i+4, i+5), \\ & \quad (i, i+1, i+5; i, i+2, i+3),\ (i, i+1, i+4; i, i+2, i+5), \\ & \quad (i, i+1, i+5; i, i+3, i+4) \mid 0 \le i \le 10\}. \end{aligned}$$

□

EXAMPLE 3.19 There is a B-wheel system (V, W) of $24K_{14}$, where $V = \{\infty\} \cup \mathbb{Z}_{13}$ and W is:

$$\begin{aligned} W = \{& \infty\text{–}(i, i+3, i+8, i+5),\ \infty\text{–}(i, i+3, i+10, i+5), \\ & \infty\text{–}(i, i+4, i+9, i+5),\ \infty\text{–}(i, i+4, i+9, i+5), \\ & \infty\text{–}(i, i+4, i+9, i+5),\ \infty\text{–}(i, i+4, i+10, i+5), \\ & i\text{–}(i+4, i+6, i+5, i+7),\ i\text{–}(i+8, i+3, i+10, i+4), \\ & i\text{–}(i+12, i+5, i+6, i+7),\ i\text{–}(i+6, i+7, i+9, i+8), \\ & i\text{–}(i+3, i+4, i+6, i+5),\ i\text{–}(i+9, i+10, i+11, i+12), \\ & i\text{–}(i+9, i+10, i+11, i+12),\ i\text{–}(i+9, i+10, i+11, i+12), \\ & i\text{–}(i+9, i+10, i+11, i+12),\ i\text{–}(i+5, i+7, i+9, i+11), \\ & i\text{–}(i+5, i+7, i+9, i+11),\ i\text{–}(i+5, i+7, i+9, i+11), \\ & i\text{–}(i+3, i+5, i+7, i+10),\ i\text{–}(i+3, i+6, i+10, i+7), \\ & i\text{–}(i+6, i+9, i+4, i+10) \mid 0 \le i \le 12\}. \end{aligned}$$

This has a metamorphosis into a 24-fold bowtie system (V, B), where B is as follows.

$$\begin{aligned}B = \{&(\infty, i, i+5; \infty, i+3, i+8),\ (\infty, i, i+5; \infty, i+3, i+10),\\ &(\infty, i, i+5; \infty, i+4, i+9),\ (\infty, i, i+5; \infty, i+4, i+9),\\ &(\infty, i, i+5; \infty, i+4, i+9),\ (\infty, i, i+5; \infty, i+4, i+10),\\ &(i, i+4, i+7; i, i+6, i+5),\ (i, i+8, i+4; i, i+3, i+10),\\ &(i, i+12, i+7; i, i+5, i+6),\ (i, i+6, i+8; i, i+7, i+9),\\ &(i, i+3, i+5; i, i+4, i+6),\ (i, i+9, i+12; i, i+10, i+11),\\ &(i, i+9, i+12; i, i+10, i+11),\ (i, i+9, i+12; i, i+10, i+11),\\ &(i, i+9, i+12; i, i+10, i+11),\ (i, i+5, i+11; i, i+7, i+9),\\ &(i, i+5, i+11; i, i+7, i+9),\ (i, i+5, i+11; i, i+7, i+9),\\ &(i, i+3, i+10; i, i+5, i+7),\ (i, i+3, i+7; i, i+6, i+10),\\ &(i, i+6, i+10; i, i+9.i+4) \mid 0 \le i \le 12\}\\ &\quad \cup\ (i, i+1, i+5; i, i+2, i+4),\ (i, i+3, i+7; i, i+1, i+2),\\ &\qquad (i, i+1, i+3; i, i+4, i+5),\ (i, i+3, i+6; i, i+1, i+2),\\ &\qquad (i, i+3, i+4; i, i+1.i+2),\ (i, i+1, i+2; i, i+3, i+4),\\ &\qquad (i, i+2, i+6; i, i+3, i+5) \mid 0 \le i \le 12\}\end{aligned}$$

□

EXAMPLE 3.20 There exists a 4-wheel system of $8(K_7 \setminus K_3)$ which has a metamorphosis into an 8-fold bowtie system. (Here $(K_7 \setminus K_3)$ refers to the complete graph on 7 vertices with three edges forming a triangle removed from it.)

