New infinite classes of 1-factorizations of complete graphs

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Abstract

Some classes of 1-factorizations of complete graphs are known. They are GK_{2n}, AK_{2n}, WK_{2n} and their variations, and automorphism-free 1-factorizations. In this paper, for any positive integer t, we construct new 1-factorizations N_tK_{2n} which are defined for all $2n$ with $2n \geq 6t$. They also have some variations.

1. INTRODUCTION

Let $K_{2n} = (V_{2n}, E_{2n})$ be the complete graph on $2n$ vertices. Put $m = 2n - 1$. A 1-factor of K_{2n} is a set of n edges that partition the vertex set V_{2n}. A 1-factorization of K_{2n} is a set of m 1-factors that partition the set of edges E_{2n}.

Some 1-factorizations of K_{2n} are known [1,2,3,4]. They have been dubbed GK_{2n}, WK_{2n} and AK_{2n}. They have some variations; $G'K_{2n}$, $W'K_{2n}$, $W''K_{2n}$, $W^{(1)}K_{2n}$, $W^{(1)\prime}K_{2n}$, $W^{(1)\prime\prime}K_{2n}$ (Figures 1 to 10). Among these, GK_{2n} is the most simple and famous 1-factorization of K_{2n}; it is called the patterned 1-factorization.

Moreover, it is known there exists an automorphism-free 1-factorization of K_{2n} ($n \geq 5$) [1].

These 1-factorizations are defined for every $2n$ except a few small $2n$. On the other hand, not for every $2n$, various 1-factorizations have been constructed; for example, cyclic 1-factorizations when $2n \neq 2^k$, geometric 1-factorizations when $2n = 2^k$ and affine 1-factorizations when $2n = 3^k + 1$ ([2], p604).

In this paper, for any positive integer t, we construct a new 1-factorization N_tK_{2n} which is defined for all $2n$ with $2n \geq 6t$. And we show their variations; $N_t'K_{2n}$, $N_t''K_{2n}$, $N_t^{(1)}K_{2n}$, $N_t^{(1)\prime}K_{2n}$, $N_t^{(1)\prime\prime}K_{2n}$.

