On matchability of graphs

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Abstract

A graph is h-*matchable* if G−X has a perfect matching for every subset $X \subseteq V(G)$ with $|X| = h$, and it is h-*extendable* if every matching of h edges can be extended to a perfect matching. It is proved that a graph G with even order is 2h-matchable if and only if (1) G is h-extendable; and (2) for any edge set D such that, for each $e = xy \in D$, $x,y \in V(G)$ and $e \notin E(G)$, $G \cup D$ is h-extendable. Also nine known sufficient conditions for a graph to be h-extendable are stated, and sharp analogues of them all are obtained for matchability, each of which implies the corresponding result for extendability.

1 Terminology and introduction

All graphs considered in this paper are undirected, finite and simple. In general we follow the terminology of [1].

Let G be a graph. We denote by o(G) the number of odd components of G and by $\omega(G)$ the number of the components of G. Let $v \in V(G)$ and $X \subseteq V(G)$. We define $N(v) = \{u \mid u \in V(G) \text{ and } uv \in E(G)\}$ and $N(X) = \bigcup_{v \in X} N(v)$. Let $S \subseteq V(G)$ and let H be a subgraph of G. We use the notation $N_S(v) = N(v) \cap S$, $N_H(v) = N(v) \cap V(H)$, $d_S(v) = |N_S(v)|$ and $d_H(v) = |N_H(v)|$. Let G and H be two disjoint graphs. We denote by kH the union of k copies of H, and by G+H the join of G and H, which is the graph constructed from G and H by joining each vertex of G to all vertices of H.

A graph G with n vertices is h-*matchable* where $0 \leq h \leq n-2$, if for each subset $X \subseteq V(G)$ with $|X| = h$, G−X has a perfect matching (a 1-factor). When h = 0, G has a perfect matching. When h = 1 or 2, G is known as factor-critical or bicritical respectively. G is h-*extendable* for $0 \leq h \leq (n-2)/2$ if G has a matching of size h and any matching of size h in G is contained in a perfect matching of G. When h = 0, G has a perfect matching.

The *toughness* of G is defined as:

tough(G) = min $\{ \frac{|X|}{\omega(G-X)} \mid X \subset V(G)$ and $\omega(G-X) \geq 2 \}$

if G is not a complete graph, and tough(G) = ∞ if G is a complete graph.

The *binding number* of G is defined as:

bind(G) = min $\{ \frac{|N(X)|}{|X|} \mid \emptyset \neq X \subset V(G)$ and $N(X) \neq V(G) \}$.

The concept of h-extendability was introduced by Plummer [6] in 1980. Since then, several general sufficient conditions for h-extendability have been found (see [2], [4–8] and Section 4 below). For each of these conditions, we shall obtain an analogous sharp sufficient condition for a graph to be h-matchable, and we shall see in Section 4 that each of our new theorems implies the corresponding result for extendability. Also we shall obtain a result to show the relation between matchability and extendability in Section 2.

2 A few properties of h-matchable graphs

In this Section, we show some important properties of h-matchable graphs of which we shall make frequent use in the next section.

Proposition 1: Let G be a graph with order n and h be an integer such that $0 \leq h \leq n-2$ and $h \equiv n \pmod 2$. Then G is h-matchable if and only if, for each subset $S \subseteq V(G)$ with $|S| \geq h$, $o(G-S) \leq |S| - h$.

Proof. This follows easily from Tutte's well known characterization of perfect matchings [11] (see also [10], Theorem 3.3.12.) □

Corollary 2: Let G be an h-matchable graph. Then G is j-matchable for every j such that $0 \leq j \leq h$ and $j \equiv h \pmod 2$.

Proof. We use Proposition 1. Suppose $S \subseteq V(G)$ and $|S| \geq j$. If $j \leq |S| < h$, then $S \subset T$ for some set T such that $|T| = h$, and then $o(G-S) \leq o(G-T) + (h-j) \leq (|S|-h) + (h-j) = |S|-j$; this holds because removing a vertex from a graph cannot reduce its number of odd components by more than 1. If $|S| \geq h$ then $o(G-S) \leq |S|-h \leq |S|-j$. □

Corollary 3: Let G be a graph with order n that is not h-matchable, where $0 \leq h \leq n-2$ and $h \equiv n \pmod 2$. Then there is a set $S \subset V(G)$ with $|S| \geq h$ such that $\omega(G-S) \geq o(G-S) \geq |S|-h+2 \geq 2$.

