Mod sum graph labelling of $H_{m,n}$ and K_n

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Abstract

The *mod sum number* $\rho(G)$ of a connected graph G is the minimum number of isolated vertices required to transform G into a mod sum graph. It is known that the mod sum number is greater than zero for wheels, W_n, when $n > 4$ and for the complete graphs, K_n when $n \geq 2$. In this paper we show that $\rho(H_{m,n}) > 0$ for $n > m \geq 3$. We show further that $\rho(K_2) = \rho(K_3) = 1$ while $\rho(K_n) = n$ for $n \geq 4$. We thus provide for the first time an exact nonzero mod sum number for an infinite class of graphs.

1 Introduction

A *sum graph* is a graph $G = (V, E)$ and a labelling θ of the vertices of G with distinct positive integers such that $uv \in E$ if and only if the sum of the labels assigned to u and v is the label of a vertex of G. It is obvious that a sum graph cannot be connected. There must always be at least one isolated vertex, namely the vertex with the largest label. The *sum number* $\sigma(H)$ of a connected graph H is the least number r of isolated vertices $\overline{K_r}$ such that $G = H \cup \overline{K_r}$ is a sum graph. For more information about sum graphs see [1], [4], [6], [7], [8], and [9].

A graph $G = (V, E)$ is a *mod sum graph* (MSG) if there exists a positive integer z and a labelling θ of the vertices of G with distinct elements from $\{1, 2, 3, \ldots, z - 1\}$ such that $uv \in E$ if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. Since all labels are distinct we have $z \geq |V| + 1$. Any sum graph is a MSG by using the same labelling scheme employed for the sum graph and choosing the modulus z to be sufficiently large.

The idea of a mod sum graph is related to that of a k-sum graph introduced by Chung in [3]. However, the labellings are sufficiently different and results obtained for k-sum graphs do not apply to mod sum graphs. Mod sum graphs (MSGs) were introduced by Harary [5]. Bolland, Laskar, Turner and Domke [2] provided some early results. Mod sum graphs have potential uses in the efficient storage of graphs and may be used to help design computer networks.

Contrary to the case of sum graphs, there do exist MSGs which are connected. For example, paths on $n \geq 3$ vertices, trees on $n \geq 3$ vertices, cycles on $n \geq 4$ vertices, cocktail party graphs, $H_{2,n}$, and some complete bipartite graphs have been shown to be MSGs [2].

On the other hand, there are connected graphs which are not MSGs, for example, complete graphs K_n for $n \geq 2$ [2] and wheels W_n for $n \geq 5$ [10].

The *mod sum number* $\rho(H)$ of a connected graph H is the least number r of isolated vertices $\overline{K_r}$ such that $G = H \cup \overline{K_r}$ is a mod sum graph.

In this paper we follow the graph theoretic notation and terminology of [5].

We shall use the label of each vertex to denote the vertex itself so that rather than refer to a vertex x and its corresponding label $y = \theta(x)$ we shall simply use y to denote the vertex. This is possible since the definition of a mod sum graph guarantees that each vertex label is distinct. We shall use the term *edge sum*, written as $\{a, b\}$ to mean the sum of the labels modulo z of the two vertices incident on an edge so that $\{a, b\} \in V$ is the same as $a + b \in V$. Traditionally, the same notation $\{a, b\}$ is used to denote an edge joining vertices a and b. Consequently, for a MSG, the statements $\{a, b\} \in E$ and $\{a, b\} = a + b \in V$ both indicate the same edge joining the vertices a and b.

We say that a mod sum labelling of $G = H \cup \overline{K_r}$ is *exclusive* if, for all distinct $a, b, c \in V(H)$, $a + b \neq c \bmod z$.

Bolland, Laskar, Turner and Domke [2] showed that the complete graphs, K_n, $n \geq 2$, are not MSGs. In this paper we show there exists another class of graphs, $H_{m,n}$, $n > m \geq 3$, with arbitrarily large vertex and edge sets, which are not MSGs (Theorem 1). Additionally, we give the mod sum number for all complete graphs K_n, $n \geq 2$ (Theorem 2, Theorem 3) thus providing for the first time an exact nonzero mod sum number for an infinite class of graphs.