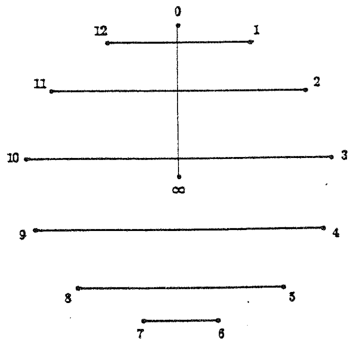

Figure 1: GK_{14}

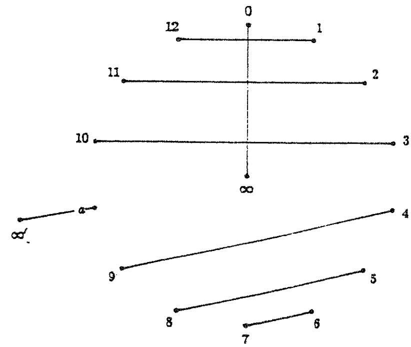

Figure 2: $G'K_{16}$

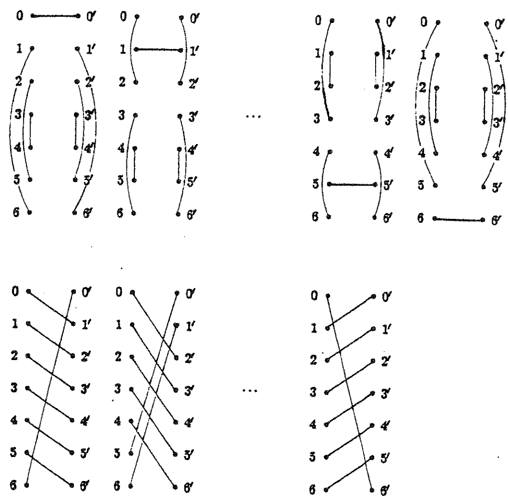

Figure 3: AK_{14}

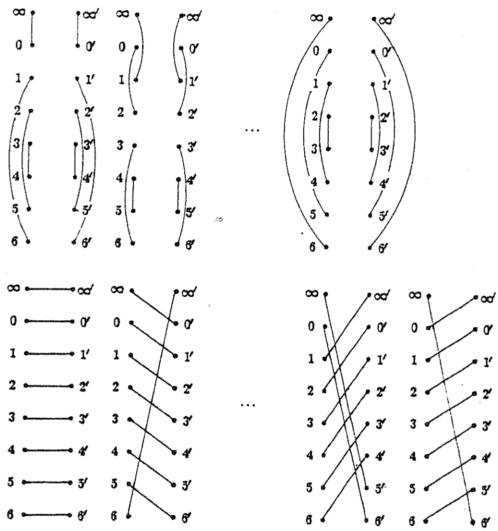

Figure 4: AK_{16}

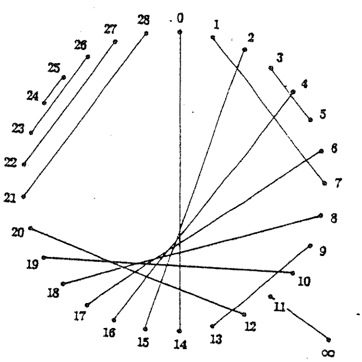

Figure 5: WK_{30}

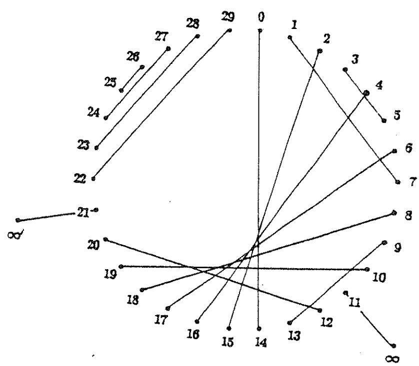

Figure 6: $W'K_{32}$

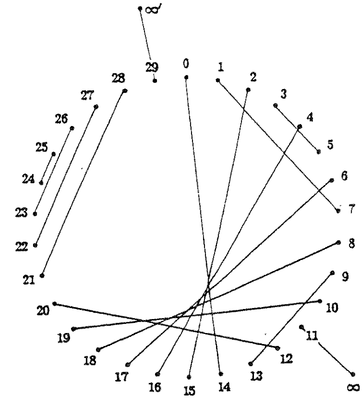

Figure 7: $W''K_{32}$

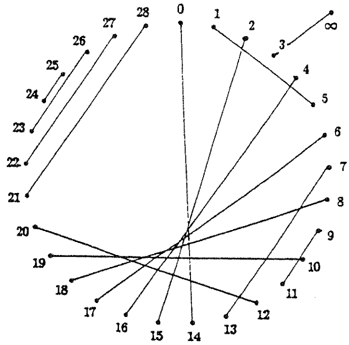

Figure 8: $W^{(1)}K_{30}$

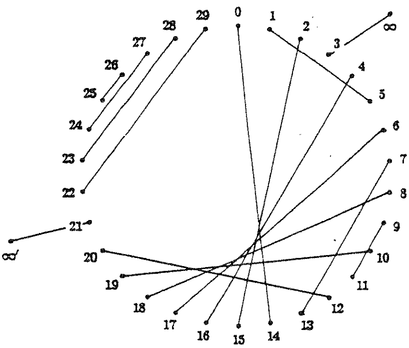

Figure 9: $W^{(1)'}K_{32}$

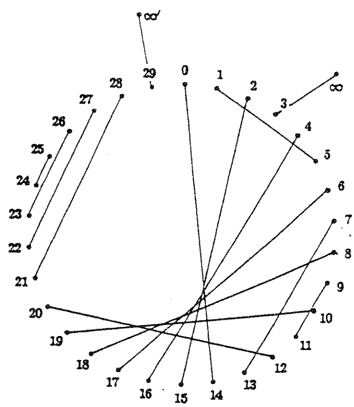

Figure 10: $W^{(1)''}K_{32}$

2. PRELIMINARIES

Put $r = n-1$. Assume the vertices are labeled as $V_{2n} = \{\infty\} \cup \{0, 1, 2, \cdots, m-1\} = \{\infty\} \cup Z_m$. Let σ be the vertex permutation $\sigma = (\infty)(0\ 1\ 2\ \cdots\ m-1)$ and put $\Sigma = \langle\sigma\rangle$. Then for any edge $\{a, b\}$ of K_{2n}, we define the length of the edge with respect to Σ:

$$d(\{a,b\}) = \begin{cases} \min\{|b-a|, m-|b-a|\} & (a, b \neq \infty) \\ \infty & \text{(otherwise)} \end{cases}$$

We note that $1 \leq d(\{a,b\}) \leq r$ when $a, b \neq \infty$.

A starter of Z_m is a set of edges $S = \{\{v_i, w_i\} \in E_{2n} \mid 1 \leq i \leq r\}$ such that the set of vertices of S is $V_{2n} \setminus \{\infty, 0\}$ and $d(S) = \{1, 2, \cdots, r\}$. If S is a starter of Z_m, $S' = S \cup \{\{\infty, 0\}\}$ is called a starter 1-factor of K_{2n}, and we obtain a 1-factorization of K_{2n} by rotating S' according to σ, that is, $\Sigma S' = \{\sigma^i S' \mid 0 \leq i \leq m-1\}$.

