ON THE EXISTENCE OF VERTEX CRITICAL REGULAR GRAPHS OF GIVEN DIAMETER

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ABSTRACT

Let G be connected simple graph with vertex set V(G) and edge set E(G). The diameter d(G), of G is defined as the maximum distance in G. G is said to be vertex diameter critical graph if $d(G - v) > d(G)$ for every $v \in V(G)$. Let $\mathcal{G}(n, r, D)$ denote the class of r-regular, vertex critical graphs of diameter D on n vertices. Plesnik [16] conjectured that $\mathcal{G}(n, r, D) \neq \phi$ for every $D \geq 2$ and $r \geq 2$. In this paper we establish this conjecture. We also consider the problem of determining, for given r and D, the minimum n for which $\mathcal{G}(n, r, D) \neq \phi$.

1. Introduction:

For our purposes a graph G is connected, undirected, loopless and finite. The vertex set and edge set of G are respectively denoted by V(G) and E(G). The ***distance*** $d_G(x, y)$ between two vertices x and y in G is the length of any shortest (x, y)-path in G. The ***eccentricity*** $e(v)$ of a vertex v in G is the distance of the furthest vertex from v, that is

$$e(v) = \max \{ d_G(v, w) : w \in V(G) \}.$$

The ***diameter*** $d(G)$ of G is defined as the maximum eccentricity in G, that is

$$d(G) = \max \{e(v) : v \in V(G) \} = \max \{ d_G(x, y) : x, y \in V(G) \}.$$

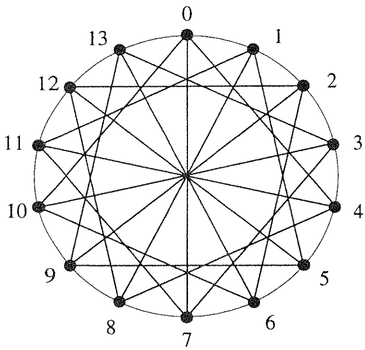
G is said to be vertex diameter critical graph or simply critical if $d(G - v) > d(G)$ for every vertex $v \in V(G)$. Let $\mathcal{G}(n, r, D)$ denote the class of r-regular, vertex critical graphs of diameter D on n vertices. Observe that C_5, the cycle of length 5 and the Petersen graph are critical graphs of diameter 2. Critical graphs have been extensively studied (see [2, 8-16]). Plesnik [16] made the following conjecture:

Conjecture 1: For any integers $D \geq 2$ and $r \geq 2$ there exists an r-regular critical graph of diameter D.

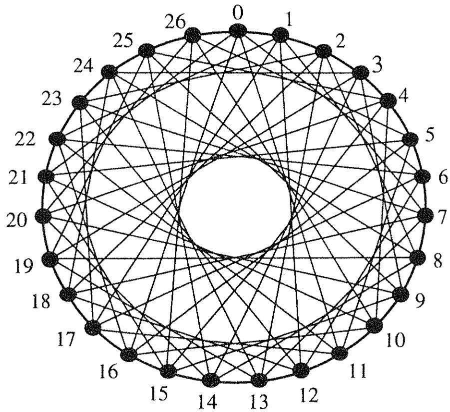
Plesnik [16] observed that the conjecture is easily established for the cases $r = 2$ (the cycle C_{2D+1} on 2D+1 vertices) and $r = 3$ (the cycle C_{4D} on 4D vertices with the main diagonals). We establish the conjecture for all r and D in Section 2.

An interesting and important class of symmetric graphs is the so called circulants graphs defined as follows. The ***circulant graph*** $C_n(a_1, a_2, \ldots, a_p)$, where $a_1 < a_2 < \ldots < a_p < \frac{1}{2}(n+1)$, has vertex set $\{0, 1, 2, \ldots, n-1\}$ and vertex i, $0 \leq i \leq n-1$, is joined to the vertices $i \pm a_1, i \pm a_2, i \pm a_3, \ldots, i \pm a_p \pmod n$. The sequence (a_j) is called the ***jump sequence*** and the a_j's are called the ***jumps***. Observe that for nr even, $r \geq 2$, the circulant $C_n(1, 2, \ldots, \lfloor \frac{1}{2} r \rfloor)$, is just the well known r-regular, r-connected graph on n vertices. For appropriate choices of the a_j's the resulting circulant yields a critical regular graph of diameter D. We now describe such a graph.

For $r \geq 2$ and $D \geq 2$ we let $n = (r-1)(2D-1) + 2$. Let $G(n, r, D) = C_n(1, 2D, 4D-1, 6D-2, \ldots, \lfloor \frac{1}{2}(r-1) \rfloor (2D-1) + 1)$. Observe that $G(2D+1, 2, D)$ is just the cycle C_{2D+1} of length $2D+1$ and the graph $G(4D, 3, D)$ is the cycle C_{4D} of length 4D with the main diagonals added. Further, note that $G(2D+1, 2, D) \in \mathcal{G}(n, 2, D)$ and $G(4D, 3, D) \in \mathcal{G}(n, 3, D)$. Figure 1.1 illustrates some other examples. That G(n, r, D) is critical of diameter D will be establish in Section 2. In Section 3 we will describe another construction based on certain building blocks. In Section 4 we will consider the important problem of finding critical r-regular graphs of minimum order.



G(14,5,2) = $C_{14}(1,4,7)$



G(27,6,3) = $C_{27}(1,6,11)$

Figure 1.1

2. Main Result:

Let G = G(n, r, D). Our objective in this section is to prove that $G \in \mathcal{G}(n, r, D)$. We achieve this through a sequence of lemmas establishing properties of G.