Proof. By Proposition 1, there is a set $S \subseteq V(G)$ with $|S| \geq h$ such that $o(G-S) \geq |S|-h+1$. But $o(G-S)$ has the same parity as $n-|S|$ and hence as $|S|-h$, and so $o(G-S) \geq |S|-h+2$. The rest is obvious. □

Lemma 4 ([2]): Let h $\geq$ 1. Then a graph G with even order is h-extendable if and only if o(G−S) $\leq$ |S|−2h for every S $\subset$ V(G) such that G[S] contains h independent edges.

The next theorem shows the relation between 2h-matchable graphs and h-extendable graphs.

Theorem 5: A graph G with even order is 2h-matchable if and only if
(1) G is h-extendable; and
(2) for any edge set D such that, for each e = xy $\in$ D, x, y $\in$ V(G) and e $\notin$ E(G), G $\cup$ D is h-extendable.
Proof. Suppose G is 2h-matchable. By Proposition 1, o(G−S) $\leq$ |S|−2h for each S $\subseteq$ V(G) with |S| $\geq$ 2h. By Lemma 4, G is h-extendable. Let D be an edge set such that, for each e = xy $\in$ D, x, y $\in$ V(G) and e $\notin$ E(G). And let G′ = G $\cup$ D. Since adding new edges to G the number of odd components in G−S does not increase for each set S $\subseteq$ V(G), we have o(G′−S) $\leq$ |S|−2h for each S $\subseteq$ V(G′) with |S| $\geq$ 2h. Hence G′ is h-extendable.

Suppose G is not 2h-matchable. By Corollary 3, there is a set S $\subseteq$ V(G) with |S| $\geq$ 2h such that o(G−S) $\geq$ |S|−2h+2. We have two cases.
Case 1: G[S] contains h independent edges.

Then by Lemma 4, G is not h-extendable. So (1) of this theorem does not hold.
Case 2: G[S] contains less than h independent edges.

However, |S| $\geq$ 2h. We can add a set D of edges such that, for each e = xy $\in$ D, x, y $\in$ S and e $\notin$ E(G), into G[S] so that G[S] contains h independent edges. Let G′ = G $\cup$ D. Since o(G−S) $\geq$ |S|−2h+2, o(G′−S) $\geq$ |S|−2h+2, by Lemma 4, G′ is not h-extendable. Hence (2) of this theorem does not hold. □

3 Some sufficient conditions for matchability

In this section, we prove nine sufficient conditions for matchability that are analogous to, and have similar proofs to, known sufficient conditions for extendability. Our first condition involves toughness.

Theorem 6: Let G be a connected graph with order n and let h be an integer with 0 $\leq$ h $\leq$ n−2 such that h $\equiv$ n (mod 2). Suppose that tough(G) > $\frac{h}{2}$, and tough(G) $\geq$ 1 if h $\leq$ 1. Then G is h-matchable.
Proof. Suppose not. Clearly tough(G) < n/2 and so h < n.

By Corollary 3, there is a set S $\subseteq$ V(G) with |S| $\geq$ h such that ω(G−S) $\geq$ $|S| - h + 2$ $\geq$2. However, if h $\leq$ 1 and tough(G) $\geq$ 1 then
ω(G−S) $\leq$ |S|/tough(G) $\leq$ |S| < $|S| - h + 2$,

a contradiction. And if $h \geq 2$ and tough(G) $> h/2$ then

$$\omega(G-S) \leq |S|/\text{tough}(G) < |S|/(h/2)$$
$$\leq \frac{2|S|}{h} + \frac{(|S|-h)(h-2)}{h} = |S| - h+2,$$

another contradiction. The result follows. □

The lower bounds on toughness in Theorem 6 are sharp. Taking $G = K_h + 2K_1$ shows the sharpness of the bound for $h \geq 2$; tough(G) $= \frac{h}{2}$ but G is not h-matchable because deleting the vertices of K_h from G, the resulting graph has no perfect matching. Taking $H = K_r + (r+1)K_1$ shows the sharpness of the bound for $h = 1$; tough(H) $= \frac{r}{r+1} \to 1-$ as $r \to \infty$. And H is not 1-matchable because deleting a vertex in K_r from H results in a graph with no perfect matching. Taking $H = K_r + (r+2)K_1$ shows the sharpness of the bound for $h = 0$ by the same reason as above. The next theorem gives a binding number condition.

