The number of 8-cycles in 2-factorizations of K_n

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Abstract

This paper gives a complete solution (with one possible exception) of the problem of constructing 2-factorizations of K_n containing a specified number of 8-cycles.

1 Introduction

A 2-factor of the complete undirected graph K_n is a collection of vertex disjoint cycles which span the vertex set of K_n. A 2-factorization of order n is a pair (S, F), where F is a collection of edge disjoint 2-factors of K_n (with vertex set S) which partitions the edge set of K_n.

Of course, a 2-factorization of K_n exists if and only if n is odd and in this case the number of 2-factors is $(n-1)/2$.

A smallest cycle in K_n is a 3-cycle and a largest cycle is a Hamiltonian cycle (a cycle of length n). The most extensively studied 2-factorizations are Kirkman Triple systems (in which all cycles have length 3) and Hamiltonian decompositions (in which all cycles have length n). It is well known that Kirkman triple systems exist precisely when $n \equiv 3 \pmod 6$ [6] and Hamiltonian decompositions exist for all odd n [5].

In [2] I. J. Dejter, F. Franek, E. Mendelsohn, and A. Rosa looked at the problem of constructing 2-factorizations of K_n containing a specified number of 3-cycles. Modulo a few exceptions they give a complete solution for $n \equiv 1$ or $3 \pmod 6$. The problem remains open for $n \equiv 5 \pmod 6$.

In [3] I.J. Dejter, C.C. Lindner, and A. Rosa gave a complete solution of the problem of constructing 2-factorizations of K_n containing a specified number of 4-cycles. In [1] P. Adams and E. J. Billington gave a complete solution of the problem of constructing 2-factorizations of K_n containing a specified number of 6-cycles.

To date, the first unsettled case of constructing 2-factorizations of K_n containing a specified number of cycles of even length is for 8-cycles. The purpose of this paper is to give a complete solution (with 3 possible exceptions) of the problem of constructing 2-factorizations of K_n containing a specified number of 8-cycles. To be specific let $Q(n)$ denote the set of all x such that there exists a 2-factorization of K_n containing x 8-cycles and let

$$FC(n) = \begin{cases} \{0,1,...,8k(2k-1)\} & \text{if } n = 16k+1, \\ \{0,1,...,2k(8k+1)\} & \text{if } n = 16k+3, \\ \{0,1,...,2k(8k+2)\} & \text{if } n = 16k+5, \\ \{0,1,...,2k(8k+3)\} & \text{if } n = 16k+7, \\ \{0,1,...,8k(2k+1)\} & \text{if } n = 16k+9, \\ \{0,1,...,(2k+1)(8k+5)\} & \text{if } n = 16k+11, \\ \{0,1,...,(2k+1)(8k+6)\} & \text{if } n = 16k+13, \text{and} \\ \{0,1,...,(2k+1)(8k+7)\} & \text{if } n = 16k+15. \end{cases}$$

We will show that $Q(n) = FC(n)$ for all odd n, with the possible exception $47 \in FC(33)$.

We will organize our results into 3 sections: a general recursive construction for $n \equiv 9, 11, 13$, and 15 (mod 16), a general recursive construction for $n \equiv 1, 3, 5$, and 7 (mod 16), and a summary followed by an appendix. The appendix contains all examples not used in the recursive constructions.

Now, let F be a 2-factor with cycles $C_1, C_2, ..., C_n$. In what follows we will denote the 2-factor F by $[C_1, C_2, ..., C_n]$.

2 n≡ 9,11,13 or 15 (mod 16)

The following construction is the principal tool used in this section.

Construction A:

Write $n = tv + r$, where t is odd and v is even and $r \in \{1,3,5,7\}$. Let $X = \{1,2,...,t\}$, $V = \{1,2,...,v\}$, and Z be a set of size r. Further, let $(X, \circ)$ be an idempotent commutative quasigroup of order t [4] and set $S = Z \cup (X \times V)$.

Define a collection F of 2-factors of K_{tv+r} as follows:

(1) Let $(Z \cup (\{1\} \times \{1,2,...,v\}), F_1)$ be a 2-factorization of K_{v+r}, where $F_1 = \{f_{1_1}, f_{1_2}, ..., f_{(v+r-1)/2}\}$.

(2) For each $x \in X \setminus \{1\}$, let $(Z \cup (\{x\} \times \{1,2,...,v\}), F_x)$ be a 2-factorization of K_{v+r} containing either 0 or max$FC(v+r)$ 8-cycles and containing a sub-2-factorization of order r, where max$FC(v+r)$ is the largest value in the set $FC(v+r)$. Let $F_x = \{f_{x_1}, f_{x_2}, ..., f_{x_{(v+r-1)/2}}\}$, where the last $(r-1)/2$ 2-factors contain the sub-2-factorization of order r.

(3) For each pair $a \neq b \in X$ such that $a \circ b = b \circ a = x$, let $(K_{a,b}, f_x(a,b))$ be any 2-factorization of $K_{v,v}$ with parts $\{a\} \times \{1,2,...,v\}$ and $\{b\} \times \{1,2,...,v\}$, where $f_x(a,b) = \{f_{x_1}(a,b), f_{x_2}(a,b), ..., f_{x_{v/2}}(a,b)\}$.

(4) Each of $\{f_{x_i}\} \cup \{f_{x_i}(a,b) \mid a \circ b = b \circ a = x\}$, where $i = 1,2,...,v/2$ is a 2-factor of K_{tv+r}.

(5) Piece together the remaining $(r-1)/2$ 2-factors of F_1, along with the remaining $(r-1)/2$ 2-factors of each F_x, for $x = 2,3,...,t$, making sure to delete the cycles belonging to the sub-2-factorization from each of the remaining 2-factors in

each F_x.

(6) For each $x \in X$, place the $v/2$ 2-factors in (4) in F as well as the 2-factors in (5).

The union of the 2-factors in (6) gives a total of $\sum_{x\in X}(v/2) + (r-1)/2 = (tv+r-1)/2$ 2-factors which form a 2-factorization of K_{tv+r} with vertex set S. □

Corollary 2.1 *Construction A gives a 2-factorization of K_{tv+r} containing exactly $\sum_{i=1}^{\binom{t}{2}} n_i + \sum_{i=1}^{t} m_i$ 8-cycles, where $n_i \in Q(K_{v,v})$, $m_1 \in Q(v+r)$, and $m_i \in \{0, \max FC(v+r)\}$ for $i = 2, 3, ..., t$.* □

It is easy to see that $Q(n) \subseteq FC(n)$ for odd n. Now, with Construction A and Corollary 2.1 we will show that $FC(n) \subseteq Q(n)$ for the cases $n \equiv 9, 11, 13,$ and 15 (mod 16). In each of the following cases we will take $t = 2k+1$ and $v = 8$.

n≡9 (mod16)

Example 2.2 $Q(9) = FC(9)$.

Since $FC(9)$ is 0, we need to construct a 2-factorization containing 0 8-cycles. Any Kirkman Triple system of order 9 will do [4]. □

Example 2.3 *$K_{8,8}$ can be 2-factorized into $\{0,1,2,3,4,5,6,7,8\}$ 8-cycles.*

Proof: Let the parts of $K_{8,8}$ be $\{1,2,3,4,5,6,7,8\}$ and $\{9,10,11,12,13,14,15,16\}$.

(i) $0 \in Q(K_{8,8})$:
[(1, 11, 3, 13, 5, 15, 7, 10, 2, 12, 4, 14, 6, 16, 8, 9)],
[(1, 10, 8, 15, 6, 13, 4, 11, 2, 9, 7, 16, 5, 14, 3, 12)],
[(1, 13, 7, 11, 5, 10, 4, 16, 2, 14, 8, 12, 6, 9, 3, 15)],
[(1, 14, 7, 12, 5, 9, 4, 15, 2, 13, 8, 11, 6, 10, 3, 16)].