1.1 $H_{m,n}$ which are not Mod Sum Graphs

Theorem 1 *$H_{m,n}$ is not a mod sum graph for $n > m \geq 3$.*

The graph $H_{m,n}$, $m, n \geq 2$ is a graph $G = (V, E)$ with a vertex set $V = \{v_1, v_2, v_3, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$, each of size m, such that $v_i v_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$, where $v_i \in I_p$, $v_j \in I_q$ and $p \neq q$. In any labelling of the vertices of $H_{m,n}$ with distict positive integers, one of the labels, s, will be the smallest. Without loss of generality we assign s to the independent set I_1 and denote by V' all the vertices not in I_1 so that $V' = V \backslash I_1$. We label the vertices of I_1 as b_i so that $I_1 = \{b_1 = s, b_2, b_3, \ldots, b_m\}$.

Lemma 1 *In a MSG labelling of $H_{m,n}$, $n > m \geq 3$, for every vertex $b_i \in I_1$ there is a corresponding vertex $a_i \in V'$ such that $a_1 + b_1 = a_2 + b_2 = a_3 + b_3 = \ldots = a_m + b_m$ modulo z.*

Proof. Consider all the edges which exist between the m vertices of I_1 and the $m(n-1)$ vertices of V'. We find there are $m^2(n-1)$ edges and hence $m^2(n-1)$ edge sums. Since there are only mn vertices in the entire graph, it is not possible for all these edge sums to be distinct and consequently, at least one label must correspond to more than one edge sum. We call the number of distinct edges represented by a particular label its *multiplicity*. The average multiplicity of all the vertex labels for edges between vertices of I_1 and V' is thus $\frac{m^2(n-1)}{mn}$ which simplifies to $m - \frac{m}{n}$. This means that at least one vertex of the graph must have a multiplicity of $\lceil m - \frac{m}{n} \rceil$ which evaluates to m when $n > m$. It is clear that each vertex of I_1 can be incident on at most one of these edges and so each of the m vertices of I_1 must be incident on exactly one edge with this edge sum.
□

We note that the vertex $T = a_1 + s = a_2 + b_2 = a_3 + b_3 = \ldots = a_m + b_m$ modulo z cannot be a vertex of the set $\{a_i\}$, $i = 1..m$, or a vertex of the set $I_1 = \{b_i\}$, $i = 1..m$. This implies

Corollary 1 $T \in V' \backslash \{a_i\}$.

Lemma 2 *If, in a MSG labelling of $H_{m,n}$, $n > m \geq 3$, the labels $v, v+s \in V$ then $v < v + s < z$.*

Proof. Since $s > 0$, it is obvious that $v + s \neq v$ and we need only show that $v + s$ cannot wrap around modulo z.
Assume that $v > v + s$ modulo z. Since zero is not a label of the graph we now have $v < z$ and $v + s > z \Rightarrow (v + s) \bmod z < s$, a contradiction.
□

If we substitute $v + s$ for v in Lemma 2 we can show that $v + s < v + 2s < z$ and repeated substitutions in Lemma 2 allow us to conclude that

Corollary 2 *In a MSG labelling of $H_{m,n}$, $n > m \geq 3$, any sequence of labels $v, v + s, v + 2s, v + 3s, \ldots, v + ks$ belonging to V is strictly increasing modulo z.*

Corollary 3 $T = a_1 + s < z$.

Lemma 3 *In a MSG labelling of $H_{m,n}$, $n > m \geq 3$, every vertex $a_i + s$, $i \in \{2..m\}$, must belong to V' unless $a_i + s = b_i$.*

Proof. Since $a_i \in V'$ and $s \in I_1$ we know that $\{a_i, s\} \in E$ and hence $\{a_i, s\} = a_i + s \in V$. We also know that $T \in V'$ and so $\{T, s\} \in E$ giving $\{T, s\} = a_i + b_i + s \in V$. This implies $\{a_i + s, b_i\} \in E$ whenever $a_i + s \neq b_i$. A contradiction occurs if $a_i + s$ and b_i are distinct labels in the independent set I_1.
□