Let the vertex set be $V = \{0, 1, 2, 3, A, B, C\}$ where the "hole" or vertices of the removed triangle is the set $\{A, B, C\}$. An 8-fold B-wheel system (V, W) with a metamorphosis into an 8-fold bowtie system (V, B) is given by:

$$\begin{aligned}W = \{&0\text{–}(A,1,B,2),\ 0\text{–}(A,3,B,2),\ 0\text{–}(B,3,C,1),\ 1\text{–}(A,2,C,3),\\ &1\text{–}(A,0,C,2),\ 1\text{–}(A,0,C,3),\ 2\text{–}(B,0,C,1),\ 2\text{–}(B,3,C,0),\\ &2\text{–}(A,1,B,3),\ 3\text{–}(A,0,B,1),\ 3\text{–}(A,2,C,0),\ 3\text{–}(B,1,C,2),\\ &A\text{–}(0,1,2,3),\ B\text{–}(0,2,3,1),\ C\text{–}(0,2,1,3),\ A\text{–}(0,1,2,3),\\ &B\text{–}(0,2,3,1),\ C\text{–}(0,2,1,3)\}.\end{aligned}$$

Then a metamorphosis into an 8-fold bowtie system is given as follows.

$$\begin{aligned}B = \{&(0,A,2;0,B,1),\ (0,A,2;0,B,3),\ (0,B,1;0,C,3),\ (1,A,3;1,C,2),\ (1,A,2;1,C,0),\\ &(1,A,3;1,C,0),\ (2,B,1;2.C,0),\ (2,B,0;2,C,3),\ (2,A,3;2,B,1),\ (3,A,1;3,B,0),\\ &(3,A,0;3,C,2),\ (3,B,2;3,C,1),\ (A,0,3;A,1,2),\ (B,0,1;B,2,3),\ (C,0,3;C,2,1),\\ &(A,0,3;A,1,2),\ (B,0,1;B,2,3),\ (C,0,3;C,2,1)\}\\ &\quad \cup\ \{(A,0,2;A,1,3),\ (1,A,0;1,B,3),\ (B,0,2;B,1,3),\\ &\qquad (2,A,0;2,B,3),\ (C,0,2;C,1,3),\ (C,0,1;C,2,3)\}.\end{aligned}$$

THEOREM 3.21 *There exists a 24-fold 4-wheel system of any order $n \geq 5$ which has a metamorphosis into a 24-fold bowtie system of the same order n.*

Proof As pointed out above, by doubling 12-fold systems or trebling 8-fold systems, we only need consider orders $n \equiv 2$ or 11 (mod 12).

First let $n = 12t+2$. We use the 3-GDD construction with $h = 0$, $\ell = 2$, $s = 6t+1$, and a 3-GDD of type $7^1 3^{2t-2}$, which exists for all $t \geq 3$. This uses B-wheel systems of $24K_{14}$ (Example 3.19), $24K_6$ (three copies of Example 3.11) and $24K_{2,2,2}$ (six copies of Example 3.6). Then Example 3.19 deals with a B-wheel system of order 14 (when $t = 1$), and a B-wheel system of order 26 (when $t = 2$) is given in the Appendix.

Now let $n = 12t + 11$. We use the 3-GDD construction with $h = 3$, $\ell = 2$, $s = 6t + 4$, and a 3-GDD of type $4^1 2^{3t}$, which exists for all $t \geq 1$. This uses a B-wheel system of $24K_{11}$ (Example 3.18), B-wheel systems of $24(K_7 \setminus K_3)$ (take three copies of Example 3.20), and of $24K_{2,2,2}$ (six copies of Example 3.6).

This completes the existence of 24-fold B-wheel systems. □

4 Concluding remarks

First consider $\lambda = 10$, 14 and 22, since there are three orders ($v = 24$, 72, 88) when $\lambda = 2$ for which existence of a B-wheel system is unknown, and the expected orders for $\lambda = 10$, 14 and 22 are the same as for $\lambda = 2$. Since $10 = 4 + 6$, $14 = 6 + 8$ and $22 = 10 + 12$, for instance, and since for orders 24, 72 and 88, B-wheel systems exist when $\lambda = 4$, 6, 8 and 12, there are no orders for these values of λ for which B-wheel systems are unknown.