(ii) $1 \in Q(K_{8,8})$:
[(1, 9, 2, 10, 3, 11, 4, 12), (5, 13, 6, 14), (7, 15, 8, 16)],
[(1, 10, 4, 13, 7, 14, 8, 9, 3, 12, 5, 15, 6, 16, 2, 11)],
[(1, 14, 2, 15, 4, 9, 7, 12, 8, 13, 3, 16), (5, 10, 6, 11)],
[(1, 13, 2, 12, 6, 9, 5, 16, 4, 14, 3, 15), (7, 10, 8, 11)].

(iii) $2 \in Q(K_{8,8})$:
[(1, 13, 2, 14, 3, 16, 4, 15), (5, 9, 7, 12, 8, 11, 6, 10)],
[(1, 14, 4, 9, 6, 12, 2, 15, 3, 13, 8, 10, 7, 11, 5, 16)],
[(1, 10, 4, 12, 3, 9, 8, 14, 7, 13, 5, 15, 6, 16, 2, 11)],
[(1, 9, 2, 10, 3, 11, 4, 13, 6, 14, 5, 12), (7, 15, 8, 16)].

(iv) $3 \in Q(K_{8,8})$:
[(1, 13, 2, 14, 3, 16, 4, 15), (5, 9, 7, 12, 8, 11, 6, 10)],
[(1, 9, 2, 10, 3, 11, 4, 12), (5, 13, 6, 14), (7, 15, 8, 16)],
[(1, 14, 4, 9, 6, 12, 2, 15, 3, 13, 8, 10, 7, 11, 5, 16)],
[(1, 10, 4, 13, 7, 14, 8, 9, 3, 12, 5, 15, 6, 16, 2, 11)].

(v) $4 \in Q(K_{8,8})$:
[(1, 13, 7, 11, 5, 9, 3, 15), (2, 14, 8, 12, 6, 10, 4, 16)],

$[(1,14,7,12,5,10,3,16),(2,13,8,11,6,9,4,15)]$,
$[(1,11,3,13,5,15,7,10,2,12,4,14,6,16,8,9)]$,
$[(1,10,8,15,6,13,4,11,2,9,7,16,5,14,3,12)]$.

(vi) $5 \in Q(K_{8,8})$:
$[(1,14,2,13,4,16,3,15),(5,10,8,9,7,12,6,11)]$,
$[(1,13,3,14,8,12,5,16),(2,11,7,10,6,9,4,15)]$,
$[(5,13,8,16,7,15,6,14),(1,10,2,9),(3,12,4,11)]$,
$[(1,11,8,15,5,9,3,10,4,14,7,13,6,16,2,12)]$.

(vii) $6 \in Q(K_{8,8})$:
$[(1,14,2,13,4,16,3,15),(5,10,8,9,7,12,6,11)]$,
$[(1,13,3,14,8,12,5,16),(2,11,7,10,6,9,4,15)]$,
$[(1,9,2,10,3,11,4,12),(5,14,6,13,8,16,7,15)]$,
$[(1,10,4,14,7,13,5,9,3,12,2,16,6,15,8,11)]$.

(vii) $7 \in Q(K_{8,8})$:
$[(1,14,2,13,4,16,3,15),(5,10,8,9,7,12,6,11)]$,
$[(1,13,3,14,8,12,5,16),(2,11,7,10,6,9,4,15)]$,
$[(1,10,4,14,7,13,8,11),(2,12,3,9,5,15,6,16)]$,
$[(1,9,2,10,3,11,4,12),(5,13,6,14),(7,15,8,16)]$.

(viii) $8 \in Q(K_{8,8})$:
$[(1,9,7,15,5,13,3,11),(2,10,8,16,6,14,4,12)]$,
$[(1,10,7,16,5,14,3,12),(2,9,8,15,6,13,4,11)]$,
$[(1,13,7,11,5,9,3,15),(2,14,8,12,6,10,4,16)]$,
$[(1,14,7,12,5,10,3,16),(2,13,8,11,6,9,4,15)]$.

Lemma 2.4 $FC(16k+9) \subseteq Q(16k+9)$.

Proof: Take $r = 1$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$ Corollary 2.1 gives $FC(16k+9) \subseteq Q(16k+9)$.

n≡11 (mod16)

Example 2.5 $Q(11) = FC(11)$, *where the 2-factorizations of* K_{11} *having* 0 *8-cycles and* 5 *8-cycles contain a cycle of length* 3.

Proof: (i) Take K_{11} to have vertex set $\{A\} \cup (\{1,2\} \times Z_5)$ and let $F = [(A,(1,2),(2,4)),((1,0),(2,0),(2,1),(2,3)),((1,1),(1,4),(1,3),(2,2))]$. If $x \in Z_5$ denote by $F+x$ the 2-factor of K_{11} obtained from F by adding x (mod 5) to the second coordinates of the ordered pairs belonging to F.Then $\{F+x \mid x \in Z_5\}$ is a 2-factorization of K_{11} containing 0 8-cycles.

(ii) The 2-factorization of K_{11} given by
$[(1,2,7),(3,10,8,4,5,6,9,11)]$, $[(5,8,9)(1,3,2,4)(6,10,11,7)]$,
$[(1,5,2,6,11,8,7,3,4,9,10)]$, $[(1,6,8,2,10,5,3,9,7,4,11)]$,
$[(1,8,3,6,4,10,7,5,11,2,9)]$
shows that $1 \in Q(11)$.

(iii) Take K_{11} to have vertex set $\{A, B, C, D, E\} \cup (\{1,2\} \times Z_3)$ and let $F = [(A,(1,2),D,(2,0),B,(1,1),E,(2,2)),((C,(1,0),(2,1))]$. Then $\{F{+}x \mid x \in Z_3\}$ with the following two 2-factors:
$F_4 = [(A,B,C,D,E),((1,0),(1,1),(1,2)),((2,0),(2,1),(2,2))]$ and
$F_5 = [(A,C,E,B,D),((1,0),(2,0),(1,1),(2,1),(1,2),(2,2))]$
is a 2-factorization of K_{11} containing 3 8-cycles.

(iv) The union of F, F_4 and F_5 can be decomposed into three 2-factors as follows:
$[(C,(1,0),(2,1)),(A,B,D,E),((1,1),(1,2),(2,2),(2,0))]$,
$[((1,0),(1,1),(2,1),(2,2)),(A,C,E,B,(2,0),D,(1,2))]$, and
$[((1,0),(1,2),(2,1),(2,0)),(A,D,C,B,(1,1),E,(2,2))]$.
This reduces the number of 8-cycles by 1. Hence $2 \in Q(11)$.

(v) Take K_{11} to have vertex set $\{A, B, C\} \cup (\{1,2\} \times Z_4)$ and let $F = [(C,(1,1),(2,0)),(A,(1,3),(1,2),B,(2,1),(1,0),(2,2),(2,3))]$. Then $\{F + x \mid x \in Z_4\}$ with the following 2-factor:
$[(A,B,C),((1,0),(1,2),(2,2),(2,0)),((1,1),(1,3),(2,3),(2,1))]$ is a 2-factorization of K_{11} containing 4 8-cycles.

(vi) Finally, take K_{11} to have vertex set $\{A\} \cup (\{1,2\} \times Z_5)$ and let $F = [(A,(1,2),(2,4)),((1,0),(2,1),(2,2),(2,0),(1,1),(1,4),(1,3),(2,3))]$. Then $\{F + x \mid x \in Z_5\}$ is a 2-factorization of K_{11} containing 5 8-cycles.

Combining all the above cases shows that $Q(11) = FC(11)$.

Lemma 2.6 $FC(16k + 11) \subseteq Q(16k + 11)$.

Proof: Take $r = 3$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$, $Q(11) = FC(11)$ and $m_i \in \{0,5\}$ for $i = 2,3,...,t$, Corollary 2.1 gives $FC(16k{+}11) \subseteq Q(16k + 11)$.

n≡13 (mod16)

Example 2.7 $Q(13) = FC(13)$, *where the 2-factorizations of* K_{13} *having* 0 *and* 6 *8-cycles contain sub-2-factorizations of order* 5.