For any positive integer t, we will construct starters of K_{2n} in sections 3 and 4.

3. $N_t K_{2n}$ WHEN t IS ODD

Assume t is odd ≥ 1 and $2n \geq 6t$. Put $s = (t-1)/2$ and put

(1) $V_I = \{a_1, a_2, \cdots, a_t, a_1', a_2', \cdots, a_t'\}$,
$E_I = \{\{a_1, a_1'\}, \{a_2, a_2'\}, \cdots, \{a_t, a_t'\}\}$,

(2) $V_{II} = \{b_1, b_2, \cdots, b_t, b_1', b_2', \cdots, b_t'\}$,
$E_{II} = \{\{b_1, b_{s+1}'\}, \{b_2, b_{s+2}'\}, \cdots, \{b_{s+1}, b_t'\}\} \cup \{\{b_{s+2}, b_1'\}, \{b_{s+3}, b_2'\}, \cdots, \{b_t, b_s'\}\}$,

(3) $V_{III} = \{\infty, 0, c_1, c_2, \cdots, c_s, c_1', c_2', \cdots, c_s'\}$,
$E_{III} = \{\{\infty, 0\}, \{c_1, c_1'\}, \{c_2, c_2'\}, \cdots, \{c_s, c_s'\}\}$,

(4) $V_{IV} = \{d_1, d_2, \cdots, d_s, d_1', d_2', \cdots, d_s'\}$,
$E_{IV} = \{\{d_1, d_1'\}, \{d_2, d_2'\}, \cdots, \{d_s, d_s'\}\}$.

(5) When $r (= n-1)$ is odd, put $u = (r - 3t + 2)/2$, $v = (r - 3t)/2$. When r is even, put $u = v = (r - 3t + 1)/2$. And put
$V_{V_1} = \{e_1, e_2, \cdots, e_u, e_1', e_2', \cdots, e_u'\}$, $E_{V_1} = \{\{e_1, e_1'\}, \{e_2, e_2'\}, \cdots, \{e_u, e_u'\}\}$,
$V_{V_2} = \{f_1, f_2, \cdots, f_v, f_1', f_2', \cdots, f_v'\}$, $E_{V_2} = \{\{f_1, f_1'\}, \{f_2, f_2'\}, \cdots, \{f_v, f_v'\}\}$.