Now let λ be any value. By combining smaller values of λ with appropriate numbers of copies of B-wheel systems of order 24, we obtain B-wheel systems of all admissible orders, for any value of λ (apart from B-wheel systems of orders 24, 72, 88 when $\lambda = 2$). We record this as follows.

THEOREM 4.1 *There exists a λ-fold 4-wheel system of any order given in Table 4.1, which has a metamorphosis into a λ-fold bowtie system, except possibly one of order 24, 72, or 88, when $\lambda = 2$.*

4-wheel system that has a metamorphosis into a bowtie system

λ (mod 24)	order
1, 5,7,11,13,17,19,23	1, 33 (mod 48)
2, 10,14,22	0,1,9,16 (mod 24)
3, 9,15,21	1 (mod 16)
4, 20	0,1,4,9 (mod 12)
6, 18	0, 1 (mod 8)
8, 16	0, 1 (mod 3)
12,	0, 1 (mod 4)
24	any $n \geq 5$

Table 4.1

APPENDIX

In each of the following examples, V is the vertex set, W is the set of 4-wheels and B is the set of bowties obtained from a metamorphosis of W.

$\boxed{\lambda = 1, \quad \textbf{order } 81}$ $\quad V = \mathbb{Z}_{81};$

$$\begin{aligned} W = \{&i\text{–}(i+18, i+24, i+78, i+76),\ i\text{–}(i+41, i+73, i+45, i+74),\\ &i\text{–}(i+1, i+10, i+14, i+31),\ i\text{–}(i+12, i+34, i+13, i+56),\\ &i\text{–}(i+15, i+26, i+42, i+61) \mid 0 \le i \le 80\}. \end{aligned}$$

$$\begin{aligned} B = \{&(i, i+18, i+24; i, i+76, i+78),\ (i, i+41, i+74; i, i+45, i+73),\\ &(i, i+1, i+31; i, i+10, i+14),\ (i, i+12, i+56; i, i+13, i+34),\\ &(i, i+15, i+26; i, i+42, i+61) \mid 0 \le i \le 80\} \cup\\ &\{(i, i+17, i+46; i, i+16, i+38) \mid 0 \le i \le 80\} \cup\\ &\{(i, i+9, i+32; i, i+27, i+54) \mid 0 \le i \le 26\} \cup\\ &\{(i+36, i+27, i+59; i+36, i+45, i+68),\\ &(i+54, i+45, i+77; i+54, i+63, i+5),\\ &(i+72, i+63, i+14; i+72, i, i+23) \mid 0 \le i \le 8\}. \end{aligned}$$

□

$\boxed{\lambda = 1, \quad \textbf{order } 97}$ $\quad V = \mathbb{Z}_{97};$

$$\begin{aligned} W = \{&i\text{–}(i+5, i+11, i+30, i+21),\ i\text{–}(i+20, i+69, i+57, i+70),\\ &i\text{–}(i+46, i+75, i+74, i+2),\ i\text{–}(i+3, i+7, i+24, i+34),\\ &i\text{–}(i+8, i+41, i+83, i+45),\ i\text{–}(i+26, i+58, i+43, i+61) \mid 0 \le i \le 96\}. \end{aligned}$$

$$\begin{aligned} B = \{&(i, i+5, i+25; i, i+11, i+30),\ (i, i+20, i+69; i, i+57, i+70),\\ &(i, i+2, i+46; i, i+74, i+75),\ (i, i+3, i+34; i, i+7, i+24),\\ &(i, i+8, i+45; i, i+41, i+83),\ (i, i+26, i+58; i, i+43, i+61) \mid 0 \le i \le 96\} \cup\\ &\{(i, i+6, i+15; i, i+10, i+35),\ (i, i+4, i+33; i, i+12, i+50) \mid 0 \le i \le 96\}. \end{aligned}$$