Lemma 4 *In a MSG labelling of $H_{m,n}$, $n > m \geq 3$, there exists at most one a_i, $i \in \{1..m\}$, such that $a_i + s \in I_1$.*

Proof. Assume the contrary so that $a_i + s, a_j + s \in I_1$. Previously we noted that $T = a_1 + s \notin \{b_i\} = I_1$ (Lemma 1, Corollary 1) and so $i, j \in \{2..m\}$.
We now have $b_i = a_i + s$ and $b_j = a_j + s$ (Lemma 3). Since $a_i + b_i = a_j + b_j$ (Lemma 1) we conclude that $2a_i + s = 2a_j + s$ and hence $2a_i = 2a_j \bmod z$ which can only occur when $a_j = a_i + \frac{z}{2}$. It is obvious that this relationship cannot hold for more than two vertices.
We now consider the vertex $2a_i + s + \frac{z}{2}$ resulting from the edge sum $\{b_i, a_j\} = (a_i + s) + (a_i + \frac{z}{2})$ and show that, if it is placed in either I_1 or V', a contradiction results.
If $2a_i + s + \frac{z}{2} \in V'$ then $\{2a_i + s + \frac{z}{2}, s\} \in V$ which implies $\{a_i + s, a_i + s + \frac{z}{2}\} = \{b_i, b_j\} \in E$, a contradiction.
If $2a_i + s + \frac{z}{2} \in I_1$ then we first note that $2a_i + s + \frac{z}{2} \neq b_j = a_i + s + \frac{z}{2}$ since $a_i \neq 0$. The edge sum $\{b_i, T\} = b_i + (a_i + b_i) = a_i + 2b_i = 3a_i + 2s$ is the same as the edge sum $\{a_i + s + \frac{z}{2}, 2a_i + s + \frac{z}{2}\}$ which implies that $\{a_i + s + \frac{z}{2}, 2a_i + s + \frac{z}{2}\} = \{b_j, 2a_i + s + \frac{z}{2}\} \in E$, a contradiction.
□

Lemma 5 *In a MSG labelling of $H_{m,n}$, $n > m \geq 3$, there exist $m - 2$ vertices a_i, $i \in \{2..m\}$, such that $a_i + s \in V'$.*

Proof. Since there are $m - 1$ vertices in the set $\{a_2 + s, a_3 + s, \ldots, a_m + s\}$ and at most one of these is in I_1 (Lemma 4), we have at least $(m - 1) - 1 = m - 2$ of these vertices in V'. We note that $m - 2 \geq 1$ when $m \geq 3$.
□

Lemma 6 *If there exists* **any** *$a_i + 2s$, $i \in \{2..m\}$ such that $a_i + 2s \in V'$ in a MSG labelling of $H_{m,n}$, $n > m \geq 3$, then the vertex with the edge sum $\{T, s\} = a_1 + 2s$ must also belong to V'.*

Proof. We first note that $T = a_i + b_i \neq a_i + s$, $i \in \{2..m\}$ as $b_i \neq s$. Since $T \in V'$ and $a_i + s \in V'$ for some $i \in \{2..m\}$ (Lemma 5), we know that $\{a_i + s, s\}, \{T, s\} \in E$ and hence $\{a_i + s, s\} = a_i + 2s$ and $\{T, s\} = T + s \in V$. If $a_i + 2s \in V'$ then $\{a_i + 2s, b_i\} \in E$ and $\{a_i + 2s, b_i\} = a_i + b_i + 2s = T + 2s \in V$, which would imply that $\{T + s, s\} \in E$ and hence $T + s$ must also belong to V' or a contradiction results.
□

The same argument may now be used to show that $T + 2s \in V'$ if any $a_i + 3s \in V'$ and then to show that $T + 3s \in V'$ if any $a_i + 4s \in V'$ and so on.