We define V_{2n} and σ as follows;

$$V_{2n} = \{\infty\} \cup V_I \cup V_{II} \cup V_{III} \cup V_{IV} \cup V_{V_1} \cup V_{V_2},$$

$$\begin{aligned}\sigma = \ & (\infty)(f_v\ f_{v-1}\ \cdots\ f_1 \\ & c_s\ c_{s-1}\ \cdots\ c_1\ 0\ c'_1\ c'_2\ \cdots\ c'_s \\ & b_1\ b_2\ \cdots\ b_t \\ & f'_1\ f'_2\ \cdots\ f'_v \\ & a_t\ a_{t-1}\ \cdots\ a_1\ a'_1\ a'_2\ \cdots\ a'_t \\ & e'_u\ e'_{u-1}\ \cdots\ e'_1 \\ & d'_s\ d'_{s-1}\ \cdots\ d'_1 \\ & b'_t\ b'_{t-1}\ \cdots\ b'_1 \\ & d_1\ d_2\ \cdots\ d_s \\ & e_1\ e_2\ \cdots\ e_u)\end{aligned}$$

and put $\Sigma = \langle\sigma\rangle$ (see Figure 11).

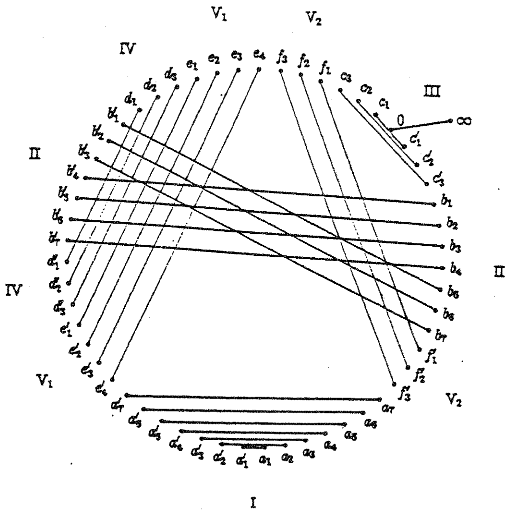

Figure 11: N_7K_{56}

Proposition 3.1 Assume t is odd ≥ 1 and $2n \geq 6t$. Then

$$S_t = E_I \cup E_{II} \cup E_{III} \cup E_{IV} \cup E_{V_1} \cup E_{V_2}$$

is a starter 1-factor of K_{2n} with respect to $\Sigma = \langle\sigma\rangle$.

Proof. We need only show that the lengths of the edges of S_t are all different.

(1) $|E_I| = t.\ d(E_I) = \{1, 3, 5, \cdots, 2t-1\}$.

(2) $|E_{II}| = t$. Since $d(\{b_1, b'_{s+1}\}) = |V_{III}| - 1 + (|V_{V_2}| + |V_{V_1}| + |V_{IV}|)/2 + s + 1 = r - t + 1$, we have $d(E_{II}) = \{r-t+1, r-t+2, \cdots, r\}$.

(3) $|E_{III}| = s + 1.\ d(E_{III}) = \{\infty, 2, 4, 6, \cdots, t-1\}$.

(4) $|E_{IV}| = s$. $d(E_{IV}) = \{t+1, t+3, t+5, \cdots, 2t-2\}$.

We have $|E_I \cup E_{II} \cup E_{III} \cup E_{IV}| = 3t$, so it doesn't depend on $2n$. (Note that we are assuming $2n \geq 6t$.) Only E_{V_1} and E_{V_2} depend on $2n$.

(5) $|E_{V_1}| = u$ and $|E_{V_2}| = v$. When r is odd, $d(E_{V_1}) = \{2t, 2t+2, 2t+4, \cdots, r-t\}$ and $d(E_{V_2}) = \{2t+1, 2t+3, 2t+5, \cdots, r-t-1\}$

When r is even, $d(E_{V_1}) = \{2t, 2t+2, 2t+4, \cdots, r-t-1\}$ and $d(E_{V_2}) = \{2t+1, 2t+3, 2t+5, \cdots, r-t\}$.