Observe that $C = 0, 1, 2, \ldots, n-1, 0$ is a hamilton cycle in G. An edge (i, j) of G with $j \neq i \pm 1$ is called a ***chord*** of G. Very often we consider the two chords (i + 1, j +1) and (i − 1, j − 1). For convenience we write these two chords as $(i \pm 1, j \pm 1)$. Further, when writing paths we adopt the convention that the "+" and the "−" go together. We now make two simple observations and then establish a number of lemmas. We begin with the following two simple observations:

Observation 2.1: If (i, j) is a chord of G, then $(i \pm 1, j \pm 1)$ are two chords of G.

Observation 2.2: If P is a shortest (a, b)-path of length $\ell(P)$ containing the chord (i, j) and the edge $(j, j \pm 1)$, then $(i, i \pm 1) \notin P$.

Proof: If $(j, j \pm 1)$ and $(i, i \pm 1)$ are in P. Then

$$P' = P - (i \pm 1, i) - (i, j) - (j, j \pm 1) + (i \pm 1, j \pm 1)$$

is an (a, b)-path of length $\ell(P) - 2$, a contradiction. □

An important property of G is given in the following lemma.

Lemma 2.1: Let (i, j) and (j, k) be two distinct chords of G. Then one of (k, i + 1) or (k, i − 1) is a chord of G.

Proof: We have

$$j = i \pm (1 + \lambda_1 (2D - 1)),\ 0 < \lambda_1 \leq \lfloor \tfrac{1}{2}(r-1) \rfloor$$

and

$$k = j \pm (1 + \lambda_2 (2D - 1)),\ 0 < \lambda_2 \leq \lfloor \tfrac{1}{2}(r-1) \rfloor.$$

Hence

$$k \equiv i \pm (1 + \lambda_1 (2D - 1)) \pm (1 + \lambda_2 (2D - 1)) \pmod{n}.$$

Observe that since $i \neq k$, $\lambda_1 + \lambda_2 < r - 1$.

We now consider two cases.

Case 1: $k \equiv i \pm (2 + (\lambda_1 + \lambda_2)(2D - 1)) \pmod{n}$.

If $\lambda = \lambda_1 + \lambda_2 \leq \lfloor \tfrac{1}{2}(r-1) \rfloor$, then $k = i \pm (2 + \lambda (2D - 1))$ and hence (k, i − 1) or (k, i + 1) is a chord of G. So we may suppose that $\lambda > \lfloor \tfrac{1}{2}(r-1) \rfloor$. If $k = i + 2 + \lambda (2D - 1)$, then we can write

$$\begin{aligned} k &= -n + k \\ &= i + 1 - (1 + (r - 1 - \lambda)(2D - 1) \\ &= i + 1 - (1 + \lambda'(2D - 1), \quad 0 < \lambda' < \lfloor \tfrac{1}{2}(r-1) \rfloor, \end{aligned}$$

and hence (k, i + 1) is a chord of G. If, on the other hand, $k = i - 2 - \lambda (2D - 1)$, then we can write

$$\begin{aligned} k &= n + k \\ &= i - 1 + (1 + (r - 1 - \lambda)(2D - 1) \\ &= i - 1 + (1 + \lambda'(2D - 1), \quad 0 < \lambda' < \lfloor \tfrac{1}{2}(r-1) \rfloor, \end{aligned}$$

and hence (k, i − 1) is a chord of G.

Case 2: $k \equiv i \pm ((\lambda_1 - \lambda_2)(2D - 1)) \pmod{n}$.

We have $k = i \pm \lambda (2D - 1)$, $0 < \lambda \leq \lfloor \tfrac{1}{2}(r-1) \rfloor$. If $k = i + \lambda (2D - 1)$, (k, i − 1) is a chord of G, whilst if $k = i - \lambda (2D - 1)$, then (k, i + 1) is a chord of G.

This completes the proof of the lemma. □

Lemma 2.2: Suppose P is a shortest (a, b)-path in $G = G(n, r, D)$ containing $t \geq 2$ chords. Then there exists a shortest (a, b)-path in G containing $t - 1$ chords.

Proof: If P has two consecutive chords (i, j) and (j, k), then, by Lemma 2.1, one of $(k, i-1)$ or $(k, i+1)$ is a chord of G. If $(k, i \pm 1)$ is a chord, then

$$P_1 = P - (i, j) - (j, k) + (i, i \pm 1) + (i \pm 1, k)$$

is a shortest (a, b)-path with $t - 1$ chords. If, on the other hand, P does not have two consecutive chords, let (i, j) be the first chord of P encountered in moving from a to b. Then one of $(j, j \pm 1) \in P$. If $i = a$, then $(a \pm 1, j \pm 1) \in E(G)$ and

$$P' = P - (a, j) - (j, j \pm 1) + (a, a \pm 1) + (a \pm 1, j \pm 1)$$

is also shortest (a, b)-path in G. Consequently we can assume without loss of generality that $i \neq a$. Then one of $(j, j \pm 1) \in P$. If $(j, j+1) \in P$, then, by Observation 2.2, $(i, i+1) \notin P$. Hence $(i, i-1) \in P$ and, by Observation 2.1, $(i+1, j+1) \in G$. Therefore,

$$P' = P - (i, j) - (j, j+1) + (i, i+1) + (i+1, j+1)$$

is a shortest (a, b)-path in G. Similarly if $(j, j-1) \in P$, then, by Observation 2.2, $(i, i-1) \notin P$. Hence $(i, i+1) \in P$ and, by Observation 2.1, $(i+1, j+1) \in G$. Therefore,

$$P'' = P - (i, j) - (j, j-1) + (i, i-1) + (i-1, j-1)$$

is a shortest (a, b)-path in G. Thus we can replace P by P' or P'' and repeat the same argument until we will get a shortest (a, b)-path in G with two consecutive chords. This completes the proof of the lemma. □

As a corollary we have:

Corollary 2.1: Let $\mathcal{P}$ be the set of shortest (a, b)-paths in $G = G(n, r, D)$ having chords. If $\mathcal{P} \neq \phi$, then there exists a $P \in \mathcal{P}$ having exactly one chord which is incident to b.