$\boxed{\lambda = 1, \quad \textbf{order } 129}$ $\quad V = \mathbb{Z}_{129};$

$$\begin{aligned} W = \{&i\text{–}(i+104, i+122, i+113, i+126),\ i\text{–}(i+6, i+37, i+67, i+82),\\ &i\text{–}(i+33, i+68, i+63, i+77),\ i\text{–}(i+1, i+11, i+19, i+21),\\ &i\text{–}(i+4, i+27, i+39, i+73),\ i\text{–}(i+17, i+49, i+100, i+57),\\ &i\text{–}(i+24, i+79, i+41, i+83),\ i\text{–}(i+26, i+84, i+36, i+101) \mid 0 \le i \le 128\}. \end{aligned}$$

$$\begin{aligned}B = \{&(i,i+104,i+126;i,i+122,i+113),\ (i,i+6,i+82;i,i+37,i+67),\\ &(i,i+33,i+68;i,i+63,i+77),\ (i,i+1,i+21;i,i+11,i+19),\\ &(i,i+4,i+73;i,i+27,i+39),\ (i,i+17,i+57;i,i+49,i+100),\\ &(i,i+24,i+79;i,i+41,i+83),\ (i,i+26,i+84;i,i+36,i+101)\ \mid 0 \le i \le 128\}\\ &\cup \{\{i,i+5,i+18;i,i+23,i+54\},\ \{i,i+15,i+59;i,i+10,i+48\} \mid 0 \le i \le 128\}\\ &\cup \{\{i,i+43,i+86;i,i+2,i+34\} \mid 0 \le i \le 14 \text{ and } 16 \le i \le 42\}\\ &\cup \{\{101,15,58;101,67,69\},\ \{17,15,49;17,112,114\}\}\\ &\cup \{\{4i+1,4i-1,4i+33;4i+1,4i+3,4i+35\},\{4i+2,4i,4i+34;4i+2,4i+4,4i+36\}\\ &\qquad \mid 11 \le i \le 16\}\\ &\cup \{\{4i-2,4i-4,4i+30;4i-2,4i,4i+32\},\{4i-1,4i-3,4i+31;4i-1,4i+1,4i+33\}\\ &\qquad \mid 18 \le i \le 28\}\\ &\cup \{\{4i-1,4i-3,4i+31;4i-1,4i+1,4i+33\},\{4i,4i-2,4i+32;4i,4i+2,4i+34\}\\ &\qquad \mid 29 \le i \le 32\}.\end{aligned}$$

$\lambda = 4$, order 21 $V = \mathbb{Z}_{21}$;

$$\begin{aligned}W = \{&i\text{-}(i+1,i+2,i+3,i+4),\ i\text{-}(i+2,i+4,i+6,i+9),\ i\text{-}(i+3,i+7,i+15,i+8),\\ &i\text{-}(i+4,i+13,i+5,i+14),\ i\text{-}(i+5,i+15,i+6,i+16) \mid 0 \le i \le 20\}.\end{aligned}$$

$$\begin{aligned}B = \{&(i,i+1,i+4;i,i+2,i+3),\ (i,i+2,i+9;i,i+4,i+6),\ (i,i+3,i+8;i,i+7,i+15),\\ &(i,i+4,i+13;i,i+5,i+14),\ (i,i+5,i+15;i,i+6,i+16) \mid 0 \le i \le 20\}\\ &\cup\ \{(i,i+3,i+20;i,i+1,i+10) \mid 0 \le i \le 20\}\\ &\cup\ \{(i,i+2,i+10;i,i+7,i+14) \mid 0 \le i \le 6\}\\ &\cup\ \{(i+18,i+8,i+10;i+18,i+5,i+16) \mid 0 \le i \le 3\}\\ &\cup\ \{(9,7,17;9,1,20),\ (14,1,12;14,3,16),\ (15,2,13;15,4,17)\}.\end{aligned}$$