Theorem 7: Let G be a connected graph with order n. Let h be an integer such that $0 \leq h \leq n-2$ and $h \equiv n \pmod 2$.
(i) If bind(G) $> \frac{h}{2}$ for $h \geq 5$, then G is h-matchable;
(ii) If bind(G) $> \frac{7h+26}{24}$ for $h = 2, 4$, then G is h-matchable;
(iii) If bind(G) $> \frac{h+3}{3}$ for $h = 1, 3$, then G is h-matchable;
(iv) If bind(G) $\geq \frac{4}{3}$, then G is h-matchable for $h = 0$.

Proof. Suppose G satisfies the hypotheses of this theorem but is not h-matchable. By Corollary 3, there is a set $S \subseteq V(G)$ with $|S| \geq h$ such that

$$o(G-S) \geq |S| - h+2 \geq 2. \qquad (1)$$

Suppose bind(G) = b and let i(G) denote the number of singleton components of G. Then we have two cases.

Case 1: $i(G-S) > 0$.

Let $X = V(G)-S$. Since $N_G(X) \neq V(G)$, $n-i(G-S) \geq |N_G(X)| \geq b|X| = bn-b|S|$. So

$$i(G-S) \leq b|S|-(b-1)n. \qquad (2)$$

By (1) and (2),

$$o(G-S)-i(G-S) \geq |S| - h+2-b|S|+(b-1)n$$
$$= (b-1)(n-|S|)-h+2. \qquad (3)$$

However, counting the vertices in $V(G)-S$ and using (3), we have

$$n-|S| \geq i(G-S)+3(o(G-S)-i(G-S))$$
$$\geq i(G-S)+3(b-1)(n-|S|)-3h+6.$$

Then

$$(3b-4)(n-|S|) \leq 3h-6-i(G-S). \qquad (4)$$

If $i(G-S) \geq 2$, since $n-|S| \geq o(G-S) \geq 2$ by (1), we deduce from (4) that $3h-6-2 \geq 2(3b-4)$. So $b \leq \frac{h}{2}$, contradicting the hypotheses for all h of this theorem.

Otherwise, $i(G-S) = 1$ and $n-|S| \geq 4$ since $o(G-S) \geq 2$ by (1). By (4), we have $3h-7 \geq 4(3b-4)$. Hence $b \leq \frac{3h+9}{12}$, contradicting the hypotheses for all h of this theorem.

Case 2: $i(G-S) = 0$.

We have two subcases.

Case (2.1): $h \geq 3$.

Let r be the order of a smallest odd component of G−S and let X be the set of a vertex from a smallest odd component of G−S and all vertices of any other $|S|-h+1$ odd components of G−S. Since $N_G(X) \neq V(G)$, we have $|S|+(|X|-1)+(r-1) \geq |N_G(X)| \geq b|X|$. So $(b-1)|X| \leq |S|+r-2$. As $|X| \geq r(|S|-h+1)+1$, we have $(b-1)[r(|S|-h+1)+1] \leq |S|+r-2$. Then

$$(r-\tfrac{1}{b-1})|S| \leq \tfrac{r-2}{b-1}+(h-1)r-1. \qquad (5)$$

Since $r \geq 3 > \frac{1}{b-1}$ and $|S| \geq h$, (5) implies that

$$b \leq \tfrac{2r+h-1}{r+1}. \qquad (6)$$

Since $h \geq 3$, the function $f(r) = (2r+h-1)/(r+1)$ attains its maximum value at $r = 3$. Thus $b \leq f(3) = \frac{h+5}{4}$, contradicting the hypotheses for $h \geq 3$.

Case (2.2): h = 0, 1, 2.

Let X be the set of all vertices of any $|S|-h+1$ odd components of G−S. Then $|S| + |X| \geq |N_G(X)| \geq b|X|$. So $|X| \leq |S|/(b-1)$. Combining this with $|X| \geq 3(|S|-h+1)$, we get

$$b \leq \tfrac{|S|}{3(|S|-h+1)} + 1. \qquad (7)$$

Since $|S| \geq h$, we have $b \leq \frac{5}{3} = \frac{7h+26}{24}$ for h = 2 and $b \leq \frac{4}{3} = \frac{h+3}{3}$ for h = 1. Also the function $f(m) = m/3(m-h+1) + 1 = m/3(m+1) + 1$ for h = 0 is a strictly monotone increasing function and $f(m) < \frac{4}{3}$ for all $m \geq 1$. Obviously, $f(m) \to \frac{4}{3}-$ as $m \to \infty$. Since $|S| \geq 1$ because G is connected, by (7), $b < \frac{4}{3}$. Thus we have contradictions to the hypotheses for h = 0, 1, 2. □