Lemma 2.3: Let $G = G(n, r, D)$ and let $C = 0, 1, 2, \ldots, n-1, 0$ be a hamilton cycle in G. If P is a shortest (0, D)-path, then P has no chords of G.

Proof: Since $0, 1, 2, \ldots, D-1, D$ is a (0, D)-path of length D, then $\ell(P) \leq D$. In view of Corollary 2.1 if P has chords, then we can assume that it has exactly one chord which is incident to D. But the only vertices along the segment $S = -D, -D+1, \ldots, n-1, 0, 1, \ldots, D-1, D$ of C that are joined to D are $-D$ and $D-1$, implying that $\ell(P) > D$, a contradiction. Hence P has no chords of G. □

As a corollary we have:

Corollary 2.2: The shortest (0, D)-path in G is the segment $0, 1, 2, \ldots, D-1, D$ of length D.

We are now ready to prove our main result.

Theorem 2.1: For $r \geq 2$ and $D \geq 2$ the graph $G(n, r, D) \in \mathcal{G}(n, r, D)$.

Proof: Let $G = G(n, r, D)$. Since G is circulant graph, it contains the hamilton cycle $C = 0, 1, 2, \ldots, n-1, 0$. Further it is vertex symmetric and transitive. Thus to show that G is an r-regular vertex critical graph of diameter D it suffices to consider one vertex, say vertex 0. Observe that

$$N_G(0) = \{\pm 1, \pm(1+(2D-1)), \pm(1+2(2D-1)), \pm(1+3(2D-1)), \ldots, \pm(1+\lfloor \tfrac{1}{2}(r-1) \rfloor (2D-1))\}$$

and thus $d_G(0) = r$, as required.

For every vertex i there exists a $\lambda \geq 0$ such that $1+(\lambda-1)(2D-1) \leq i \leq 1+\lambda(2D-1)$. Hence the vertices i and 0 are contained in the cycle $0, 1+(\lambda-1)(2D-1), 2+(\lambda-1)(2D-1), \ldots, 1+\lambda(2D-1), 0$ of length $2D+1$. Consequently $d_G(0, i) \leq D$ and hence G has diameter $\leq D$.

By Corollary 2.2, $d_G(0, D) = D$. Thus G has diameter D as required.

Finally we show that vertex 0 is critical. By Corollary 2.2, the shortest $(n-1, D-1)$-path is the segment $n-1, 0, 1, \ldots, D-2, D-1$. Hence $d_{G-0}(n-1, D-1) > D$. This completes the proof of the theorem. □

We conclude this section by establishing two further properties of the graph G(n, r, D). These further properties make the graph useful in the context of network applications.

Lemma 2.4: For $r \geq 2$ and $D \geq 2$ the graph G(n, r, D) is r-connected.

Proof: Let $G = G(n, r, D)$. Since G is vertex symmetric and transitive, to show that G is r-connected it suffices to construct r disjoint paths joining vertex 0 and any other vertex $j \in V(G)$. Observe that $C = 0, 1, 2, \ldots, n-1, 0$ is a hamilton cycle in G and

$$N_G(0) = \{\pm 1, \pm(1+(2D-1)), \pm(1+2(2D-1)), \pm(1+3(2D-1)), \ldots, \pm(1+\lfloor \tfrac{1}{2}(r-1) \rfloor (2D-1))\}$$
$$= \{n_1, n_2, n_3, \ldots, n_r\}, \text{ with } n_1 = 1 < n_2 < n_3 < \ldots < n_r = (r-1)(2D-1)+1.$$

Also, note that if $j \notin [n_i, n_{i+1}-1]$ then it is adjacent to only one vertex along the segment $n_i, n_i+1, n_i+2, \ldots, n_{i+1}-1$ of C.

We construct the r disjoint paths as follows: Start with an edge $0n_i$, $1 \leq i \leq r$, and proceed as follows: if $n_i < j$ we move forward along C till the first vertex joining j; if $n_i = j$ we stop; if $n_i > j$ we move backward along C till the first vertex joining j. By repeating the previous procedure we get r disjoint paths joining the vertices 0 and j. Hence the result. □

Lemma 2.5: For $r \geq 3$ and $D \geq 2$, let the graph $G = G(n, r, D)$. Then $d(G - v) = D + 1$ for every $v \in V(G)$.

Proof: Since G is critical and vertex symmetric and transitive, it suffices to show that there are two disjoint paths of length $\leq D + 1$ joining vertex 0 and any other vertex $i \in V(G)$. We now consider a number of cases according to the value of r.

Case 1: $r = 3$.

We have the two cycles $0, 1, 2, \ldots, 2D, 0$ and $0, 2D, 2D + 1, \ldots, 4D - 1, 0$ of length $2D + 1$. If $i = 2D$, then we have the two (0, 2D)-paths $0, 2D$ and $0, 1, 2D + 1, 2D$ of length $\leq D + 1$. If, on the other hand, $i \neq 2D$, then i is contained in one of these cycles and is adjacent to one vertex only of the other. Consequently, there exist two disjoint (0, i)-paths length $\leq D + 1$.