Proof: (i) The 2-factorization of K_{13} given by
$[(1,2,3,4,5),(6,10,7,11),(8,12,9,13)],[(1,3,5,2,4),(6,12,7,13),(8,10,9,11)]$,
$[(1,6,7,8,9),(2,10,3,11),(4,12,5,13)],[(1,7,9,6,8),(2,12,3,13),(4,10,5,11)]$,
$[(1,10,11,12,13),(2,6,3,7),(4,8,5,9)],[(1,11,13,10,12),(2,8,3,9),(4,6,5,7)]$ has 0 8-cycles and contains a sub-2-factorization of order 5.

(ii) The 2-factorization of K_{13} given by
$[(1,2,3,4,5),(6,10,7,11,8,12,9,13)],[(1,3,5,2,4),(6,11,9,10,8,13,7,12)]$,
$[(1,6,7,8,9),(2,10,3,11,4,12,5,13)],[(1,7,9,6,8),(2,11,5,10,4,13,3,12)]$,
$[(1,10,11,12,13),(2,6,3,7,4,8,5,9)],[(1,11,13,10,12),(2,7,5,6,4,9,3,8)]$ has 6 8-cycles and contains a sub-2-factorization of order 5.

(iii) For $\{2,4\} \subseteq Q(K_{13})$ take $r = 1$, $t = 3$, and $v = 4$ in Construction A. Since $Q(K_{4,4}) = \{0,2\}$, it follows immediately that $\{2,4\} \subseteq Q(K_{13})$.

Now take K_{13} to have vertex set $\{A, B, C\} \cup (\{1,2\} \times Z_5)$ in (iv), (v), and (vi).

(iv) Let $F = [(A, (1,3), (2,1), (2,4), (2,0), (1,4), (1,2), (1,1), (2,3), B, (1,0), C, (2,2))]$. Then $\{F + x \mid x = 0,1,2,3\}$ with the following 2-factors
$[(A, B, (1,4), C, (2,1)), ((1,0), (1,1), (2,0), (1,2), (2,2), (1,3), (2,3), (2,4))]$ and
$[(A, C, B, (2,2), (1,0), (2,0), (2,3), (1,4), (2,4), (1,3), (1,1), (2,1), (1,2))]$
is a 2-factorization of K_{13} containing 1 8-cycle.

(v) Now let
$F = [(A, (1,4), (1,1), (1,0), (2,1)), (B, (1,3), C, (2,0), (1,2), (2,4), (2,2), (2,3))]$.
Then $\{F + x \mid x \in Z_5\}$ with the following 2-factor
$[(A, B, C), ((1,0), (2,0), (1,1), (2,1), (1,2), (2,2), (1,3), (2,3), (1,4), (2,4))]$
is a 2-factorization of K_{13} containing 5 8-cycles.

(vi) Finally, the union of F and $F + 1$ in (v) can be decomposed into 2 2-factors as follows:
$[(A, (1,0), (1,2), (1,1), (1,4), B, (2,4), (2,2), (2,3), (2,0), C, (1,3), (2,1))]$ and
$[(A, (1,4), C, (2,1), (1,0), (1,1), (2,2)), (B, (1,3), (2,0), (1,2), (2,4), (2,3))]$.

This reduces the number of 8-cycles by 2. Hence $3 \in Q(13)$.

Lemma 2.8 $FC(16k + 13) \subseteq Q(16k + 13)$.

Proof: Take $r = 5$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$, $Q(13) = FC(13)$ and $m_i \in \{0,6\}$ for $i = 2,3,...,t$, Corollary 2.1 gives $FC(16k+13) \subseteq Q(16k + 13)$.

n≡15 (mod16)

Example 2.9 $Q(15) = FC(15)$, *where the 2-factorizations of* K_{15} *having* 0 *or* 7 *8-cycles contain a sub-2-factorization of order* 7.

Proof: (i) The 2-factorization of K_{15} given by
$[(1,4,3,6,7,2,5), (8,12,9,13), (10,14,11,15)]$,
$[(1,6,2,4,5,3,7), (8,14,9,15), (10,12,11,13)]$,
$[(1,8,3,10,11,2,9), (4,13,5,12), (6,14,7,15)]$,
$[(1,10,2,8,9,3,11), (4,14,5,15), (6,12,7,13)]$,
$[(1,12,3,14,15,2,13), (4,8,5,9), (6,10,7,11)]$,
$[(1,14,2,12,13,3,15), (4,10,5,11), (6,8,7,9)]$, and
$[(1,2,3), (4,6,5,7), (8,10,9,11), (12,14,13,15)]$ has 0 8-cycles and contains a sub-2-factorization of order 7.

(ii) For $\{2,4,6\} \subseteq Q(15)$ take $r = 3$, $t = 3$, and $v = 4$ in Construction A. It follows that $\{2,4,6\} \subseteq Q(15)$.

(iii) $1 \in Q(15)$:
$F_1 = [(1,4,3,6,7,2,5), (8,15,13,9,11,14,10,12)]$,
$F_2 = [(1,6,2,4,5,3,7), (8,10,13,14), (9,12,11,15)]$,
$F_3 = [(1,8,3,10,11,2,9), (4,14,5,12), (6,13,7,15)]$,
$F_4 = [(1,10,2,8,9,3,11), (4,13,5,15), (6,12,7,14)]$,
$F_5 = [(1,12,3,14,15,2,13), (4,10,5,8), (6,9,7,11)]$,
$F_6 = [(1,14,2,12,13,3,15), (4,9,5,11), (6,8,7,10)]$,
$F_7 = [(1,2,3), (4,6,5,7), (8,11,13), (10,9,14,12,15)]$.

(iv) The union of F_3 and F_4 in (iii) can be decomposed into the following two 2-factors:
$F_3' = [(1,8,3,10,11,2,9),(4,12,5,14,6,13,7,15)]$ and
$F_4' = [(1,10,2,8,9,3,11),(4,14,7,12,6,15,5,13)]$.
This increases the number of 8-cycles by 2. Hence $3 \in Q(15)$.

(v) The union of F_5 and F_6 in (iii) can be decomposed into the following two 2-factors:
$F_5' = [(1,12,3,14,15,2,13),(4,8,5,10,6,9,7,11)]$ and
$F_6' = [(1,14,2,12,13,3,15),(4,10,7,8,6,11,5,9)]$
Then $\{F_1, F_2, F_3', F_4', F_5', F_6', F_7\}$ is a 2-factorization of K_{15} containing 5 8-cycles.

(vi)Finally replace the two 2-factors F_2 and F_7 in (v) by the following two 2-factors:
$F_2' = [(1,6,2,4,5,3,7),(8,11,13,14,12,15,9,10)]$ and
$F_7' = [(1,2,3),(4,6,5,7),(8,13,10,15,11,12,9,14)]$. Hence $7 \in Q(15)$.

Lemma 2.10 $FC(16k+15) \subseteq Q(16k+15)$.

Proof: Take $r = 7$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$, $Q(15) = FC(15)$ and $m_i \in \{0,7\}$ for $i = 2,3,...,t$, Corollary 2.1 gives $FC(16k+15) \subseteq Q(16k+15)$.

3 n≡ 1,3,5 or 7 (mod 16)

We will begin with the following construction.

Construction B:

Write $n = tv + r$, where v and t are even and $r \in \{1,3,5,7\}$. Let $X = \{1,2,...,t\}$, $V = \{1,2,...,v\}$, and Z be a set of size r. Further, let $(X, \circ)$ be a commutative quasigroup of order $t \geq 6$ with holes $H = \{h_1, h_2, ..., h_{t/2}\}$ of size 2 [4] and set $S = Z \cup (X \times V)$.

Define a collection F of 2-factors of K_{tv+r} as follows:

(1) For the hole $h_1 \in H$, let $(Z \cup (h_1 \times \{1,2,...,v\}), F_1)$ be any 2-factorization of K_{2v+r}, where $F_1 = \{f_{1_1}, f_{1_2}, ..., f_{1_{v+(r-1)/2}}\}$.

(2) For each hole $h_i \in H \setminus \{h_1\}$, let $(Z \cup (h_i \times \{1,2,...,v\}), F_i)$ be any 2-factorization of K_{2v+r} having either 0 or max$FC(2v+r)$ 8-cycles and containing a sub-2-factorization of order r, where max$FC(2v+r)$ is the largest value in the set $FC(2v+r)$. Let $F_i = \{f_{i_1}, f_{i_2}, ..., f_{i_{v+(r-1)/2}}\}$, where the last $(r-1)/2$ 2-factors contain the sub-2-factorization of order r.