Therefore $d(S_t) = \{\infty, 1, 2, \cdots, r\}$, from which it follows that S_t is a starter 1-factor of K_{2n}.

4. N_tK_{2n} WHEN t IS EVEN

Assume t is even ≥ 2 and $2n \geq 6t$. Put $s = t/2$ and put

(1) $V_I = \{a_1, a_2, \cdots, a_t, a'_1, a'_2, \cdots, a'_t\}$,
$E_I = \{\{a_1, a'_1\}, \{a_2, a'_2\}, \cdots, \{a_t, a'_t\}\}$,

(2) $V_{II} = \{b_0, b_1, b_2, \cdots, b_{t-1}, b'_0, b'_1, b'_2, \cdots, b'_{t-1}\}$,
$E_{II} = \{\{b_0, b'_0\}\} \cup \{\{b_1, b'_s\}, \{b_2, b'_{s+1}\}, \cdots, \{b_s, b'_{t-1}\}\}$
$\cup \{\{b_{s+1}, b'_1\}, \{b_{s+2}, b'_2\}, \cdots, \{b_{t-1}, b'_{s-1}\}\}$,

(3) $V_{III} = \{\infty, 0, c_1, c_2, \cdots, c_{s-1}, c'_1, c'_2, \cdots, c'_{s-1}\}$,
$E_{III} = \{\{\infty, 0\}, \{c_1, c'_1\}, \{c_2, c'_2\}, \cdots, \{c_{s-1}, c'_{s-1}\}\}$,

(4) $V_{IV} = \{d_1, d_2, \cdots, d_s, d'_1, d'_2, \cdots, d'_s\}$,
$E_{IV} = \{\{d_1, d'_1\}, \{d_2, d'_2\}, \cdots, \{d_s, d'_s\}\}$.

(5) When r is odd, put $u = v = (r-3t+1)/2$. When r is even, put $u = (r-3t)/2$, $v = (r-3t+2)/2$. Put
$V_{V_1} = \{e_1, e_2, \cdots, e_u, e'_1, e'_2, \cdots, e'_u\}$, $E_{V_1} = \{\{e_1, e'_1\}, \{e_2, e'_2\}, \cdots, \{e_u, e'_u\}\}$,
$V_{V_2} = \{f_1, f_2, \cdots, f_v, f'_1, f'_2, \cdots, f'_v\}$, $E_{V_2} = \{\{f_1, f'_1\}, \{f_2, f'_2\}, \cdots, \{f_v, f'_v\}\}$.

We define V_{2n} and σ as follows;

$$V_{2n} = \{\infty\} \cup V_I \cup V_{II} \cup V_{III} \cup V_{IV} \cup V_{V_1} \cup V_{V_2}$$

$$\begin{aligned}\sigma = \ & (\infty)(f_v\ f_{v-1}\ \cdots\ f_1 \\ & c_{s-1}\ c_{s-2}\ \cdots\ c_1\ 0\ c'_1\ c'_2\ \cdots\ c'_{s-1} \\ & b_0\ b_1\ b_2\ \cdots\ b_{t-1} \\ & f'_1\ f'_2\ \cdots\ f'_v \\ & a_t\ a_{t-1}\ \cdots\ a_1\ a'_1\ a'_2\ \cdots\ a'_t \\ & e'_u\ e'_{u-1}\ \cdots\ e'_1 \\ & b'_0 \\ & d'_s\ d'_{s-1}\ \cdots\ d'_1 \\ & b'_{t-1}\ b'_{t-2}\ \cdots\ b'_1 \\ & d_1\ d_2\ \cdots\ d_s \\ & e_1\ e_2\ \cdots\ e_u)\end{aligned}$$

and put $\Sigma = \langle \sigma \rangle$ (see Figure 12).