Case 2: $r = 4$.

We have the two cycles $0, 1, 2, \ldots, 2D, 0$ and $0, 4D - 1, 4D, \ldots, 6D - 2, 0$ of length $2D + 1$. If $1 \leq i \leq 2D$ or $4D - 1 \leq i \leq 6D - 2$, then i is contained in one of these cycles and adjacent to one vertex only of the other. If, on the other hand, $2D < i < 4D - 1$, then i is adjacent to one vertex only of each of the two cycles. In either case there exist two disjoint (0, i)-paths length $\leq D + 1$.

Case 3: $r \geq 4$.

For every vertex i there exists a $\lambda \geq 0$ such that $1+(\lambda - 1)(2D - 1) \leq i \leq 1+\lambda(2D-1)$. Hence the vertex i is contained in the cycle $0, 1+(\lambda - 1)(2D - 1), 2+(\lambda - 1)(2D - 1), \ldots, 1+\lambda(2D-1), 0$ of length $2D + 1$. Furthermore, i is adjacent to one vertex only of the cycle $0, 1+(\lambda+1)(2D-1), 2+(\lambda+1)(2D-1), \ldots, 1+(\lambda+2)(2D-1), 0$ of length $2D + 1$. Therefore, there exist two disjoint (0, i)-paths length $\leq D + 1$.

This completes the proof of the lemma. □

3. Another Construction:

We have observed that C_5 and the Petersen graphs are critical graphs of diameter 2. The Petersen graph can be obtained by adding an appropriate matching between two C_5's (see Figure 3.1). In this section we describe a construction, based on building blocks, that generates a member of $\mathcal{G}(n, r, D)$ for $r \geq 3$. We begin with $D = 2$ where our building block is the 5-cycle C_5.

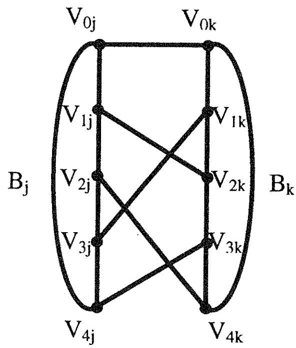


Figure 3.1

For r = 2 or 3, we have already described members of $\mathcal{G}(n, r, 2)$. For $r \geq 4$, we construct a graph $G(n, r, 2) \in \mathcal{G}(n, r, 2)$ as follows. Let $G(n, r, 2) = G(V, E)$ have $r - 1$ copies $B_1, B_2, \ldots, B_{r-1}$ of the 5-cycle $B_j = v_{0j}\, v_{1j}\, v_{2j}\, v_{3j}\, v_{4j}\, v_{0j}$ and we define $E(G)$ as follows:

$$E(G) = \bigcup_{1 \leq j \leq r-1} E(B_j) \quad \bigcup_{1 \leq j < k \leq r-1} \{v_{0j}v_{0k},\ v_{1j}v_{2k},\ v_{2j}v_{4k},\ v_{3j}v_{1k},\ v_{4j}v_{3k}\}.$$

We consider B_j to be in level j. Figure 3.2 displays $G(5(r-1), r, 2)$.

Lemma 3.1: $G(5(r-1), r, 2) \in \mathcal{G}(n, r, 2)$.

Proof: Observe that G is r-regular since each vertex v_{ij}, $0 \leq i \leq 4$, $1 \leq j \leq r-1$ has two neighbours in B_j and one neighbour in B_k, $1 \leq k \leq r-1$, $k \neq j$. Observe also that every vertex v_{ij} in B_j is adjacent to only one vertex $v_{\ell k}$ in B_k and the subgraph induced by the vertices in any two levels is the Petersen graph P. Therefore $d_G(v_{ij}, v_{\ell k}) \leq 2$, $d(G) = 2$ and $d_{G-v_{ij}}(v_{i-1,j}, v_{i+1,j}) = 3$, for every v_{ij}, where the subscripts are read modulo 5. Hence $G(5(r-1), r, 2) \in \mathcal{G}(n, r, 2)$, as required. □

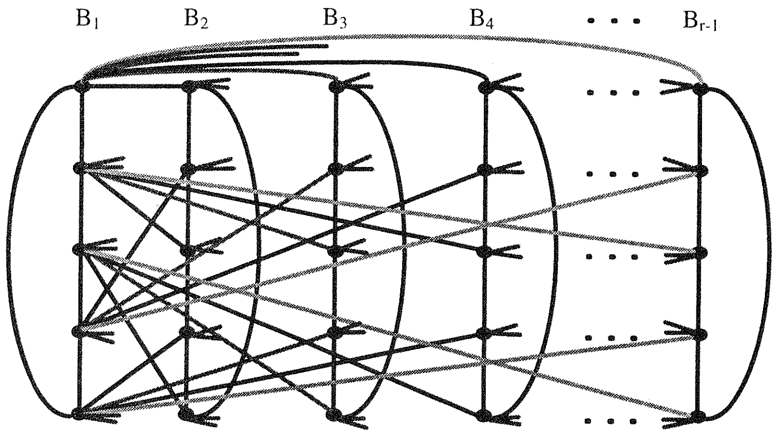


Figure 3.2: G(5(r – 1), r, 2)

Now for $D \geq 3$, our building block (level j) is the graph $H_j(V_{H_j}, E_{H_j})$, where

$$V_{H_j} = \{a_{ij} : 1 \leq i \leq 2D-2\} \bigcup \{b_{1j}, b_{2j}\} \bigcup \{c_{ij} : 1 \leq i \leq 2D-2\},$$

$$E_{H_j} = E(C_j) \bigcup \{ b_{1j}\, b_{2j}, a_{2D-2,j}\, c_{1,j}\} \bigcup \{ a_{ij}\, c_{i+1,j} : 1 \leq i \leq 2D-3\} \text{ and}$$

$$C_j = b_{1j}\, a_{1j}\, a_{2j} \ldots a_{2D-3,j}\, a_{2D-2,j}\, b_{2j}\, c_{2D-2,j}\, c_{2D-3,j} \ldots c_{2j}\, c_{1j}\, b_{1j}$$

Figure 3.3 displays H_j.