Corollary 4 *If there exist labels $a_i + s, a_i + 2s, a_i + 3s, \ldots, a_i + ks$, $i \in \{2..m\}$, for some integer k satisfying $1 \leq k \leq \lfloor \frac{z}{s} \rfloor$, all of which belong to V' then the labels $T, T + s, T + 2s, T + 3s, \ldots, T + (k - 1)s$ must also belong to V'.*

Proof of Theorem 1.
There exists at least one set of labels in V' with elements of the form $a_i + s, a_i + 2s, a_i + 3s, \ldots, a_i + ks$ (Lemma 5) and a corresponding set of labels $T, T + s, T + 2s, T + 3s, \ldots, T + (k-1)s$ also in V' (Lemma 6, Corollary 4) for some integer k satisfying $1 \leq k \leq \lfloor \frac{z}{s} \rfloor$. We also assume that k is the largest such value.
Since we know that the label $\{T + (k-1)s, s\} = T + ks \in V$ (and possibly also $T + (k+1)s, T + (k+2)s, \ldots$), we conclude that a vertex $T + ls$ must belong to I_1 for some $l \geq k$ since there is a finite number of vertices in V' (Lemma 2, Corollary 2). We know that $T + ls \neq s$ (Lemma 2, Corollary 2) and assume, without loss of generality, that $T + ls = b_j,\ j \in \{2 \ldots m\}$. Substituting $a_j + b_j$ for T we find that $a_j + ls = 0 \bmod z$.
If we now consider the labels $T, T+s, T+2s, \ldots, T+ls$, we see that $T < z$ (Lemma 2, Corollary 3) but $T + ls = a_j + b_j + ls > z$ since $a_j + ls = 0 \bmod z$. This is a direct contradiction of Lemma 2, Corollary 2. We conclude that $H_{m,n}$ is not a mod sum graph for $n > m \geq 3$.
□

Unlike the case considered above, for $n \leq m$ it is not known if graphs of the class $H_{m,n}$ are MSGs. This class includes many graphs of particular interest such as $H_{m,2}$, the complete symmetric bipartite graphs.

1.2 The mod sum number of K_n

In this section we present a complete solution to the problem of determining the mod sum number of complete graphs, K_n. Recall that the mod sum number is the minimum number of isolated vertices needed so that the union of K_n and the isolated vertices may be labelled as a mod sum graph.

For the complete graphs K_2 and K_3 we have

Theorem 2 $\rho(K_2) = \rho(K_3) = 1$

Proof of Theorem 2. We label K_2 as a and b and $\overline{K_1}$ as $a + b$ where $z \geq a + b + 1$. We label K_3 as $a, 2a + b$ and $3a + b$ and $\overline{K_1}$ as $4a + b$ where $z = 4a + 2b$. Since it has been shown that K_n, $n \geq 2$ cannot be labelled as a MSG [2], the result follows.
□

Theorem 3 $\rho(K_n) = n$ *for* $n \geq 4$

It is known that K_n are not MSGs [2], and so we have $\rho(K_n) \geq 1$. The proof of the theorem consists of determining a lower bound for $\rho(K_n)$ and providing a labelling which achieves this lower bound. To simplify notation we set $r = \rho(K_n)$.

Lemma 7 *Any MSG labelling of* $K_n \cup \overline{K_r}$ *is an exclusive labelling for* $n \geq 4$.