(3) For each $x \in X$, set $F(x) = \{\{a,b\} \mid a \neq b, a \circ b = b \circ a = x$, and a and b do not belong to the hole containing $x\}$. Denote by $(K_{a,b}, f_x(a,b))$, $\{a,b\} \in F(x)$, any 2-factorization of $K_{v,v}$ with parts $\{a\} \times \{1,2,...,v\}$ and $\{b\} \times \{1,2,...,v\}$, where $f_x(a,b) = \{f_{x_1}(a,b), f_{x_2}(a,b), ..., f_{x_{v/2}}(a,b)\}$.

(4) For each hole $h_i = \{x,y\} \in H$, each of the following is a 2-factor of K_{tv+r}:

$$\begin{cases} \{f_{i_j}\} \cup \{f_{x_j}(a,b) \mid \{a,b\} \in F(x)\}, & j = 1,2,...,v/2, \\ \{f_{i_k}\} \cup \{f_{y_j}(c,d) \mid \{c,d\} \in F(y)\}, & j = 1,2,...,v/2 \text{ and } k = v/2, (v/2)+1, ..., v. \end{cases}$$

(5) Piece together the remaining $(r-1)/2$ 2-factors of F_1, along with the remaining $(r-1)/2$ 2-factors of each F_x, for $x = 2, 3, ..., t$, making sure to delete the cycles belonging to the sub-2-factorization from each of the remaining 2-factors in each F_x.

(6) For each hole in H, place the v 2-factors in (4) in F as well as the 2-factors in (5).

The union of the 2-factors in (6) gives a total of $\sum_{h \in H}(v)+(r-1)/2 = (tv+r-1)/2$ 2-factors which form a 2-factorization of K_{tv+r} with vertex set S. □

Corollary 3.1 *Construction B gives a 2-factorization of K_{tv+r} containing exactly $\sum_{i=1}^{t(t-2)/2} n_i + \sum_{i=1}^{t/2} m_i$ 8-cycles, where $n_i \in Q(K_{v,v})$, $m_1 \in Q(2v+r)$, and $m_i \in \{0, \max FC(2v+r)\}$ for $i = 2, 3, ..., t/2$.* □

We will now use Construction B and Corollary 3.1 to show that $FC(n) \subseteq Q(n)$ for the cases $n \equiv 1, 3, 5$ and 7 (mod 16).

n≡1 (mod16)

Example 3.2 $Q(17) = FC(17)$.

Proof: (i) Take K_{17} to have vertex set $\{A, B, C, D, E, F, G\} \cup (\{1, 2\} \times Z_5)$ and let $F = [(A, (1,0), B, (2,0), (1,3), C, (2,3), F, (1,4), G, (2,4), (1,1), (2,2), E, (1,2), D, (2,1))]$. Then $\{F + x \mid x \in Z_5\}$ with the following three 2-factors

$F_1 = [(A, B, C, D, E, F, G), ((1,0), (1,1), (1,2), (1,3), (1,4)), ((2,0), (2,1), (2,2), (2,3), (2,4))]$,

$F_2 = [(A, C, E, G, B, D, F), ((1,0), (1,2), (1,4), (1,1), (1,3)), ((2,0), (2,2), (2,4), (2,1), (2,3))]$,

$F_3 = [(A, D, G, C, F, B, E), ((1,0), (2,0), (1,1), (2,1), (1,2), (2,2), (1,3), (2,3), (1,4), (2,4))]$

is a 2-factorization of K_{17} containing 0 8-cycles.

(ii) The union of F and F_1 in (i) can be decomposed into the following two 2-factors:

$F' = [(A, B, C, D, (1,2), E, F, G), ((1,0), (1,1), (2,4), (2,3), (2,2), (2,1), (2,0), (1,3), (1,4))]$,

$F_1' = [(A, (1,0), B, (2,0), (2,4), G, (1,4), F, (2,3), C, (1,3), (1,2), (1,1), (2,2), E, D, (2,1))]$.

This increases the number of 8-cycles by 1. Hence $1 \in Q(17)$.

(iii) $2 \in Q(17)$:

$F_1 = [(9, 11, 13, 15, 17, 10, 12, 14, 16), (1, 6, 4, 5, 3, 8, 2, 7)]$,

$F_2 = [(9, 12, 15), (10, 13, 16), (11, 14, 17), (1, 5, 2, 6, 3, 7, 4, 8)]$,

$F_3 = [(9, 13, 17, 12, 16, 11, 15, 10, 14), (1, 2, 3, 4)(5, 6, 7, 8)]$,

$F_4 = [(9, 1, 10, 2, 11, 3, 12, 4, 13, 5, 14, 6, 15, 7, 16, 17, 8)]$,

$F_5 = [(9, 2, 17, 1, 16, 15, 8, 14, 7, 13, 6, 12, 5, 11, 4, 10, 3)]$,

$F_6 = [(9, 4, 17, 3, 16, 2, 15, 14, 1, 13, 8, 12, 7, 11, 6, 10, 5)]$,

$F_7 = [(9, 6, 17, 5, 16, 4, 15, 3, 14, 13, 2, 12, 1, 11, 8, 10, 7)]$,

$F_8 = [(9, 10, 11, 12, 13, 3, 1, 15, 5, 7, 17), (14, 2, 4), (16, 6, 8)]$.

(iv) $3 \in Q(17)$:

The union of F_2 and F_3 in (iii) can be decomposed into the following two 2-factors:
$F_2' = [(9,12,15),(10,13,16),(11,14,17),(1,8,5,2,6,3,7,4)]$ and
$F_3' = [(9,13,17,12,16,11,15,10,14),(1,5,6,7,8,4,3,2)]$.
This increases the number of 8-cycles by 1. Hence $3 \in Q(17)$.

(v) $4 \in Q(17)$:
$F_1 = [(1,2,3,16,15,14,13,17,9,8,12,11,10,4,5,6,7)]$,
$F_2 = [(1,3,5,7,2,4,6),(8,10,12,9,11),(13,15,17,14,16)]$,
$F_3 = [(1,4,7,3,6,2,5),(8,13,9,14,10,15,11,16,12,17)]$,
$F_4 = [(1,15,2,12,3,4,16,17,6,9,10),(8,5,14),(11,13,7)]$.
$F_5 = [(1,16,2,8,3,17,4,11),(9,5,15),(10,6,13),(12,14,7)]$,
$F_6 = [(1,17,2,9,3,13,4,12),(10,5,16),(11,6,14),(8,15,7)]$,
$F_7 = [(1,13,2,10,3,14,4,8),(11,5,17),(12,6,15),(9,16,7)]$,
$F_8 = [(1,14,2,11,3,15,4,9),(12,5,13),(8,6,16),(10,17,7)]$.

(vi) $5 \in Q(17)$:
The union of F_1 and F_4 in (v) can be decomposed into the following two 2-factors:
$F_1' = [(1,2,3,4,5,6,7),(8,9,10,11,12),(13,14,15,16,17)]$ and
$F_4' = [(1,15,2,12,3,16,4,10),(8,5,14),(9,6,17),(11,13,7)]$,
This increases the number of 8-cycles by 1. Hence $5 \in Q(17)$.

(vii) $6 \in Q(17)$:
Take K_{17} to have vertex set $\{A,B,C,D,E\} \cup (\{1,2\} \times Z_6)$ and let
$F = [(C,(1,2),(2,0),(2,5),(1,3),(1,1),(1,0),(2,3)),(A,(1,4),B,(2,1),D,(1,5),E,(2,4),(2,2))]$.
Then $\{F + x \mid x \in Z_6\}$ with the following two 2-factors:
$[(A,B,C,D,E),((1,0),(2,0),(2,3),(1,3)),((1,1),(2,1),(2,4),(1,4)),((1,2),(2,2),(2,5),(1,5))]$ and
$[(A,C,E,B,D),((1,0),(2,1),(1,2),(2,3),(1,4),(2,5)),((1,1),(2,0),(1,5),(2,4),(1,3),(2,2))]$.
is a 2-factorization of K_{17} containing 6 8-cycles.