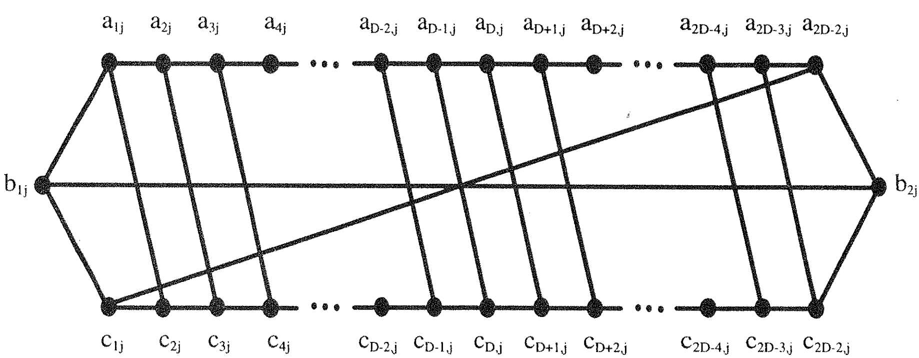


Figure 3.3: H_j

Lemma 3.2: $H_j \in \mathcal{G}(4D-2, 3, D)$.

Proof: It is clear from the definition of H_j that H_j is 3-regular and $|V(H_j)| = 4D-2$. Observe that $d(a_{1j}, a_{D+1,j}) = D$ and every pair of vertices $v_{ij}, v_{\ell j} \in V(H_j)$ are contained in a cycle of length $\leq 2D$. Hence $d(H_j) = D$. Further, observe that

$$d_{H_j - a_{1j}}(b_{1j}, a_{D-1,j}) = D+1$$

$$d_{H_j - a_{ij}}(a_{i-1,j}, a_{i+D-2,j}) = D+1 \quad \text{for} \quad 1 < i \leq D$$

$$d_{H_j - a_{ij}}(a_{i-1,j}, c_{i-D,j}) = D+1 \quad \text{for} \quad D+1 \leq i \leq 2D\text{-}2.$$

Hence $d(H_j - a_{ij}) > D$. Similarly, $d(H_j - c_{ij}) > D$. Finally for $D > 3$, $d_{H_j - b_{ij}}(a_{1j}, c_{D+2,j}) = D+1$ and for $D = 3$, $d_{H_j - b_{1j}}(a_{1j}, b_{2j}) = D+1$ and $d_{H_j - b_{2j}}(b_{1j}, c_{D+1,j}) = D+1$. Therefore $d(H_j - v_{ij}) > D$ for every $v_{ij} \in V(H_j)$. Hence $H_j \in \mathcal{G}(4D-2, 3, D)$, as required. □

Now we will construct a $H(n, r, D) \in \mathcal{G}(n, r, D)$ for $D \geq 3$ and $r \geq 3$ which contains $r-2$ levels, each level j has the block H_j as an induced subgraph. More specifically, let $H(n, r, D) = H(V, E)$, where

$$V(H) = \bigcup_{1 \leq j \leq r-2} V(H_j) \qquad n = |V(H)| = (4D-2)(r-2)$$

$$E(H) = \bigcup_{1 \leq j \leq r-2} E(H_j) \quad \bigcup_{1 \leq j < k \leq r-2} \{b_{1j}\, b_{2k},\ b_{2j}\, b_{1k}\}$$

$$\bigcup_{1 \leq j < k \leq r-2} \{a_{2i-1,j}\, a_{2i-1,k},\ c_{2i-1,j}\, c_{2i-1,k},\ a_{2i,j}\, c_{2i,k},\ c_{2i,j}\, a_{2i,k} : 1 \leq i \leq D-1\}.$$

Figure 3.4 displays $H(4(2D-1), 4, D)$.

Lemma 3.3: $H(n, r, D) \in \mathcal{G}(n, r, D)$.

Proof: Clearly, H is r-regular since each vertex in H is adjacent to three vertices in its level j and to one vertex in the other $r-3$ levels. Observe that $d(a_{1j}, a_{D+1,j}) = D$ and every pair of vertices $v_{ij}, v_{\ell k} \in V(H)$ are contained in a cycle of length $\leq 2D$. Hence $d(H) = D$. Further, observe that

$$d_{H-a_{1j}}(b_{1j}, a_{D-1,j}) = D+1$$

$$d_{H-a_{ij}}(a_{i-1,j}, a_{i+D-2,j}) = D+1 \quad \text{for} \quad 1 < i \leq D$$

$$d_{H-a_{ij}}(a_{i-1,j}, c_{i-D,j}) = D+1 \quad \text{for} \quad D+1 \le i \le 2D-2.$$

Hence $d(H - a_{ij}) > D$. Similarly, $d(H - c_{ij}) > D$. Finally observe that $d_{H-b_{1j}}(a_{1j}, c_{D+1,k}) = D+1$ and $d_{H-b_{2j}}(b_{1j}, c_{D+1,k}) = D+1$. Therefore $d(H - v_{ij}) > D$ for every $v_{ij} \in V(H)$. □

Hence $H(n, r, D) \in \mathcal{G}(n, r, D)$, as required.