Taking $G = K_h+2K_1$ shows the sharpness of the bound on binding number for $h \geq 5$ and $H = K_h+2K_3$ shows the sharpness of the bound for $1 \leq h \leq 4$ in Theorem 7. Let $F = K_{m+1}+(m+3)K_3$ $(m \geq 0)$. Then $\text{bind}(F) = \frac{7+4m}{6+3m} \to \frac{4}{3}-$ as $m \to \infty$, where we choose X to be the set of all vertices of m+2 copies of K_3 in F such that $\text{bind}(F) = |N(X)|/|X|$. But $\text{bind}(F) < \frac{4}{3}$ for all m. Then F shows the sharpness of the bound for h = 0. Theorems 8 and 9 give a neighbourhood union condition and a degree sum condition for h-matchable graphs.

Theorem 8: Let G be a k-connected graph with order n and h an integer such that $0 \leq h \leq n-2$ and $h \equiv n \pmod 2$. Suppose there is an integer t, $1 \leq t \leq k-h+2$, such that for each independent set $I = \{w_1, w_2, \cdots, w_t\}$, $|N(I)| \geq n+h-1-k$. Then G is h-matchable.

Proof. Suppose not. By Corollary 3, there is a set $S \subset V(G)$ with $|S| \geq h$ such that $\omega(G-S) \geq |S|-h+2 \geq 2$. Since G is k-connected, $|S| \geq k$ and so $\omega(G-S) \geq k-h+2 \geq t$. Let $C_1, C_2, \cdots, C_{\omega(G-S)}$ be the components of G−S; choose a $w_i \in V(C_i)$ for each i, and let $I = \{w_1, w_2, \cdots, w_t\}$. Then I is an independent set.

Since $|V(C_i)| \geq 1$ for $t+1 \leq i \leq \omega(G-S)$, it follows that

$$n \geq |S| + \sum_{i=1}^{t} |V(C_i)| + \omega(G-S)-t$$

$$\geq |S| + \sum_{i=1}^{t} |V(C_i)| + |S|-h+2-t$$

so that

$$\sum_{i=1}^{t}(|V(C_i)|-1) \leq n+h-2-2|S|. \qquad (8)$$

Thus

$$|N(I)| \leq \sum_{i=1}^{t}(|V(C_i)|-1)+|S| \leq n+h-2-|S| \leq n+h-2-k,$$

contrary to an hypothesis. □

Theorem 9: Let G be a k-connected graph with order n and h an integer such that $0 \leq h \leq n-2$ and $h \equiv n \pmod 2$. Suppose there is an integer t, $1 \leq t \leq k-h+2$, such that for each independent set $I = \{w_1, w_2, \cdots, w_t\} \subseteq V(G)$,

$$\sum_{i=1}^{t} d(w_i) \geq t(n+h-2)/2 + 1.$$

Then G is h-matchable.

Proof. Suppose not. By Corollary 3, there is a set $S \subseteq V(G)$ with $|S| \geq h$ such that

$$\omega(G-S) \geq |S|-h+2 \geq 2. \qquad (9)$$

Suppose first that $t \geq 2$. Construct I exactly as in the proof of Theorem 8, and note that, since $|V(C_i)| \geq 1$ for all i, (8) gives

$$|S| \leq (n+h-2)/2. \qquad (10)$$

Hence

$$\sum_{i=1}^{t} d(w_i) \leq \sum_{i=1}^{t}(|V(C_i)|-1)+t|S| \leq n+h-2+(t-2)|S| \text{ by } (8) \leq t(n+h-2)/2 \text{ by } (10),$$

contrary to an hypothesis.