$\lambda = 4$, order 24 $V = \{\infty\} \cup \mathbb{Z}_{23}$;

$$\begin{aligned}W = \{&\infty\text{-}(i,i+7,i+20,i+10),\ i\text{-}(i+15,i+16,i+17,i+3),\\ &i\text{-}(i+2,i+17,i+9,i+18),\ i\text{-}(i+17,i+18,i+19,i+21),\\ &i\text{-}(i+11,i+13,i+16,i+19),\ i\text{-}(i+11,i+14,i+18,i+6) \mid 0 \le i \le 22\}.\end{aligned}$$

$$\begin{aligned}B = \{&(\infty,i,i+10;\infty,i+7,i+20),\ (i,i+15,i+3;i,i+16,i+17),\\ &(i,i+2,i+17;i,i+9,i+18),\ (i,i+17,i+21;i,i+18,i+19),\\ &(i,i+11,i+19;i,i+13,i+16),\ (i,i+11,i+14;i,i+18,i+6) \mid 0 \le i \le 22\}\\ &\cup\ \{(i,i+9,i+19;i,i+7,i+8),\ (i,i+2,i+3;i,i+5,i+7) \mid 0 \le i \le 22\}.\end{aligned}$$

$\boxed{\lambda = 4, \quad \textbf{order } 25}$ $\quad V = \mathbb{Z}_{25};$

$$W = \{i\text{–}(i+13, i+1, i+19, i+2),\ i\text{–}(i+12, i+18, i+13, i+20),\ i\text{–}(i+1, i+2, i+3, i+5),\ i\text{–}(i+2, i+5, i+8, i+11),\ i\text{–}(i+4, i+14, i+10, i+21),\ i\text{–}(i+6, i+15, i+9, i+16) \mid 0 \le i \le 24\}.$$

$$B = \{(i, i+13, i+1; i, i+19, i+2),\ (i, i+12, i+18; i, i+13, i+20),\ (i, i+1, i+2; i, i+3, i+5),\ (i, i+2, i+5; i, i+8, i+11),\ (i, i+4, i+21; i, i+14, i+10),\ (i, i+6, i+16; i, i+15, i+9) \mid 0 \le i \le 24\} \cup \{(i, i+10, i+18; i, i+9, i+16),\ (i, i+11, i+22; i, i+1, i+5) \mid 0 \le i \le 24\}.$$

$\boxed{\lambda = 4, \quad \textbf{order } 28}$ $\quad V = \{\infty\} \cup \mathbb{Z}_{27};$

$$W = \{\infty\text{–}(i, i+12, i+25, i+13),\ i\text{–}(i+5, i+24, i+22, i+2),\ i\text{–}(i+23, i+24, i+13, i+5),\ i\text{–}(i+20, i+21, i+22, i+23),\ i\text{–}(i+15, i+17, i+19, i+23),\ i\text{–}(i+12, i+18, i+7, i+21),\ i\text{–}(i+11, i+20, i+10, i+21) \mid 0 \le i \le 26\}.$$

$$B = \{(\infty, i, i+13; \infty, i+12, i+25),\ (i, i+5, i+2; i, i+24, i+22),\ (i, i+23, i+24; i, i+13, i+5),\ (i, i+20, i+23; i, i+21, i+22),\ (i, i+15, i+23; i, i+17, i+19),\ (i, i+12, i+21; i, i+18, i+7),\ (i, i+11, i+21; i, i+20, i+10) \mid 0 \le i \le 26\} \cup \{(i, i+1, i+7; i, i+4, i+12) \mid 0 \le i \le 26\} \cup \{(i, i+9, i+18; i, i+1, i+12),\ (i, i+9, i+18; i, i+2, i+13) \mid 0 \le i \le 8\} \cup \{(i, i+1, i+12; i, i+2, i+13) \mid 9 \le i \le 26\}.$$