(viii) $7 \in Q(17)$:
Now take K_{17} to have vertex set $\{A,B,C\} \cup (\{1,2\} \times Z_7)$ and let
$F = [(B,(1,0),(2,2),(2,3),(1,5),(1,6),(1,2),(2,5)),(A,(1,4),(2,1),(2,6)),(C,(1,1),(1,3),(2,4),(2,0))]$.
Then $\{F + x \mid x \in Z_7\}$ with the following 2-factor:
$[((1,0),(2,0),(1,1),(2,1),(1,2),(2,2),(1,3),(2,3),(1,4),(2,4),(1,5),(2,5),(1,6),(2,6)),(A,B,C)]$
is a 2-factorization of K_{17} containing 7 8-cycles.

(ix) $8 \in Q(17)$:
Take K_{17} to have vertex set $\{A\} \cup Z_{16}$ and let
$F = [(2,5,7,6,10,13,15,14),(1,8,3,9,0,11),(A,4,12)]$. Then $\{F + x \mid x = 0,1,2,3,4,5,6,7\}$ (mod16) is a 2-factorization of K_{17} containing 8 8-cycles.

Example 3.3 *$K_{10,10}$ can be 2-factorized into 0 or 10 8-cycles.*

Proof: See Appendix

Example 3.4 *K_{33} can be 2-factorized into $FC(33) \setminus \{47\}$ 8-cycles.*

Proof: See Appendix

Lemma 3.5 *$FC(16k+1) \subseteq Q(16k+1)$, with the possible exception of $47 \in FC(33)$.*

Proof: Take $r = 1$, $t = 2k$ and $v = 8$ in Construction B. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$ and $Q(17) = FC(17)$, Corollary 3.1 gives $FC(16k+1) \subseteq Q(16k+1)$ for $k \geq 3$. Examples 3.2 and 3.4 complete the proof.

n≡3 (mod16)

Example 3.6 *$K_{6,6}$ can be 2-factorized into $0, 1,$ or 3 8-cycles.*

Proof: See Appendix

Example 3.7 $Q(19) = FC(19)$.

Proof: See Appendix

Lemma 3.8 $FC(16k+3) \subseteq Q(16k+3)$.

Proof: Take $r = 3$, $t = 4k$ and $v = 4$ in Construction B. Since $n_i \in \{0,2\}$, $m_1 \in Q(11)$ and $m_i \in \{0,5\}$ for $i = 2,3,...,2k$, Corollary 3.1 gives $FC(16k+3) \subseteq Q(16k+3)$ for $k \geq 2$. Example 3.7 completes the proof.

n≡5 (mod16)

Example 3.9 $Q(21) = FC(21)$.

Proof: See Appendix

Lemma 3.10 $FC(16k+5) \subseteq Q(16k+5)$.

Proof: Take $r = 5$, $t = 4k$ and $v = 4$ in Construction B. Since $n_i \in \{0,2\}$, $m_1 \in Q(13)$ and $m_i \in \{0,6\}$ for $i = 2,3,...,2k$, Corollary 3.1 gives $FC(16k+5) \subseteq Q(16k+5)$ for $k \geq 2$. Example 3.9 completes the proof.

n≡7 (mod16)

Example 3.11 $Q(23) = FC(23)$, *where the 2-factorizations of K_{23} having 0 and 22 8-cycles contain sub-2-factorizations of order 7.*

Proof: (i) The following 2-factorization of K_{23} gives $0 \in Q(23)$.
$[(1,4,3,6,7,2,5),(8,22,10,20),(9,21,11,23),(12,16,14,18),(13,17,15,19)]$,
$[(1,6,2,4,5,3,7),(8,21,10,23),(9,20,11,22),(12,17,14,19),(13,16,15,18)]$,
$[(1,8,3,10,11,2,9),(12,22,14,20),(13,21,15,23),(4,18,5,16),(6,17,7,19)]$,
$[(1,10,2,8,9,3,11),(12,21,14,23),(13,20,15,22),(4,17,5,19),(6,16,7,18)]$,
$[(1,12,3,14,15,2,13),(16,22,18,20),(17,21,19,23),(4,8,5,10),(6,9,7,11)]$,
$[(1,14,2,12,13,3,15),(16,21,18,23),(17,20,19,22),(4,9,5,11),(6,8,7,10)]$,
$[(1,16,3,18,19,2,17),(8,12,10,14),(9,13,11,15),(4,20,5,22),(6,21,7,23)]$,
$[(1,18,2,16,17,3,19),(8,13,10,15),(9,12,11,14),(4,21,5,23),(6,20,7,22)]$,
$[(1,20,3,22,23,2,21),(8,18,10,16),(9,17,11,19),(4,14,5,12),(6,13,7,15)]$,
$[(1,22,2,20,21,3,23),(8,17,10,19),(9,16,11,18),(4,13,5,15),(6,12,7,14)]$,
$[(1,2,3),(4,6,5,7),(8,10,9,11),(12,14,13,15),(16,18,17,19),(20,22,21,23)]$.

(ii) Take $r = 5$, $t = 3$ and $v = 6$ in Construction A. It follows that $FC(23) \setminus \{21,22\} \subseteq Q(23)$.

(iii) The 2-factorization of K_{23} given by
$F_1 = [(1,6,2,4,5,3,7),(8,10,9,11,23,21,22,20),(12,16,14,18,13,17,15,19)]$,
$F_2 = [(1,4,3,6,7,2,5),(8,22,11,20,9,23,10,21),(12,18,15,16,13,19,14,17)]$,
$F_3 = [(1,8,3,10,11,2,9),(12,22,15,20,13,23,14,21),(4,18,7,16,6,19,5,17)]$,
$F_4 = [(1,10,2,8,9,3,11),(12,20,14,22,13,21,15,23),(4,16,5,18,6,17,7,19)]$,
$F_5 = [(1,12,3,14,15,2,13),(16,22,19,20,17,23,18,21),(4,10,7,8,6,11,5,9)]$,
$F_6 = [(1,14,2,12,13,3,15),(16,20,18,22,17,21,19,23),(4,8,5,10,6,9,7,11)]$,
$F_7 = [(1,16,3,18,19,2,17),(8,14,11,12,9,15,10,13),(4,22,7,20,6,23,5,21)]$,
$F_8 = [(1,18,2,16,17,3,19),(8,12,10,14,9,13,11,15),(4,20,5,22,6,21,7,23)]$,
$F_9 = [(1,20,3,22,23,2,21),(8,18,11,16,9,19,10,17),(4,14,7,12,6,15,5,13)]$,
$F_{10} = [(1,22,2,20,21,3,23),(8,16,10,18,9,17,11,19),(4,12,5,14,6,13,7,15)]$,
$F_{11} = [(1,2,3),(4,6,5,7),(12,14,13,15),(16,18,17,19),(8,23,20,10,22,9,21,11)]$.
shows that $21 \in Q(23)$.

(iv) Finally the union of F_1 and F_{11} in (iii) can be decomposed into two 2-factors as follows:
$F_1' = [(1,6,2,4,5,3,7),(8,10,9,11,23,21,22,20),(12,14,13,15,19,17,18,16)]$ and
$F_{11}' = [(1,2,3),(4,6,5,7),(8,23,20,10,22,9,21,11),(12,19,16,14,18,13,17,15)]$.
This increases the number of 8-cycles by 1. Hence $22 \in Q(23)$.

Example 3.12 *$K_{12,12}$ can be 2-factorized into 0 or 18 8-cycles.*

Proof: See Appendix

Example 3.13 $Q(39) = FC(39)$.

Proof: See Appendix

Lemma 3.14 $FC(16k+7) \subseteq Q(16k+7)$.

Proof: Take $r = 7$, $t = 2k$ and $v = 8$ in Construction B. Since $n_i \in \{0,1,2,3,4,5,6,7,8\}$, $m_1 \in Q(23)$ and $m_i \in \{0,22\}$ for $i = 2,3,...,k$, Corollary 3.1 gives $FC(16k+7) \subseteq Q(16k+7)$ for $k \geq 3$. Examples 3.11 and 3.13 complete the proof.