This completes the proof when $t \geq 2$, so suppose $t = 1$. Then the hypotheses of the theorem imply $d(w) \geq (n+h)/2$ for each $w \in V(G)$, so that $|V(C_i)| \geq (n+h)/2-|S|+1$ for each i. Let $x := \min_i |V(C_i)|$ and $\omega := \omega(G-S)$. Then we have just seen

$$2x \geq n+h-2|S|+2. \qquad (11)$$

Counting the vertices in G gives

$$n \geq |S| + \omega x. \qquad (12)$$

Adding (9), (11) and (12) and rearranging gives $(\omega-2)x \leq \omega-4$, which is impossible since $\omega \geq 2$ from (9), and x is a positive integer. This contradiction completes the proof of Theorem 9. □

For any integers h, k, t such that $0 \leq h \leq k$ and $1 \leq t \leq k-h+2$, $G = K_k+(k-h+2)K_1$ shows that the bounds in Theorems 8 and 9 are sharp. For any independent set $I = \{w_1, w_2, \cdots, w_t\} \subseteq V(G)$, $|N(I)| \geq k = n+h-2-k$ and $\sum_{i=1}^{t} d(w_i) \geq tk = t(n+h-2)/2$. But G is not h-matchable.

Now we introduce the following definitions. Let $\kappa(G)$, $\alpha(G)$ and $\delta(G)$ denote the connectivity, independence number and minimum degree of G. If $u, v \in V(G)$, let $d(u,v)$ denote the distance between u and v, let $N_2(v) = \{ u \mid u \in V(G)$ and

$d(u,v) = 2$ }, and let $G_v = G[\{v\} \cup N_G(v)]$. If $d(u,v) = 2$, let $n_{u,v}(w) = \max\{ |S| \mid S$ is independent and $\{u,v\} \subseteq S \subseteq N(w)$ for a vertex $w \in N(u) \cap N(v)$ } and $\alpha^*(u,v) = \max_w\{n_{u,v}(w) \mid w \in N(u) \cap N(v)$ }. We can now define the following five conditions.

$C_1(h)$: For each $v \in V(G)$, $\kappa(G_v) \geq \alpha(G_v)+h-1$.
$C_2(h)$: For each $v \in V(G)$ and each independent set $R \subseteq N_2(v)$, $|N(v) \cap N(R)| \geq |R|+h$.
$C_3(h)$: For each $u, v \in V(G)$ such that $d(u,v) = 2$, $|N(u) \cap N(v)| \geq \alpha^*(u,v)+h-1$.
$C_4(h)$: For each $v \in V(G)$ and nonadjacent vertices $u, w \in N(v)$, $d_{G_v}(u)+d_{G_v}(w) \geq d_G(v)+h$.
$C_5(h)$: $\delta(G) \geq (n+h)/2$.

Now we prove the following theorem.

Theorem 10: Let G be a connected graph with order n and h an integer such that $0 \leq h \leq n-2$ and $h \equiv n \pmod 2$. If G satisfies any of the conditions $C_i(h)$ $(1 \leq i \leq 5)$, then G is h-matchable.

Proof. By [4] Theorem 9 and [5] Theorem 1, $C_1(h)$ and $C_2(h)$ hold for $h \geq 1$. By [4] and [5], $C_i(h)$ implies $C_{i+1}(h)$ for $h \geq 0$ and $i = 2, 3, 4$.

Now we prove the result of $C_2(h)$ for $h = 0$.

Suppose G is not h-matchable. By Corollary 3, there is a set $S \subseteq V(G)$ with $|S| \geq 1$ (since G is connected) such that

$$\omega(G-S) \geq o(G-S) \geq |S|-h+2 \geq 2. \qquad (13)$$

We choose S such that |S| is as small as possible subject to (13). Then we have the following claim.

Claim 1: For each vertex v in an odd component such that $N_S(v) \neq \emptyset$, there is an independent set $R \subseteq N_2(v)$ such that $|N(v) \cap N(R)| < |R| = |R|+h$ for $h = 0$, (contradicting the hypothesis of $C_2(h)$ for $h = 0$).

Suppose not. Then there is a vertex v in an odd component C with $N_S(v) \neq \emptyset$ such that the vertices in $N_S(v)$ are adjacent to at least $|N_S(v)|+2$ odd component C, $C_1, C_2, \cdots, C_t$ ($t \geq |N_S(v)|+1$). (Otherwise, let $S' = S \backslash N_S(v)$, we have $|S'| < |S|$ and $|S'|-o(G-S') \leq |S|-o(G-S)$, contrary to the choice of S). Now we choose a vertex w_i in C_i which is adjacent to $N_S(v)$ for $1 \leq i \leq t$. Then $R = \{w_1, w_2, \cdots, w_t\}$ satisfies the inequality $|N(R) \cap N(v)| = |N_S(v)| < |R|$, as claimed.