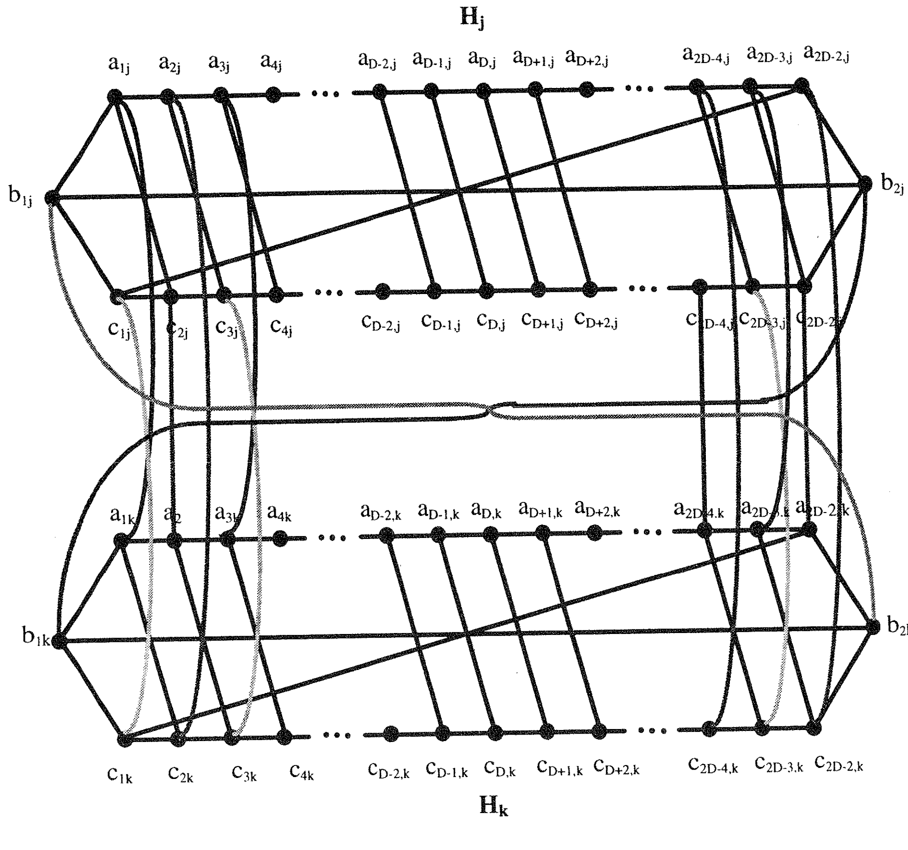


Figure 3.4: $H(4(2D-1), 4, D)$.

4. Minimum Order Critical Graphs.

For $r \geq 2$ and $D \geq 2$, let

$$f(r, D) = \min \{n: \mathcal{G}(n, r, D) \neq \phi\}$$

Observe that $f(2, 2) = 5$ and $f(2, D) = 2D$ for $D \geq 3$.

Caccetta [6] posed the problem of determining $f(r, D)$. In this section we consider this problem.

The circulant graphs constructed in the introduction established that $f(r, D) \leq (r-1)(2D-1) + 2$. This upper bound is far from best possible. Our consideration of many constructions suggests that the following is true:

Conjecture 4.1:

$$f(r,D) = \begin{cases} 4D-2, & \text{for } r = 3 \text{ and } D \geq 3 \\ rD+1, & \text{for even } r > 3 \\ (r+1)D, & \text{otherwise.} \end{cases}$$

We will start by establishing the conjecture for $D = 2$. After that we will consider some special cases for r and D. Then we conclude this section by presenting some constructions to show that this bound is achievable.

In studying the diameter of a graph it is convenient to consider the level structure of the graph. More specifically, let $G(n, r, D) = G(V, E)$ be a graph of diameter D. Then for a vertex $v \in V(G)$, $V(G)$ can be partitioned into non empty subsets $L_0(v) = \{v\}$, $L_1(v)$, $L_2(v)$, …, $L_t(v)$, $t = e(v)$, such that $L_i(v)$, $1 \leq i \leq t$, consists of those vertices of $G - v$ that are at distance i from v.

Now for $D = 2$ we begin with the following lemma which establishes a lower bound on $f(r, 2)$.

Lemma 4.1: For $r \geq 2$

$$f(r, 2) \geq \begin{cases} 2r+1, & \text{for even } r \\ 2r+2, & \text{otherwise.} \end{cases}$$

Proof: Let $G(V, E) \in \mathcal{G}(n, r, 2)$ be a graph with vertex decomposition $\{v\} \cup L_1(v) \cup L_2(v)$. Boals et al. [2] show that if $G \in \mathcal{G}(n, r, 2)$ then $d(G - v) = 3$ for every $v \in V(G)$. Then there exists a pair of vertices x and y such that $d_{G-v}(x, y) = 3$. Hence $V(G)\setminus\{v\}$ can be partitioned into non empty subsets $L_0(x) = \{x\}$, $L_1(x)$, $L_2(x)$ and $L_3(x)$. Furthermore, $|L_0(x)| = 1$, $|L_1(x)| = r - 1$ and $|L_2(x)| + |L_3(x)| \geq r$. Hence $|V(G - v)| \geq 2r$ and thus $f(r, 2) \geq 2r + 1$. The result follows since $r\,f(r, 2)$ is even. □

We now describe some constructions which show that the bounds in Lemma 4.1 are in fact sharp. We consider four cases according to the value of $r \pmod 4$. In all cases the vertex set is $V(G_t) = \{0, 1, 2, \ldots, n-1\}$, $0 \leq t \leq 3$.