4 Summary

We summarize our results with the following theorem.

Theorem 4.1 *$Q(n) = FC(n)$ for all odd n with the possible exception of $47 \in FC(33)$.* □

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Appendix

Example 3.3 $K_{10,10}$ can be 2-factorized into 0 or 10 8-cycles.
Proof: (i) $0 \in Q(K_{10,10})$:

Take a Hamilton decomposition of $K_{10,10}$.

(ii) $10 \in Q(K_{10,10})$:

Let $\{1,2,3,4,5,6,7,8,9,10\}$ and $\{11,12,13,14,15,16,17,18,19,20\}$ be the parts of $K_{10,10}$. The following 2-factors form a 2-factorization of $K_{10,10}$ containing 10 8-cycles.
$[(1,11,2,12,3,13,4,14),(5,15,6,16,7,17,8,18),(9,19,10,20)]$,
$[(1,13,2,14,3,15,4,16),(5,17,6,18,7,19,8,20),(9,11,10,12)]$,
$[(1,15,2,16,3,17,4,18),(5,12,8,11,7,20,6,19),(9,13,10,14)]$,
$[(1,17,2,18,3,19,4,20),(5,11,6,12,7,13,8,14),(9,15,10,16)]$,
$[(1,12,4,11,3,20,2,19),(5,13,6,14,7,15,8,16),(9,17,10,18)]$.

Example 3.4 K_{33} can be 2-factorized into $FC(33)\setminus\{47\}$ 8-cycles.
Proof: (i) Take $r=3$, $t=3$ and $v=10$ in Construction A. Since a 2-factorization of K_{13} containing a cycle of length 3 can not have 6 8-cycles, in step (2) of the construction for each $x \in \{2,3\}$ place a 2-factorization of K_{13} having either 0 or 5 8-cycles and containing a cycle of length 3. (The 2-factorization of K_{13} having 5 8-cycles in Example 2.7 contains a 3-cycle. For a 2-factorization of K_{13} having 0 8-cycles and containing a cycle of length 3, replace F in Example 2.7(v) by $[(A,(1,2),(2,0),(2,3),(2,4),(1,3),(1,1),(1,0),(2,2),B,(1,4),C,(2,1))]$. Then it follows that $FC(33)\setminus\{47,48\} \subseteq Q(33)$.

(ii) Now, take $r=1$, $t=8$ and $v=4$ in Construction B. In step (3) for each $x \in X$, let $(K_{a,b}, f_x(a,b))$ be any 2-factorization of $K_{4,4}$ containing 2 8-cycles. This gives $48 \in Q(33)$.

Example 3.6 $K_{6,6}$ can be 2-factorized into 0, 1, or 3 8-cycles.
Proof: Let $\{1,2,3,4,5,6\}$ and $\{7,8,9,10,11,12\}$ be the parts of $K_{6,6}$.

(i) $0 \in Q(K_{6,6})$:

$[(1,7,2,8,3,9,4,10,5,11,6,12)]$, $[(1,8,6,7,5,12,4,11,3,10,2,9)]$,
$[(1,10,6,9,5,8,4,7,3,12,2,11)]$.

(ii) $1 \in Q(K_{6,6})$:

$[(3,7,4,8,5,9,6,10),(1,11,2,12)]$, $[(1,9,3,11,4,10),(2,7,5,12,6,8)]$,
$[(1,7,6,11,5,10,2,9,4,12,3,8)]$.

(iii) $3 \in Q(K_{6,6})$:

$[(3,9,4,10,5,11,6,12),(1,7,2,8)]$, $[(3,8,6,7,5,12,4,11),(1,9,2,10)]$,
$[(3,7,4,8,5,9,6,10),(1,11,2,12)]$.

Example 3.7 $Q(19)=FC(19)$.
Proof: (i) Take $r=1$, $t=3$ and $v=6$ in Construction A. It follows that $\{0,1,2,3,4,5,6,7,9\} \subseteq Q(19)$.

(ii) Now, take K_{19} to have vertex set $(\{1,2\}\times Z_8)\cup\{A,B,C\}$ and let
$F=[(B,(2,2),(2,5),(2,7),(1,4),(1,3),(2,1),(1,0))$,

$(A,(2,4),(2,3),C,(1,6),(2,0),(1,1),(2,6),(1,2),(1,7),(1,5))]$.
Then $\{F+x \mid x \in Z_8\}$ with the following 2-factor
$[(A,B,C),((1,0),(2,0),(2,4),(1,4)),((1,1),(2,1),(2,5),(1,5)),$
$((1,2),(2,2),(2,6),(1,6))((1,3),(2,3),(2,7),(1,7))]$
is a 2-factorization of K_{19} containing 8 8-cycles.

(iii)Take K_{19} to have vertex set $\{A,B,C,D,E,F,G\} \cup (\{1,2\} \times Z_6)$ and let
$F=[(A,(2,2),(1,0),B,(2,3),(2,1),(1,3),(1,5)),((1,1),G,(2,4)),$
$(C,(2,5),F,(1,4),E,(2,0),D,(1,2))]$.
Then $\{F+x \mid x \in Z_6\}$ with the following 3 2-factors:
$F_1=[(A,B,C,D,E,F,G),((1,0),(1,1),(1,2),(1,3),(1,4),(1,5)),$
$((2,0),(2,1),(2,2),(2,3),(2,4),(2,5))]$,
$F_2=[((1,0),(2,0),(1,1),(2,1),(1,2),(2,2),(1,3),(2,3),(1,4),(2,4),(1,5),(2,5)),$
$(A,E,B,F,C,G,D)]$,
$F_3=[(A,C,E,G,B,D,F),((1,0),(2,1),(2,4),(1,3)),((1,1),(2,2),(2,5),(1,4)),$
$((1,2),(2,3),(2,0)(1,5))]$
is a 2-factorization of K_{19} containing 12 8-cycles.

(iv) The union of F and F_1 in (iii) can be decomposed into 2 2-factors as follows:
$[(A,B,C,(2,5),(2,4),(2,3),(2,2),(2,1),(2,0),D,E,F,G),$
$((1,0),(1,1),(1,2),(1,3),(1,4),(1,5))]$ and
$[(A,(2,2),(1,0),B,(2,3),(2,1),(1,3),(1,5)),(C,D,(1,2)),$
$((E,(1,4),F,(2,5),(2,0)),((1,1),G,(2,4))]$.
This reduces the number of 8-cycles by 1. Hence $11 \in Q(19)$.

(v) Now consider again F and F_1 in (iii). Their union can be decomposed into 2 2-factors as follows:
$[(A,(2,2),(1,0),(1,1),G,(2,4),(2,5),F,(1,4),E,(2,0),D,(1,2),C,B,(2,3),(2,1),$
$(1,3),(1,5))]$ and
$[(A,B,(1,0),(1,5),(1,4),(1,3),(1,2),(1,1),(2,4),(2,3),(2,2),(2,1),(2,0),(2,5),C,$
$D,E,F,G)]$.
This reduces the number of 8-cycles by 2. Hence $10 \in Q(19)$.

(vi) Now take K_{19} to have vertex set $\{A,B,C,D,E\} \cup (\{1,2\} \times Z_7)$. Let
$F=[(A,(1,6),(2,2),(1,0),(1,2),(2,3),(2,5),(2,1)),(E,(2,0),(1,3)),$
$(B,(1,4),(1,1),(2,6),D,(1,5),C,(2,4))]$.
Then $\{F+x \mid x \in Z_7\}$ with the following 2 2-factors:
$F_1=[(A,D,E,B,C),((1,0),(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)),$
$((2,0),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6))]$ and
$F_2=[((1,0),(2,0),(1,1),(2,1),(1,2),(2,2),(1,3),(2,3),(1,4),(2,4),(1,5),(2,5),$
$(1,6),(2,6)),(A,B,D,C,E)]$.
is a 2-factorization of K_{19} containing 14 8-cycles.