In the following, we shall prove $C_1(h)$ holds for $h = 0$.

Suppose G is not h-matchable.. By Corollary 3, there is a set $S \subseteq V(G)$ with $|S| \geq 1$ such that

$$o(G-S) \geq |S|-h+2 = |S|+2 > 2 \qquad (14)$$

We choose S to be a minimum set subject to the inequality (14).

Let $|S| = s$, $o(G-S) = t$ and $C_1, C_2, \cdots, C_t$ be the odd components of $G-S$. Let $S = \{v_1, v_2, \cdots, v_s\}$ and k_i be the number of odd components in $G-S$ which are adjacent to v_i. Without loss of generality, assume $k_1 \leq k_2 \leq \cdots \leq k_s$.

Let $k_{m_j} = \max\{ k_i \mid v_i$ is adjacent to C_j and $1 \leq i \leq s\}$ $(j = 1,2,\cdots,t)$. Without loss of generality, assume $k_{m_1} \leq k_{m_2} \leq \cdots \leq k_{m_t}$.

Claim 2: $k_i \geq 3$ for all i such that $1 \leq i \leq s$.

Suppose $k_i \leq 2$ for some i. Then we use $S' = S\backslash\{v_i\}$ to replace S. We have $o(G-S') \geq o(G-S)-1 \geq |S|+2-1 = |S'|+2$, contradicting the choice of S.

Claim 3: For each $v_i \in S$, if v_i is adjacent to C_j, then C_j is adjacent to at least k_i vertices in S.

Since v_i is adjacent to k_i odd components in G−S, there is an independent set of order k_i in $N(v_i)$. Let $u \in V(C_j)$ such that $v_iu \in E(G)$. By condition $C_1(0)$, there are at least k_i-1 internally disjoint paths from u to $w \in V(C_k)$, where $v_iw \in E(G)$ and $k \neq j$. These paths must go through vertices of S. So C_j is adjacent to at least k_i-1 vertices in S.

Suppose C_j is adjacent to exactly k_i-1 vertices $v_i, u_1, u_2, \cdots, u_{k_i-2}$ in S. Let C_j, $D_1, D_2, \cdots, D_{k_i-1}$ be the odd components in G−S which are adjacent to v_i. Since there are k_i-1 internally disjoint paths from u to each of $D_1, D_2, \cdots, D_{k_i-1}$ and these paths must go through $v_i, u_1, u_2, \cdots, u_{k_i-2}$, each of $C_j, D_1, D_2, \cdots, D_{k_i-1}$ is adjacent to all of $v_i, u_1, \cdots, u_{k_i-2}$.

If u_k is only adjacent to $C_j, D_1, \cdots, D_{k_i-1}$ for $k = 1,2,\cdots,k_i-2$, then let $S' = S\backslash\{v_i, u_1, \cdots, u_{k_i-2}\}$ and we have $o(G-S') = o(G-S)-(k_i-1) \geq |S|+2-(k_i-1) = |S'|+2$, contradicting the choice of S.

Hence a u_k $(1 \leq k \leq k_i-2)$ is adjacent to at least k_i+1 odd components in G−S. But u_k is adjacent to C_j, so C_j is adjacent to at least $(k_i+1)-1 = k_i$ vertices in S by the above argument. Hence Claim 3 is proved.

Considering all vertices in S adjacent to C_j, by Claim 3, C_j is adjacent to at least k_{m_j} vertices in S. For the convenience of explanation, if a vertex in S is adjacent to k odd components of G−S, then we say that it sends k edges to the odd components. If an odd component C of G−S has k neighbours in S, then we say that C sends k edges to S. Now the vertices in S send $k_1+k_2+\cdots+k_s$ edges to the odd components of G−S. And the odd components of G−S send at least $k_{m_1}+k_{m_2}+\cdots+k_{m_t}$ edges to S. So we have

$$k_1+k_2+\cdots+k_s \geq k_{m_1}+k_{m_2}+\cdots+k_{m_t} \qquad (15)$$

Claim 4: $\sum_{i=1}^{s} k_i \leq \sum_{i=1}^{s} k_{m_i}$

By induction, we shall prove that $k_{m_i} \geq k_i$ $(i = 1,2,\cdots,s)$. Then the claim holds. By the definition of k_{m_i}, $k_{m_1} \geq k_1$.