(a) For $r \equiv 0 \pmod 4$, define the graph G_0 as

$G_0 = C_n(1, 4, 5, 8, 9, 12, 13, 16, 17, \ldots, r-8, r-7, r-4, r-3, r)$, $n = 2r + 1$.

(b) For $r \equiv 1 \pmod 4$, let $n = 2r + 2$ and define the graph G_1 in which the vertex i, $0 \leq i \leq n-1$, is joined to the vertices $i + 1, i + 4, i + 5, i + 8, i + 9, i + 12, i + 13, \ldots, i + 2r - 6, i + 2r - 5, i + 2r - 2 \pmod n$ for i even and $i + 1, i + 4, i + 7, i + 8, i + 11, i + 12, \ldots, i + 2r - 6, i + 2r - 3, i + 2r - 2 \pmod n$ for i odd.

(c) For $r \equiv 2 \pmod 4$, define the graph G_2 as

$G_2 = C_n(1, 4, 5, 8, 9, 12, 13, \ldots, r-6, r-5, r-2, r-1)$, $n = 2r + 1$.

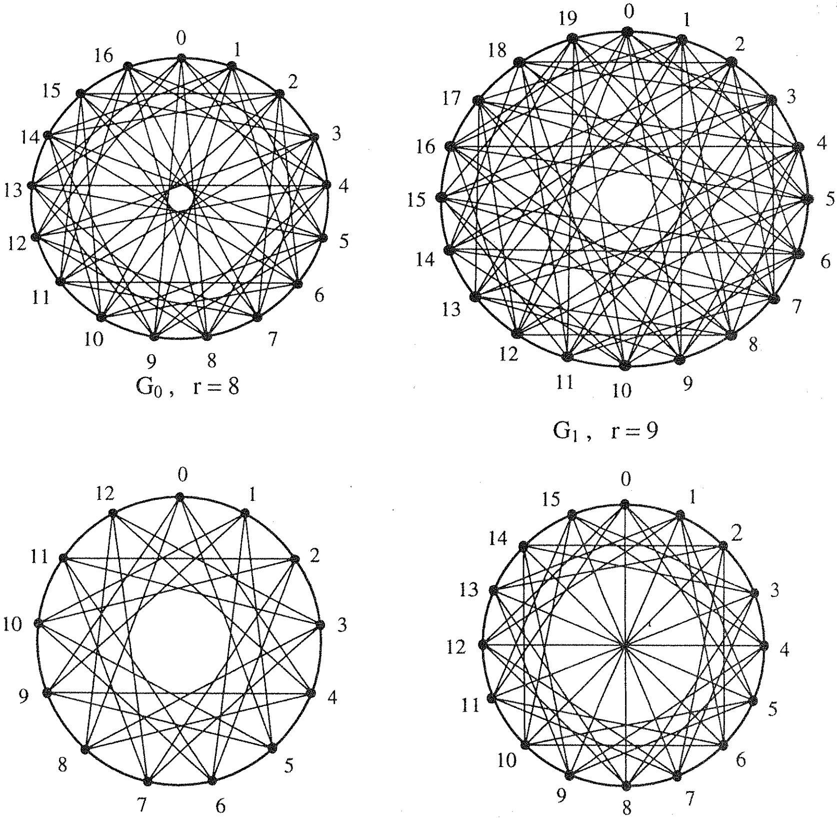
(d) For $r \equiv 3 \pmod 4$, define the graph G_3 as

$G_3 = C_n(1, 4, 5, 8, 9, 12, 13, \ldots, r-7, r-6, r-3, r-2, r+1)$, $n = 2r + 2$.

Figure 4.1 shows some examples of these constructions.

Lemma 4.2: The graphs in (a) - (d) $G_t \in \mathcal{G}(n, r, 2)$, $0 \leq t \leq 3$.

Proof: For any pair of vertices $i, j \in V(G_t)$, $0 \leq t \leq 3$, if $(i, j) \notin E(G_t)$ then from the definition of G_t the vertex i is joined to one of the vertices $j - 1$ or $j + 1 \pmod n$. If vertex i is joined to $j - 1$ then we have the cycle $i, j-1, j, j+1, j+2, i$. On the other hand, if vertex i is joined to $j + 1$ then we have the cycle $i, j-2, j-1, j, j+1, i$. Hence $d_{G_t}(i, j) \leq 2$ for any $i, j \in V(G_t)$. It is easy to see from the definition that the graph G_t is r-regular and $N\{i\} \cap N\{i+2\} = \{i + 1\}$ for every vertex $i \in V(G_t)$. Therefore $d_{G_t}(i, i+2) = 2$ and $d_{G_t-(i+1)}(i, i+2) = 3$. Hence the result. □



G_0, $r = 8$ G_1, $r = 9$ G_2, $r = 6$ G_3, $r = 7$

Figure 4.1

Lemmas 4.1 and 4.2 together yield:

Theorem 4.1: For $r \geq 2$

$$f(r, 2) = \begin{cases} 2r+1, & \text{for even } r \\ 2r+2, & \text{otherwise.} \end{cases}$$

A useful property for $D \geq 3$ is given in the following lemma.