$\boxed{\lambda = 8, \quad \textbf{order } 22}$ $\quad V = \{\infty\} \cup \mathbb{Z}_{21};$

$$W = \{\infty\text{–}(i, i+6, i+16, i+10),\ \infty\text{–}(i, i+8, i+18, i+10),\ i\text{–}(i+13, i+18, i+4, i+1),\ i\text{–}(i+4, i+6, i+5, i+8),\ i\text{–}(i+9, i+10, i+21, i+2),\ i\text{–}(i+20, i+7, i+8, i+9),\ i\text{–}(i+16, i+17, i+18, i+20),\ i\text{–}(i+13, i+15, i+17, i+19),\ i\text{–}(i+9, i+12, i+15, i+18),\ i\text{–}(i+8, i+12, i+17, i+13),\ i\text{–}(i+10, i+15, i+7, i+16) \mid 0 \le i \le 20\}.$$

$$B = \{(\infty, i, i+10; \infty, i+6, i+16),\ (\infty, i, i+8; \infty, i+18, i+10),\ (i, i+14, 1+19; i, i+4, i+1),\ (i, i+4, i+6; i, i+5, i+8),\ (i, i+9, i+10; i, i+21, i+2),\ (i, i+20, i+7; i, i+8, i+9),\ (i, i+16, i+177; i, i+18, i+20),\ (i, i+13, i+19; i, i+15, i+17),\ (i, i+9, i+12; i, i+15, i+18),\ (i, i+8, i+12; i, i+17, i+13),\ (i, i+10, i+15; i, i+7, i+16) \mid 0 \le i \le 20\} \cup \{(i, i+4, i+10; i, i+1, i+6),\ (i, i+4, i+10; i, i+2, i+11),\ (i, i+1, i+7; i, i+2, i+5) \mid 0 \le i \le 20\} \cup \{(i, i+7, i+14; i, i+1, i+10) \mid 0 \le i \le 6\} \cup \{(2i, 2i-1, 2i+9; 2i, 2i+1, 2i+10) \mid 4 \le i \le 10\}.$$

$\boxed{\lambda = 8, \ \ \textbf{order } 27}$ $\quad V = \mathbb{Z}_{27};$

$$\begin{aligned} W = \; & i\text{-}(i+21, i+5, i+25, i+9),\ i\text{-}(i+23, i+8, i+3, i+9),\ i\text{-}(i+21, i+4, i+13, i+10), \\ & i\text{-}(i+17, i+23, i+22, i+24),\ i\text{-}(i+3, i+8, i+11, i+10),\ i\text{-}(i+1, i+2, i+3, i+4), \\ & i\text{-}(i+1, i+2, i+4, i+6),\ i\text{-}(i+2, i+4, i+7, i+11),\ i\text{-}(i+4, i+9, i+17, i+12), \\ & i\text{-}(i+5, i+17, i+6, i+18),\ i\text{-}(i+6, i+18, i+8, i+19),\ i\text{-}(i+6, i+19, i+7, i+20), \\ & i\text{-}(i+7, i+18, i+8, i+20) \mid 0 \le i \le 26\}. \end{aligned}$$

$$\begin{aligned} B = \; & \{(i, i+21, i+5; i, i+25, i+9),\ (i, i+23, i+8; i, i+3, i+9),\ (i, i+21, i+4; i, i+13, i+10), \\ & (i, i+17, i+23; i, i+22, i+24),\ (i, i+3, i+8; i, i+11, i+10),\ (i, i+1, i+4; i, i+2, i+3), \\ & (i, i+1, i+6; i, i+2, i+4),\ (i, i+2, i+4; i, i+7, i+11),\ (i, i+4, i+9; i, i+17, i+12), \\ & (i, i+5, i+17; i, i+6, i+18),\ (i, i+6, i+18; i, i+8, i+19),\ (i, i+6, i+19; i, i+7, i+20), \\ & (i, i+7, i+18; i, i+8, i+20) \mid 0 \le i \le 26\} \\ & \quad \cup \ \{(i, i+1, i+8; i, i+5, i+12),\ (i, i+3, i+14; i, i+13, i+26), \\ & \quad\quad (i, i+1, i+11; i, i+3, i+13) \mid 0 \le i \le 26\} \\ & \quad \cup \ \{(i, i+9, i+18; i, i+1, i+8),\ (i, i+9, i+18; i, i+2, i+14) \mid 0 \le i \le 8\} \\ & \quad \cup \ \{(i, i+2, i+14; i, i+1, i+8) \mid 9 \le i \le 26\}. \end{aligned}$$