(vii) The union of F and F_1 in (vi) can be decomposed into 2 2-factors as follows:
$[(A,(1,6),(2,2),(1,0),(1,2),(2,3),(2,5),(2,1)),$
$((1,1),(1,4),(1,3),(2,0),(2,6)),(B,E,D,(1,5),C,(2,4))]$ and
$[(A,D,(2,6),(2,5),(2,4),(2,3),(2,2),(2,1),(2,0),E,(1,3),(1,2),(1,1),(1,0),(1,6),$
$(1,5),(1,4),B,C)]$.
This reduces the number of 8-cycles by 1. Hence $13 \in Q(19)$.

(viii) Take K_{19} to have vertex set $(\{1,2\} \times Z_8) \cup \{A, B, C\}$ and let
$F = [(A,(1,1),(1,3),(2,5),C,(1,2),(1,5),(2,2)),((1,4),(2,7),(2,0)),$
$(B,(1,0),(2,1),(2,3),(2,6),(1,7),(1,6),(2,4))]$.
Then $\{F + x \mid x \in Z_8\}$ with the following 2-factor
$[(A,B,C),((1,0),(2,0),(2,4),(1,4)),((1,1),(2,1),(2,5),(1,5)),$
$((1,2),(2,2),(2,6),(1,6)),((1,3),(2,3),(2,7),(1,7))]$
is a 2-factorization of K_{19} containing 16 8-cycles.

(ix) The union of F and F_1 in (viii) can be decomposed into the following 2 2-factors:
$[(A,B,(2,4),(1,6),(1,7),(2,6),(2,2),(1,2),(1,5),(2,5),C),$
$((1,0),(2,0),(1,4),(2,7),(2,3),(1,3),(1,1),(2,1))]$ and
$[(B,C,(1,2),(1,6),(2,6),(2,3),(2,1),(2,5),(1,3),(1,7),(2,7),(2,0),(2,4),(1,4),$
$(1,0)),(A,(1,1),(1,5),(2,2))]$.
This reduces the number of 8-cycles by 1. Hence $15 \in Q(19)$.

(x) Now take K_{19} to have vertex set $\{A\} \cup (\{1,2\} \times Z_9)$ and let
$F = [(A,(2,2),(1,7)),((1,0),(2,1),(2,3),(1,4),(2,6),(1,8),(1,5),(2,5)),$
$((1,1),(1,3),(1,2),(1,6),(2,0),(2,8),(2,4),(2,7))]$.
Then $\{F + x \mid x \in Z_9\}$ is a 2-factorization of K_{19} containing 18 8-cycles.

(xi) $17 \in FC(19)$:
$[(1,2,3,4,5,6,7,19),(8,9,10,11,12,13,14,15),(16,17,18)]$,
$[(2,4,1,3,5,7,8,19),(6,9,11,13,10,16,14,17),(12,15,18)]$,
$[(3,6,1,5,2,7,4,19),(8,10,12,9,13,16,15,17),(11,14,18)]$,
$[(5,8,1,7,3,9,14,19),(2,6,10,15,11,16,12,17),(4,13,18)]$,
$[(6,4,8,2,9,1,11,19),(3,15,13,17,10,18,5,16),(7,12,14)]$,
$[(9,4,10,1,12,2,13,19),(3,14,5,15,6,16,8,18),(7,11,17)]$,
$[(10,2,11,3,12,4,15,19),(1,13,5,17,9,18,7,16),(6,8,14)]$,
$[(12,5,10,7,15,9,16,19),(1,14,2,18,6,11,4,17),(3,8,13)]$,
$[(5,9,7,13,6,12,8,11),(19,17,3,10,14,4,16,2,15,1,18)]$.

Example 3.9 $Q(21) = FC(21)$.
Proof: (i) Take $r = 1$, $t = 5$ and $v = 4$ in Construction A. It follows that $\{0,2,4,6,8,10,12,14,16,18,20\} \subseteq Q(21)$.

(ii) Now take $r = 3$, $t = 3$ and $v = 6$ in Construction A. It follows that $\{1,3,5,7,9\} \subseteq Q(21)$.

(iii) Now take K_{21} to have vertex set $\{A,B,C,D,E,F,G\} \cup (\{1,2\} \times Z_7)$. Let
$F = [(A,(1,4),(2,1),(1,5),(2,4),(2,3),(1,1),(2,6)),(B,(1,2),(1,3),C,(2,2)),$
$(D,(1,6),G,(2,0),F,(1,0),E,(2,5))]$.
Then $\{F + x \mid x = 0,1,2,3,4,5\}$ with the following 4 2-factors:
$F_1 = [(A,F,D,B,G,E,C),((1,0),(1,2),(1,4),(1,6),(1,1),(1,3),(1,5)),$
$((2,0),(2,2),(2,4),(2,6),(2,1),(2,3),(2,5))]$,
$F_2 = [(A,E,B,F,C,G,D),((1,0),(2,0),(1,5),(2,5),(1,3),(2,3),(1,1),$
$(2,1),(1,6),(2,6),(1,4),(2,4),(1,2),(2,2))]$
$F_3 = [(A,(1,3),(2,0),(1,4),(2,3),(2,2),(1,0),(2,5)),$
$((B,C,(2,1),(2,4),D,E,(1,6),F,(2,6),G,(1,5),(1,2),(1,1))]$, and

$F_4 = [(A, B, (2,1), (2,5), (2,2), (2,6), (2,3), (2,0), (2,4), E, F, G),$
$((1,0), (1,3), (1,6), (1,2), C, D, (1,5), (1,1), (1,4))]$
is a 2-factorization of K_{21} containing 13 8-cycles.

(iv) Now consider $\{F + x \mid x = 1,2,3,4,5\}, F_1, F_2, F_4$ in (iii) and the following 2 2-factors:
$[(A, (1,3), (1,2), (1,5), (2,4), (2,1), (1,4), (2,3), (1,1), (2,6)), (B, C, (2,2)),$
$(D, (1,6), G, (2,0), F, (1,0), E, (2,5))]$ and
$[(A, (1,4), (2,0), (1,3), C, (2,1), (1,5), G, (2,6), F, (1,6), E, D, (2,4), (2,3), (2,2),$
$(1,0), (2,5)), (B, (1,1), (1,2))]$.
This gives a 2-factorization of K_{21} containing 11 8-cycles.

(v) Take K_{21} to have the vertex set $\{A, B, C, D, E\} \cup (\{1,2\} \times Z_8)$. Let
$F = [(A, (1,7), E, (2,7), D, (1,5), C, (2,3)), ((1,2), (2,1), (2,4), (1,0), (2,5)),$
$(B, (2,0), (2,6), (1,4), (2,2), (1,1), (1,3), (1,6))]$.
Then $\{F + x \mid x = 0,1,2,3,4,5,6\}$ with the following 3 2-factors:
$[(B, (1,5), (1,2), (1,0), (1,1), (2,4), (1,7), (2,3), (2,0), (2,1), (1,3), (2,5), (2,7)),$
$(A, (1,6), E, (2,6), D, (1,4), C, (2,2))]$,
$[((1,0), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,0), (1,1), (1,2), (1,3), (1,4),$
$(1,5), (1,6), (1,7)), (A, C, E, B, D)]$, and
$[((1,0), (2,0), (2,4), (1,4)), ((1,1), (2,1), (2,5), (1,5)), ((1,2), (2,2), (2,6), (1,6)),$
$((1,3, (2,3), (2,7), (1,7)), (A, B, C, D, E)]$,
is a 2-factorization of K_{21} containing 15 8-cycles.

(vi) Now take K_{21} to have the vertex set $\{A, B, C\} \cup (\{1,2\} \times Z_9)$. Let
$F = [(B, (1,1), (2,1), (1,2), (2,4), (2,5), (1,7), (2,2)), (C, (1,4), (2,0), (1,6), (2,7)),$
$(A, (1,3), (1,5), (1,8), (1,0), (2,6), (2,3), (2,8))]$.
Then $\{F + x \mid x = 0,1,2,3,4,5,6,7\}$ with the following 2 2-factors:
$[(B, (1,0), (2,0), (1,1), (2,3), (2,4), (1,6), (2,1)), ((2,2), (2,5), (2,7)),$
$(A, (1,2), (1,4), (1,8), (1,7), (1,3), (2,8), (1,5), (2,6), C)]$ and
$[(A, B, C, (1,3), (1,8), (2,5), (2,3), (2,1), (2,8), (2,6), (2,4), (2,2), (2,0), (2,7)),$
$((1,0), (1,5), (1,1), (1,6), (1,2), (1,7), (1,4))]$
is a 2-factorization of K_{21} containing 17 8-cycles.