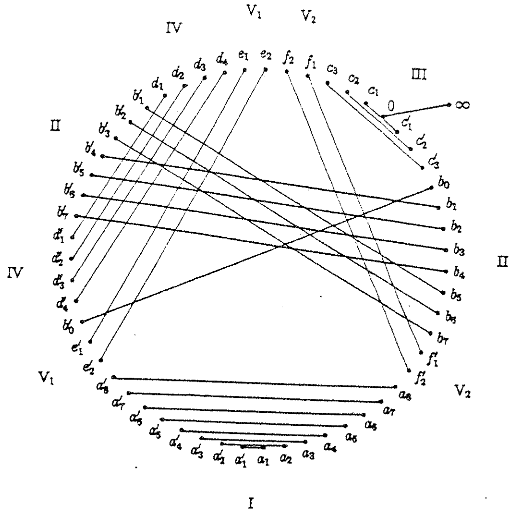

Figure 12: N_8K_{56}

Proposition 4.1 Assume t is even ≥ 2 and $2n \geq 6t$. Then

$$S_t = E_I \cup E_{II} \cup E_{III} \cup E_{IV} \cup E_{V_1} \cup E_{V_2}$$

is a starter 1-factor of K_{2n} with respect to $\Sigma = \langle \sigma \rangle$.

Proof. We need only show that the lengths of the edges of S_t are all different.

(1) $|E_I| = t$. $d(E_I) = \{1, 3, 5, \cdots, 2t-1\}$.

(2) $|E_{II}| = t$. $d(E_{II}) = \{r-t+1, r-t+2, \cdots, r\}$.

(3) $|E_{III}| = s$. $d(E_{III}) = \{\infty, 2, 4, 6, \cdots, t-2\}$.

(4) $|E_{IV}| = s$. $d(E_{IV}) = \{t, t+2, t+4, \cdots, 2t-2\}$.

(5) $|E_{V_1}| = u$ and $|E_{V_2}| = v$.
When r is odd, $d(E_{V_1}) = \{2t+1, 2t+3, 2t+5, \cdots, r-t\}$ and $d(E_{V_2}) = \{2t, 2t+2, 2t+4, \cdots, r-t-1\}$.
When r is even, $d(E_{V_1}) = \{2t+1, 2t+3, 2t+5, \cdots, r-t-1\}$ and $d(E_{V_2}) = \{2t, 2t+2, 2t+4, \cdots, r-t\}$.

Therefore $d(S_t) = \{\infty, 1, 2, \cdots, r\}$, from which it follows that S_t is a starter 1-factor of K_{2n}.

We denote by N_tK_{2n} the 1-factorization induced by S_t in Proposition 3.1 and 4.1.

5. EVEN STARTERS

When the vertices are labeled as $V_{2n} = \{\infty, \infty'\} \cup \{0, 1, 2, \cdots, 2n-3\} = \{\infty, \infty'\} \cup Z_{2n-2}$, we define an even starter.

Let σ_1 be the vertex permutation $\sigma_1 = (\infty)(\infty')(0\ 1\ 2\ \cdots\ 2n-3)$ and put $\Sigma_1 = \langle \sigma_1 \rangle$. For any edge $\{a, b\}$ of K_{2n} we define the length of the edge with respect to Σ_1:

$$d(\{a,b\}) = \begin{cases} \min\{|b-a|, (2n-2)-|b-a|\} & (a, b \neq \infty, \infty') \\ \infty & (\text{otherwise}) \end{cases}$$

We note that $1 \leq d(\{a, b\}) \leq n-1$ when $a, b \neq \infty, \infty'$.

An even starter of Z_{2n-2} is a set of edges $T = \{\{v_i, w_i\} \in E_{2n} \mid 1 \leq i \leq n-2\}$ such that the set of vertices of T is $V_{2n} \setminus \{\infty, \infty', 0, a\}$ for some $a \in V_{2n}, a \neq \infty, \infty', 0$, and $d(T) = \{1, 2, \cdots, n-2\}$.