Assume that $k_{m_i} \geq k_i$ for all $i < j$. Now $i = j$. If there is an odd component $C_p \in \{C_1, C_2, \cdots, C_j\}$ such that C_p is adjacent to v_q for some $q \geq j$, then $k_{m_j} \geq k_{m_p} \geq k_q \geq k_j$. Otherwise, $C_1, C_2, \cdots, C_j$ are only adjacent to $v_1, v_2, \cdots, v_{j-1}$. Then $k_1+k_2+\cdots+k_{j-1} \geq k_{m_1}+k_{m_2}+\cdots+k_{m_j}$. By induction hypothesis, $k_{m_i} \geq k_i$ $(i = 1,2,\cdots,j-1)$, and $k_{m_j} \geq 1$. So $k_{m_1}+k_{m_2}+\cdots+k_{m_{j-1}}+k_{m_j} > k_1+k_2+\cdots+k_{j-1}$, a contradiction.

Since $k_{m_{s+1}} \geq 1$, by Claim 4, $k_{m_1}+k_{m_2}+\cdots+k_{m_t} > k_1+k_2+\cdots+k_s$, contradicting (15). This last contradiction completes the proof of Theorem 10. □

Taking $G = K_h+2K_1$ shows that the results of Theorem 10 for $C_1(h)$ and $C_2(h)$ for $h \geq 1$ are sharp. Taking $H = K_r+(r+2)K_1$ shows that the results of Theorem 10 for $C_1(h)$ and $C_2(h)$ for $h = 0$ to be sharp. The above counterexamples also show the sharpness of the results of Theorem 10 for $C_i(h)$ for $i = 3,4,5$ and $h \geq 0$.

4 Sufficient conditions for extendability

In view of Theorem 5, Theorems 6−10 immediately imply the following known results (and Theorems 8 and 9 gives short proofs of Corollaries 13 and 14).

Corollary 11 ([7]): Let G be a connected graph with even order. If tough(G) > h for $h \geq 1$, then G is h-extendable.

Corollary 12 ([2]): Let G be a connected graph with even order. If bind(G) > max $\{h, (7h+13)/12\}$ for $h \geq 1$, then G is h-extendable.

Corollary 13 ([8]): Let G be a k-connected graph with even order n. Further, suppose there is an integer t, $1 \leq t \leq k-2h+2$, such that for each independent set $I = \{w_1, w_2, \cdots, w_t\}$, $|N(I)| \geq n-k+2h-1$. Then if
(a) $h = 1$, G is bicritical (and hence 1-extendable) and if
(b) $h \geq 2$, G is h-extendable.

Corollary 14 ([8]): Let G be a k-connected graph with even order n. Further, suppose there is an integer t, $1 \leq t \leq k-2h+2$, such that for each independent set $I = \{w_1,w_2, \cdots ,w_t\} \subseteq V(G)$, $\sum_{i=1}^{t} d(w_i) \geq t((n-2)/2+h)+1$. Then if
(a) $h = 1$, G is bicritical (and hence 1-extendable) and if
(b) $h \geq 2$, G is h-extendable.

Corollary 15 ([4,5,6]): Let G be a connected graph with even order n and h an integer such that $1 \leq h \leq (n-2)/2$. If G satisfies any of the conditions $C_i(2h)$ ($1 \leq i \leq 5$), then G is h-extendable.

Remark 1: We notice that when we prove a graph to be h-extendable we seldom use the edges in G[S] for the S in Lemma 4. So "almost all" sufficient conditions for a graph to be h-extendable actually force the graph to be 2h-matchable. Hence we can obtain analogous conditions for a graph to be h-matchable.
Remark 2: Akira Saito [9] raised a problem about adding new edges to an h-extendable graph to obtain new h-extendable graphs. However, Gyori and Plummer [3] showed that adding any new edge to some h-extendable graphs, which are neither K_n nor $K_{m,m}$, cannot keep h-extendability.

By Theorem 5, when we prove that some sufficient conditions for a graph to be h-extendable actually force the graph to be 2h-matchable, then adding any new edges to the graph results in many new h-extendable graphs which may not satisfy the original sufficient conditions. For example, conditions $C_i(2h)$ $(1 \leq i \leq 4)$ in Corollary 15 can apply to graphs with arbitrary large diameter (see [4]), adding new edges to the graphs, we can obtain many new h-extendable graphs which do not satisfy $C_i(2h)$.

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