(vii) $19 \in FC(21)$:
[(1, 2, 3, 4, 5, 6, 7, 21), (8, 9, 10, 11, 12, 13, 14, 15), (16, 17, 18, 19, 20)],
[(2, 4, 1, 3, 5, 7, 8, 21), (6, 9, 11, 13, 10, 12, 14, 16), (15, 18, 20, 17, 19)],
[(3, 6, 1, 5, 2, 7, 4, 21), (8, 10, 14, 9, 12, 17, 11, 18), (13, 19, 16, 15, 20)],
[(5, 8, 1, 7, 3, 9, 13, 21), (2, 6, 4, 10, 15, 17, 14, 19), (11, 16, 18, 12, 20)],
[(6, 8, 2, 9, 1, 10, 16, 21), (3, 11, 4, 15, 12, 19, 5, 17), (7, 13, 18, 14, 20)],
[(9, 4, 8, 3, 10, 2, 14, 21), (1, 15, 11, 19, 6, 18, 5, 20), (7, 12, 16, 13, 17)],
[(10, 5, 9, 15, 13, 1, 12, 21), (2, 16, 3, 18, 4, 17, 6, 20), (7, 11, 14, 8, 19)],
[(11, 1, 14, 3, 12, 2, 17, 21), (4, 13, 5, 16, 8, 20, 9, 19), (6, 10, 18, 7, 15)],
[(15, 3, 13, 6, 14, 4, 20), (2, 11, 5, 12, 8, 17, 9, 18), (1, 16, 7, 10, 19)],
[(18, 1, 17, 10, 20, 3, 19, 21), (2, 13, 8, 11, 6, 12, 4, 16, 9, 7, 14, 5, 15)].

Example 3.12 $K_{12,12}$ can be 2-factorized into 0 or 18 8-cycles.

Proof: (i) $0 \in Q(K_{12,12})$:

Take a Hamilton decomposition of $K_{12,12}$.

(ii) $18 \in Q(K_{12,12})$:

Let $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ and $\{13,14,15,16,17,18,19,20,21,22,23,24\}$ be the parts of $K_{12,12}$. The following 2-factors form a 2-factorization of $K_{12,12}$ containing 18 8-cycles.

$[(1,13,2,14,3,15,4,16),(9,21,10,22,11,23,12,24),(5,17,6,18,7,19,8,20)]$,
$[(1,14,4,13,3,16,2,15),(5,18,8,17,7,20,6,19),(9,22,12,21,11,24,10,23)]$,
$[(1,17,2,18,3,19,4,20),(5,21,6,22,7,23,8,24),(9,13,10,14,11,15,12,16)]$,
$[(1,18,4,17,3,20,2,19),(5,22,8,21,7,24,6,23),(9,14,12,13,11,16,10,15)]$,
$[(1,21,2,22,3,23,4,24),(5,13,6,14,7,15,8,16),(9,17,10,18,11,19,12,20)]$,
$[(1,22,4,21,3,24,2,23),(5,14,8,13,7,16,6,15),(9,18,12,17,11,20,10,19)]$.

Example 3.13 $Q(39) = FC(39)$.

Proof: (i) Take $r = 3$, $t = 3$ and $v = 12$ in Construction A. It follows that $FC(39) \setminus \{76\} \subseteq Q(39)$.

(ii) The 2-factorization of K_{39} given by
$[(1,4,3,6,7,2,5),(8,38,11,36,9,39,10,37),(12,34,15,32,13,35,14,33),$
$(16,29,18,31,17,28,19,30),(20,26,23,24,21,27,22,25)]$,
$[(1,6,2,4,5,3,7),(8,10,9,11,39,37,38,36),(12,14,13,15,35,33,34,32),$
$(16,18,17,19,31,29,30,28),(20,22,21,23,27,25,26,24)]$,
$[(1,8,3,10,11,2,9),(12,38,15,36,13,39,14,37),(16,34,19,32,17,35,18,33),$
$(20,30,23,28,21,31,22,29),(4,26,7,24,6,27,5,25)]$,
$[(1,10,2,8,9,3,11),(12,36,14,38,13,37,15,39),(16,32,18,34,17,33,19,35),$
$(20,28,22,30,21,29,23,31),(4,24,5,26,6,25,7,27)]$,
$[(1,12,3,14,15,2,13),(16,38,19,36,17,39,18,37),(20,34,23,32,21,35,22,33),$
$(24,30,27,28,25,31,26,29),(4,10,7,8,6,11,5,9)]$,
$[(1,14,2,12,13,3,15),(16,36,18,38,17,37,19,39),(20,32,22,34,21,33,23,35),$
$(24,28,26,30,25,29,27,31),(4,8,5,10,6,9,7,11)]$,
$[(1,16,3,18,19,2,17),(8,14,11,12,9,15,10,13),(20,38,23,36,21,39,22,37),$
$(24,34,27,32,25,35,26,33),(4,30,7,28,6,31,5,29)]$,
$[(1,18,2,16,17,3,19),(8,12,10,14,9,13,11,15),(20,36,22,38,21,37,23,39),$
$(24,32,26,34,25,33,27,35),(4,28,5,30,6,29,7,31)]$,
$[(1,20,3,22,23,2,21),(8,18,11,16,9,19,10,17),(4,14,7,12,6,15,5,13),$
$(24,38,27,36,25,39,26,37),(28,34,31,32,29,35,30,33)]$,
$[(1,22,2,20,21,3,23),(8,16,10,18,9,17,11,19),(4,12,5,14,6,13,7,15),$
$(24,36,26,38,25,37,27,39),(28,32,30,34,29,33,31,35)]$,
$[(1,24,3,26,27,2,25),(8,22,11,20,9,23,10,21),(12,18,15,16,13,19,14,17),$
$(28,38,31,36,29,39,30,37),(4,34,7,32,6,35,5,33)]$,
$[(1,26,2,24,25,3,27),(8,20,10,22,9,21,11,23),(12,16,14,18,13,17,15,19),$
$(28,36,30,38,29,37,31,39),(4,32,5,34,6,33,7,35)]$,
$[(1,28,3,30,31,2,29),(8,26,11,24,9,27,10,25),(12,22,15,20,13,23,14,21),$
$(32,38,35,36,33,39,34,37),(4,18,7,16,6,19,5,17)]$,

$[(1,30,2,28,29,3,31),(8,24,10,26,9,25,11,27),(12,20,14,22,13,21,15,23),$
$(32,36,34,38,33,37,35,39),(4,16,5,18,6,17,7,19)],$
$[(1,35,3,33,32,2,34),(8,30,11,28,9,31,10,29),(12,26,15,24,13,27,14,25),$
$(16,22,19,20,17,23,18,21),(4,38,7,36,6,39,5,37)],$
$[(1,33,2,35,34,3,32),(8,28,10,30,9,29,11,31),(12,24,14,26,13,25,15,27),$
$(16,20,18,22,17,21,19,23),(4,36,5,38,6,37,7,39)]$
$[(1,36,3,38,39,2,37),(8,34,11,32,9,35,10,33),(12,30,15,28,13,31,14,29),$
$(16,26,19,24,17,27,18,25),(4,22,7,20,6,23,5,21)],$
$[(1,38,2,36,37,3,39),(8,32,10,34,9,33,11,35),(12,28,14,30,13,29,15,31),$
$(16,24,18,26,17,25,19,27),(4,20,5,22,6,21,7,23)],$
$[(1,2,3),(4,6,5,7),(8,39,36,10,38,9,37,11),(12,35,32,14,34,13,33,15),$
$(16,31,28,18,30,17,29,19),(20,27,24,22,26,21,25,23)]$
shows that $76 \in Q(39)$.

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