If T is an even starter of Z_{2n-2}, $T' = T \cup \{\{\infty, 0\}, \{\infty', a\}\}$ is called an even starter 1-factor of K_{2n}. We may call $\sigma_1^i T'$ $(1 \leq i \leq 2n-3)$ an even starter 1-factor. For an even starter 1-factor T', we obtain a 1-factorization of K_{2n} by rotating T' according to σ_1 and adding the pinwheel P,

$$P = \{\{0, n-1\}, \{1, n\}, \cdots, \{n-2, 2n-3\}\},$$

that is, $\Sigma_1 T' \cup \{P\}$ is a 1-factorization of K_{2n}.

Some starter 1-factors of K_{2n} can be extended to even starter 1-factors of K_{2n+2} ([1], p48). Our starter 1-factors constructed in sections 3 and 4 can be extended to even starter 1-factors, also (Figures 13,14). We denote the induced 1-factorizations by $N_t'K_{2n}$ and $N_t''K_{2n}$, respectively.

When $t \geq 3$, N_tK_{2n} has more variations $N_t^{(1)}K_{2n}$, ${N_t}^{(1)\prime}K_{2n}$ and ${N_t}^{(1)\prime\prime}K_{2n}$ (Figures 15 to 18).

Finally, we should mention whether the 1-factorizations constructed in this paper are new, i.e., not isomorphic to known 1-factorizations. For example, when $2n = 20, 22$, the t satisfying $6t \leq 2n$ are $t = 1, 2, 3$; so K_{2n} has GK_{2n}, $G'K_{2n}$, AK_{2n}, WK_{2n}, $W'K_{2n}$, $W''K_{2n}$, $W^{(1)}K_{2n}$, $W^{(1)\prime}K_{2n}$, $W^{(1)\prime\prime}K_{2n}$, N_tK_{2n}, $N_t'K_{2n}$, $N_t''K_{2n}$ $(t = 1, 2, 3)$, ${N_3}^{(1)}K_{2n}$, ${N_3}^{(1)\prime}K_{2n}$, ${N_3}^{(1)\prime\prime}K_{2n}$. It is shown that these 1-factorizations are not isomorphic each other with the aid of a computer.

It is not easy to demonstrate in general that the 1-factorizations constructed in this paper are new, but it is clear that there are new 1-factorizations among them because the number of the 1-factorizaions of K_{2n} constructed in this paper increases as $2n$ increases.