$\boxed{\lambda = 12, \ \ \textbf{order } 20}$ $\quad V = \{\infty\} \cup \mathbb{Z}_{19};$

$$\begin{aligned} W = \; & \{\infty\text{-}(i, i+6, i+16, i+7),\ \infty\text{-}(i, i+7, i+16, i+9), \\ & \infty\text{-}(i, i+7, i+17, i+8),\ i\text{-}(i+11, i+16, i+15, i+1), \\ & i\text{-}(i+4, i+6, i+5, i+8),\ i\text{-}(i+9, i+12, i+14, i+2), \\ & i\text{-}(i+10, i+18, i+16, i+2),\ i\text{-}(i+15, i+16, i+17, i+18), \\ & i\text{-}(i+15, i+16, i+17, i+18),\ i\text{-}(i+11.i+13, i+15, i+17), \\ & i\text{-}(i+8, i+10, i+13, i+16),\ i\text{-}(i+4, i+7, i+11, i+15), \\ & i\text{-}(i+5, i+9, i+14, i+10),\ i\text{-}(i+8, i+13, i+6, i+14), \\ & i\text{-}(i+7, i+12, i+6, i+13) \mid 0 \le i \le 18\}. \end{aligned}$$

$$\begin{aligned} B = \; & \{(\infty, i, i+7; \infty, i+6, i+16),\ (infty, i, i+9; \infty, i+7, i+16), \\ & (\infty, i, i+8; \infty, i+7, i+17),\ (i, i+11, i+1; i, i+16, i+15), \\ & (i, i+4, i+8; i, i+6, i+5),\ (i, i+9, i+2; i, i+12, i+14), \\ & (i, i+10, i+2; i, i+18, i+16),\ (i, i+15, i+18; i, i+16, i+17), \\ & (i, i+15, i+18; i, i+16, i+17),\ (i, i+11, i+17; i, i+13, i+15), \\ & (i, i+8, i+16; i, i+10, i+13),\ (i, i+4, i+15; i, i+7, i+11), \\ & (i, i+5, i+10; i, i+9, i+14),\ (i, i+8, i+14; i, i+13, i+6), \\ & (i, i+7, i+13; i, i+12, i+6) \ \mid \ 0 \le i \le 18\} \\ & \quad \cup \ \{(i, i+5, i+10; i, i+7, i+14),\ (i, i+5, i+12; i, i+4, i+8), \\ & \quad\quad (i, i+3, i+6; i, i+2, i+4),\ (i, i+3, i+8; i, i+1, i+2), \\ & \quad\quad (i, i+1.i+8; i, i+2, i+3) \ \mid \ 0 \le i \le 18\}. \end{aligned}$$

$\boxed{\lambda = 24, \quad \textbf{order 26}}$ $\quad V = \{\infty\} \cup \mathbb{Z}_{25};$