Lemma 4.3: Let $G(n, r, D) = G(V,E) \in \mathcal{G}(n, r, D)$, $D \geq 3$, such that for a vertex $v \in V$ $d_{G-v}(x, y) > D$ and $x, y \in L_1(v)$. Then $|L_2(v)| \geq r + 1$.

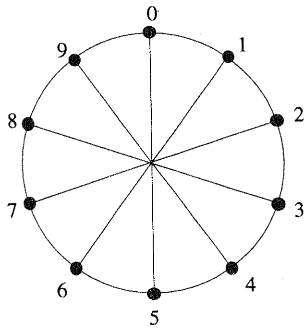
Proof: Let $X_i = N_G(x) \cap L_i(v)$, $Y_i = N_G(y) \cap L_i(v)$, $A = L_1(v) \setminus (\{x, y\} \cup X_1 \cup Y_1)$, $|X_i| = m_i$, $|Y_i| = n_i$ and $|A| = a$. Then $m_1 + m_2 = r-1$, $n_1 + n_2 = r-1$ and $m_1 + n_1 + a = r-2$.

Now we have $m_2 + n_2 = 2(r-1) - (m_1 + n_1) = 2r - 2 - r + 2 + a = r + a$. Thus $|L_2(v)| \geq r + a$. Therefore, we need only to consider the case $a = 0$. Observe that there is no edge joining a vertex of X_i to Y_i. If $w \in X_1$, then $N_G(w) \subseteq \{x, y\} \cup (X_1 \setminus \{w\}) \cup X_2 = S$ and $|S| = 2 + m_1 - 1 + m_2 = r$, and so $N_G(w) = S$ for $d_G(w) = r$. But then x and w have the same closed neighbour set, a contradiction. Hence the result. □

Theorem 4.2: $f(3, 3) = 10$.

Proof: Let $G(n, 3, 3) = G(V,E) \in \mathcal{G}(n, 3, 3)$. Then there exist vertices v, x and $y \in V$ such that $e(v) = 3$ and $d_{G-v}(x, y) > 3$. Let $x \in L_i(v)$ and $y \in L_j(v)$. Clearly $i + j \leq 3$. Therefore we can assume without loss of generality that $x \in L_1(v)$. Suppose that $y \in L_1(v)$. Then by Lemma 4.2 $|L_2(v)| \geq 4$. Hence $|V| = |L_0(v)| + |L_1(v)| + |L_2(v)| + |L_3(v)| \geq 1 + 3 + 4 + 1 = 9$.

Now we consider the case $y \in L_2(v)$. Here $V(G) \setminus \{v\}$ can be partitioned into non empty subsets $L_0(x) = \{x\}$, $L_1(x)$, $L_2(x)$, …, $L_t(x)$, where $t > 3$ and $y \in L_t(x)$ such that $L_i(x)$, $1 \leq i \leq t$, consists of those vertices of $G - v$ that are at distance i from x. Furthermore, $|L_0(x)| = 1$, $|L_1(x)| = 2$, $|L_2(x)| \geq 1$ and $|L_{t-1}(x)| + |L_t(x)| \geq 4$. Hence $|V(G-v)| \geq 8$ and thus $|V| \geq 9$. Now since $r = 3$, $f(3, 3) \geq 10$. The graph $G(10, 3, 3) = C_{10}(1, 5)$ depicted in Figure 4.2 shows that $f(3, 3) = 10$, as required. □



$G(10, 3, 3) = C_{10}(1, 5)$

Figure 4.2

Remark: Using a lengthy case analysis we have established that f(3, 4) = 14 and f(4, 3) = 13.

Lemma 4.4: For $r \geq 3$, $r \neq 1 \pmod 4$, $D \geq 3$

$$f(r,D) \leq \begin{cases} 4D-2, & \text{for } r=3 \text{ and } D \geq 3 \\ rD+1, & \text{for even } r>3 \\ (r+1)D, & \text{otherwise.} \end{cases}$$

Proof: We establish our upper bounds by construction. For $r = 3$ and $D \geq 3$ the graph H_j depicted in Figure 3.3 has the required property. For $r \geq 4$ we consider three cases according to the value of $r \pmod 4$. In all cases the vertex set is $V(G_t) = \{0, 1, 2, \ldots, n-1\}$, $t = 0, 2, 3$ and the constructed graph $G_t \in \mathcal{G}(n, r, D)$.

(a) For $r \equiv 0 \pmod 4$, define the graph G_0 as

$G_0 = C_n(1, 2D, 2D+, 4D, 4D+1, \ldots, (\frac{1}{2}r-2)D, (\frac{1}{2}r-2)D+1, \frac{1}{2}rD)$, $n = rD + 1$.

(b) For $r \equiv 2 \pmod 4$, define the graph G_2 as

$G_2 = C_n(1, 2D, 2D+1, 4D, 4D+1, \ldots, (\frac{1}{2}r-1)D, (\frac{1}{2}r-1)D+1)$, $n = rD + 1$.

(c) For $r \equiv 3 \pmod 4$, define the graph G_3 as

$G_3 = C_n(1, 2D, 2D+1, 4D, 4D+1, \ldots, (\frac{1}{2}(r+1)-2)D, (\frac{1}{2}(r+1)-2)D+1, \frac{1}{2}(r+1)D)$,

$n = (r + 1)D$

This completes the proof of the lemma. □

Remark: For $r \equiv 1 \pmod 4$ we believe the upper bound for f(r, D) is (r +1) D. However, we have not been able to construct graphs having this bound except for some special cases (r = 5, D = 3 and 4).

Acknowledgement :

This project was supported, in part, by the Australian Research Council Grant No. A49532661

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(Received 15/12/98)