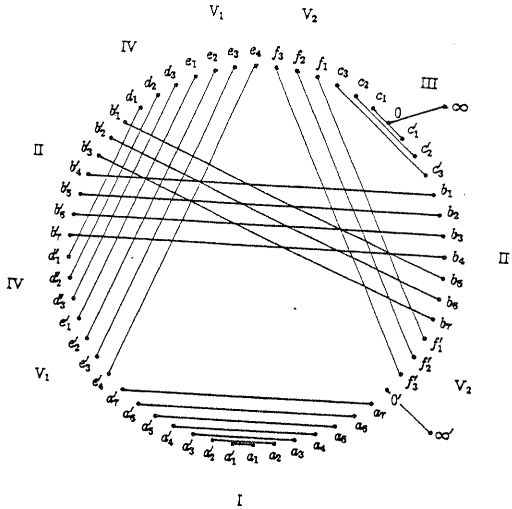

Figure 13: $N_7'K_{58}$

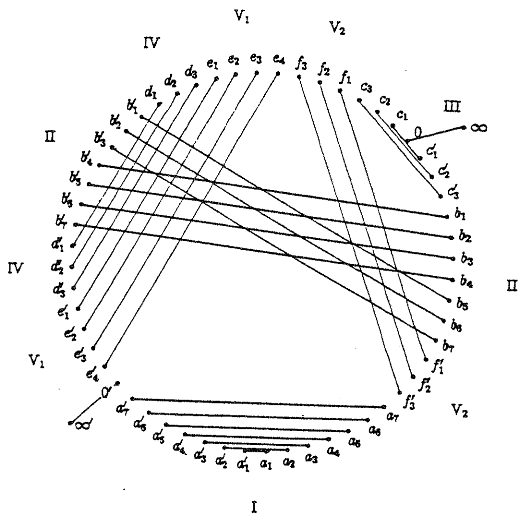

Figure 14: $N_7''K_{58}$

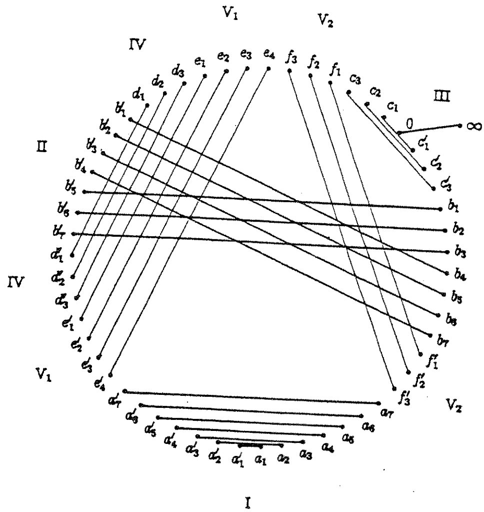

Figure 15: $N_7^{(1)}K_{56}$

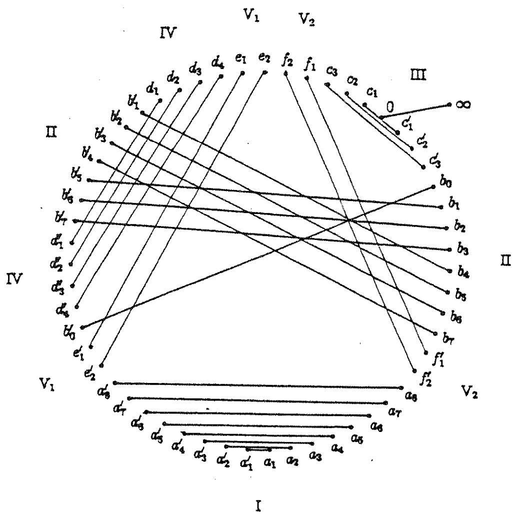

Figure 16: $N_8^{(1)}K_{56}$

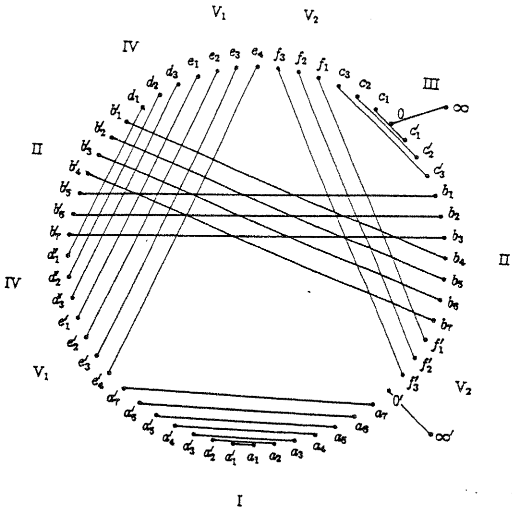

Figure 17: $N_7^{(1)\prime}K_{58}$

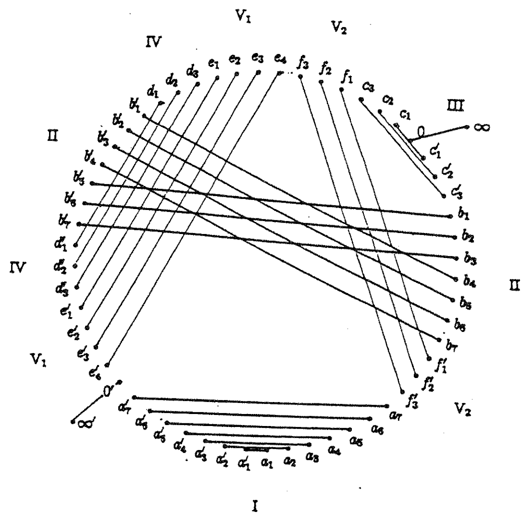

Figure 18: $N_7^{(1)\prime\prime}K_{58}$

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