Proof. Assume the contrary so that a, b and $a+b$ are three distinct labels of K_n. We search for a fourth distinct label $c \in K_n$ with the property that an edge exists from c to another label $d \in K_n$ so that the resulting edge sum, $\{c, d\} = c + d$, is a label in $\overline{K_r}$.
Case 1.
There exists such a label $c \in K_n$ such that $\{c, d\} = c + d \in \overline{K_r}$ for some $d \in K_n \backslash c$.
Since $\{a, c\}, \{b, c\}$ and $\{a+b, c\} \in E$ we note that $a+c, a+b, a+b+c \in V$ and also that $a + c \neq b + c$.
The label d cannot be equal to a as then $c + d = c + a$ and the label $a + b + c$ would imply the edge $\{b, c+a\}$.
Similarly $d \neq b$ otherwise $c + d = c + b$ and the label $a + b + c$ would imply the edge$\{a, c+b\}$.
Now d cannot be simultaneously equal to both $a+c$ and $b+c$ and we assume, without loss of generality, that $d \neq a + c$ and note that $a + c \neq c + d$. The label $a + c$ must be in $\overline{K_r}$ otherwise the edge $\{a+c, d\}$ implies the edge $\{a, c+d\}$. But now the label $a + b + c$ implies the edge $\{b, a+c\}$, a contradiction, whatever the value of d.
Case 2.
There does not exist any fourth label $c \in K_n$ such that $\{c, d\} = c + d \in \overline{K_r}$ for any $d \in K_n \backslash c$.
We take c to be *any* fourth label in K_n and note that, since $\{a, c\}, \{b, c\}$ and $\{a + b, c\} \in E$, the labels $a+c, b+c$ and $a+b+c$ must all be in K_n. We also note that $a + b \neq a + c \neq b + c$.
Since $\rho(K_n) \geq 1$, either the label $\{a, a+b\} = 2a + b \in \overline{K_r}$ or the label $\{b, a+b\} = a + 2b \in \overline{K_r}$. If $2a + b \in \overline{K_r}$ then the edge $\{a+c, a+b\}$ implies the edge $\{2a+b, c\}$, a contradiction. Similarly if $a + 2b \in \overline{K_r}$ then the edge $\{b+c, a+b\}$ implies the edge $\{a+2b, c\}$, a contradiction.
□

Lemma 8 *The lower bound of $\rho(K_n)$, $n \geq 4$ is n.*

Proof. The smallest label of K_n in a MSG labelling of $K_n \cup \overline{K_r}$ is incident on $(n-1)$ edges resulting in $(n-1)$ distinct edge sums which must all belong to $\overline{K_r}$ (Lemma 7). The edge connecting the second smallest label of K_n to the largest label of K_n produces an edge sum which cannot be in $\overline{K_r}$ since it clearly lies (modulo z) between the two edge sums resulting from connecting the smallest vertex to the largest vertex and connecting the two smallest vertices. Thus the minimum value of $\rho(K_n)$ is n.
□

Proof of Theorem 3. We label the vertices of K_n as $1, 1+a, 1+2a, \ldots, 1+(n-1)a$ where $a \geq 4$ and the graph modulus is na. The n vertices of $\overline{K_r}$ (Lemma 8) are labelled as $2, 2+a, 2+2a, \ldots, 2+(n-1)a$.

To prove the theorem we need to show that this labelling has the following three properties.

- There is a label with the edge sum of every edge in K_n.

- There are no labels which imply an edge between a vertex of K_n and a vertex of $\overline{K_r}$.
- There are no labels which imply an edge between two vertices of $\overline{K_r}$.

Every edge of K_n has an edge sum, modulo na, of the form $2+ka$ where $0 \leq k \leq n-1$. There are only n distinct labels of this type and all are present in $\overline{K_r}$.
Any edge between a vertex of K_n and a vertex of $\overline{K_r}$ would have an edge sum, modulo na, of the form $3+ka$ where $0 \leq k \leq n-1$. It is not possible for $3+ka$ to equal one of the labels of the graph, which are all of the form $1+ia$ or $2+ja$, when $a \geq 4$ and the graph modulus is na.
Any edge between two vertices of $\overline{K_r}$ would have an edge sum, modulo na, of the form $4+ka$ where $0 \leq k \leq n-1$. It is not possible for $4+ka$ to equal one of the labels of the graph, which are all of the form $1+ia$ or $2+ja$, when $a \geq 4$ and the graph modulus is na.
□

It remains an open problem to determine the mod sum number of most graphs. The table below is a summary of the current state of knowledge with respect to mod sum numbers.

Graph Class	Restrictions	Mod Sum Number
Trees	$p \geq 3$	0
C_p	$p \geq 4$	0
$K_{p+1,q}$	$q \geq 1$ and $p \geq r_q + r_{q-1} - 1$	0
$K_{2,p}$	$p \geq 0$	0
W_4		0
W_n	$n \geq 5$	> 0
$H_{2,n}$	$n \geq 0$	0
$H_{m,n}$	$n > m \geq 3$	> 0
K_2		1
K_3		1
K_n	$n \geq 4$	n

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