$$W = \{i\text{–}(i+13, i+16, i+14, i+17),\ i\text{–}(i+11, i+16, i+13, i+19),\ i\text{–}(i+7, i+10, i+8, i+12),$$
$$i\text{–}(i+22, i+3, i+15, i+5),\ i\text{–}(i+19, i+8, i+4, i+10),\ i\text{–}(i+14, i+15, i+21, i+20),$$
$$i\text{–}(i+13, i+17, i+19, i+18),\ i\text{–}(i+23, i+9, i+8, i+11),\ i\text{–}(i+24, i+6, i+1, i+9),$$
$$i\text{–}(i+11, i+20, i+12, i+22),\ i\text{–}(i+21, i+22, i+23, i+24),$$
$$i\text{–}(i+21, i+22, i+23, i+24),\ i\text{–}(i+21, i+22, i+23, i+24),$$
$$i\text{–}(i+21, i+22, i+23, i+24),\ i\text{–}(i+19, i+20, i+21, i+23),$$
$$i\text{–}(i+17, i+19, i+21, i+23),\ i\text{–}(i+17, i+19, i+21, i+23),$$
$$i\text{–}(i+17, i+19, i+21, i+23),\ i\text{–}(i+15, i+17, i+19, i+22),$$
$$i\text{–}(i+13, i+16, i+19, i+22),\ i\text{–}(i+11, i+14, i+17, i+21),$$
$$i\text{–}(i+9, i+13, i+17, i+21),\ i\text{–}(i+8, i+12, i+16, i+20),\ i\text{–}(i+5, i+10, i+15, i+20),$$
$$i\text{–}(i+5, i+10, i+15, i+20),\ i\text{–}(i+5, i+10, i+15, i+20),\ i\text{–}(i+6, i+12, i+19, i+13),$$
$$i\text{–}(i+12, i+18, i+7, i+19),\ i\text{–}(i+10, i+17, i+7, i+18),\ i\text{–}(i+10, i+17, i+7, i+18),$$
$$i\text{–}(i+10, i+17, i+7, i+18),\ i\text{–}(i+9, i+16, i+7, i+18),\ i\text{–}(i+9, i+16, i+7, i+18),$$
$$\infty\text{–}(i, i+11, i+24, i+12),\ \infty\text{–}(i, i+11, i+24, i+12),\ \infty\text{–}(i, i+11, i+23, i+12),$$
$$\infty\text{–}(i, i+9, i+20, i+11),\ \infty\text{–}(i, i+9, i+20, i+11),\ \infty\text{–}(i, i+8, i+17, i+9) \mid 0 \le i \le 24\}.$$

Let B' be the set of bowties $(x, a, d; x, b, c)$ for each 4-wheel x–(a, b, c, d) written as oriented in W above. Then

$$B = B' \cup \{(i, i+3, i+14; i, i+7, i+16),\ (i, i+11, i+14; i, i+9, i+16),$$
$$(i, i+5, i+12; i, i+4, i+11),\ (i, i+5, i+12; i, i+3, i+8),$$
$$(i, i+6, i+12; i, i+1, i+2),\ (i, i+4, i+8; i, i+1, i+2),$$
$$(i, i+1, i+5; i, i+2.i+4),\ (i, i+5, i+6; i, i+1, i+2),$$
$$(i, i+1, i+11; i, i+3, i+14),\ (i, i+2, i+6; i, i+7, i+11),$$
$$(i, i+10, i+11; i, i+3, i+5),\ (i, i+3, i+9; i, i+5, i+8),$$
$$(i, i+1, i+3; i, i+5, i+6)\}.$$

Acknowledgements

Many thanks to the Department of Discrete and Statistical Sciences at Auburn University, where work on this paper was started, and especially to Curt Lindner for suggesting this type of problem.

References

[1] P. Adams, *autogen*, a graph decomposition package (in PhD thesis, *Combinatorial designs and related computational constructions*, University of Queensland, 1995).

[2] J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combinatoria **10** (1980), 211–254.

[3] E.J. Billington and C.C. Lindner, *The spectrum for 2-perfect bowtie systems*, Discrete Mathematics **135** (1994), 61–68.

[4] E.J. Billington and C.C. Lindner, *Maximum packings of bowtie designs*, Journal of Combinatorial Mathematics and Combinatorial Computing **27** (1998), 227–249.

[5] E.J. Billington and C.C. Lindner, *The metamorphosis of lambda-fold 4-wheel systems into lambda-fold 4-cycle systems*, Utilitas Mathematica (to appear).

[6] C.C. Lindner and A. Rosa, *The metamorphosis of λ-fold block designs with block size four into λ-fold triple systems*, Journal of Statistical Planning and Inference (to appear).

[7] C.C. Lindner and A.P. Street, *The metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems*, Bulletin of the Institute of Combinatorics and its Applications **28** (2000), 7–18; *Corrigendum*, ibid. **29** (2000), 88.

[8] R.C. Mullin and H.-D.O.F. Gronau, *PBDs and GDDs: the basics*, in CRC Handbook of Combinatorial Designs (eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton 1996, pp. 185–193.

[9] P. Horák and A. Rosa, *Decomposing Steiner triple systems into small configurations*, Ars Combinatoria **26** (1988), 91–105.

(Received 2/12/99)