Strongly 2-perfect trail systems and related quasigroups

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Abstract

In a recent paper of the authors the question of for which values of m the quasigroups arising from strongly two 2-perfect m-cycle systems are the finite members of a variety was considered. In comparison with existing results on similar problems, it seems likely that for m prime, we shall fail to get varieties only for those values of m where strongly 2-perfect closed m-trail systems which are not strongly 2-perfect m-cycle systems exist and we give a necessary and sufficient condition for their existence. We show that if there exist strongly 2-perfect closed m-trail systems satisfying an additional condition, then the quasigroups arising from strongly 2-perfect m-cycle systems are not the finite members of a variety. We give reasons why the existence of such systems seems plausible. We also show that, the quasigroups corresponding to strongly 2-perfect 9-cycle systems are not the finite members of a variety.

1. Introduction

The idea of cycle systems and their associated groupoids seems to have first appeared in the work of Kotzig [5] and several authors have since worked with this concept ([7] gives an excellent survey). Additional structure may be required of cycle systems. One such property is that of being *i-perfect*. The case where $i = 2$ is of particular interest, because then, as was shown by Keedwell [3], [4], the associated groupoid is a quasigroup, which he called a *P-quasigroup*. The question as to when these are the finite members of a variety has been intensively studied (see [6]). When the quasigroups arising from a class of cycle systems are precisely the finite members of a variety, the class is said to be *equationally defined*.

In their paper [1], Bryant and Lindner showed that 2-perfect m-cycle systems are equationally defined for $m = 3$, 5 and 7 only. In [2], we looked at the question as to when *strongly* 2-perfect cycle systems are equationally defined. We noted that

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$m = 127$ was the first prime for which our technique failed to prove that strongly 2-perfect m-cycle systems are equationally defined. In this paper we show that this is no accident, as $m = 127$ is the first prime for which a strongly 2-perfect closed m-trail system exists. We show that the existence of strongly 2-perfect closed m-trail systems is almost certainly enough to guarantee that strongly 2-perfect m-cycle systems are not equationally defined. We also prove that strongly 2-perfect 9-cycle systems are not equationally defined.

Some definitions are in order. A *closed m-trail* is any connected simple graph (undirected and without loops) with m edges and all vertices of even degree. All trails considered in this paper will be closed and so from here on we may refer to them simply as trails with the understanding that they are closed. An *m-cycle* is an m-trail in which all vertices have degree 2. A trail which is not a cycle will be called a *proper* trail.

A pair v_i and v_j of vertices of a cycle are said to occur at distance i if they are joined by a path of length i in the cycle. We take more care to define distance between pairs of vertices in a proper trail. Indeed, we need to first associate an Euler circuit with the trail. The vertices v_i and v_j are said to occur at distance i in the m-trail, with associated Euler circuit $(v_1, v_2, \dots, v_m)$, if and only if v_i and v_j are joined by a path of length i in the Euler circuit $(v_1, v_2, \dots, v_m)$. Throughout this paper, m-trails will have an associated Euler circuit $(v_1, v_2, \dots, v_m)$ and will be denoted by this m-tuple.

The *distance i* graph of an m-trail $(v_1, v_2, \dots, v_m)$, is the graph, denoted by $\Lambda_i(v_1, v_2, \dots, v_m)$, with vertices $v_1, v_2, \dots, v_m$ and having the edge $\{v_i, v_j\}$ if and only if v_i and v_j occur at distance i in $(v_1, v_2, \dots, v_m)$. Note that the distance i graph of a proper trail is not necessarily a simple graph.

An m-cycle (m-trail) system of order n is an ordered pair (V, S) where V is the vertex set of the complete graph K_n on n vertices and S is a set of edge disjoint cycles (trails) of length m whose edges partition the edge set of K_n. It is *i-perfect* if every pair of vertices occurs at distance i in a unique cycle (trail). A 2-perfect m-cycle (m-trail) system (V, S) is said to be *strongly 2-perfect* if $\Lambda_2(v_1, v_2, \dots, v_m) \in S$ for each $(v_1, v_2, \dots, v_m) \in S$. (Note that this implies that m is odd, and from now on we shall assume this to be so.)

We can define a groupoid (of order n and with underlying set V) from any m-trail system (V, S) of order n in the following manner. Let $v^2 = v$ for all $v \in V$ and let $u * v = w$ if and only if $(\dots, u, v, w, \dots)$ or $(\dots, w, v, u, \dots)$ is in S. From now on, we shall refer to this groupoid as the groupoid *arising* from the m-trail system.

2. The existence of strongly 2-perfect proper trail systems

The problem in constructing strongly 2-perfect proper trail systems lies in the fact that the repeated vertices may cause iterated cycles to have edges in common without being identical. Since the problem is obviously exacerbated by having many repeated vertices, from now on we shall assume that our trail systems have only two subloops, so that there is one of odd length and one of even length. Once we have a set of edge disjoint trails, their union forms a graph, and by Wilson's Theorem, there will be complete graphs decomposable into unions of such graphs.

Thus, to prove the existence of strongly 2-perfect proper trail systems, it is sufficient to prove that the iterated systems are edge disjoint

In [2] we made use of the following mapping:

Definition 2.1. Let m be odd and let $S = \{a \in \mathbb{N} \mid 0 < a \leq \frac{m-1}{2}, a \text{ odd }\}$. Define the map $\sigma_m : S \to S$ as follows. If $m - a = 2^k b$, where b is odd, then $a\sigma_m = b$.

We showed that σ_m is a permutation of S, and it played an important rôle in the proof of our main result.

In this paper we use a similar map:

Definition 2.2. Let m be odd and let $T = \{a \in \mathbb{N} \mid 0 < a \leq m, a \text{ odd }\}$. Define the map $\tau_m : T \to T$ as follows: $a\tau_m = b$ where b is the odd member of the pair $\{\frac{m-a}{2}, \frac{m+a}{2}\}$.

It is easy to see that the inverse of τ_m is given by:

$$a\tau_m^{-1} = \begin{cases} m - 2a & \text{if } a < \frac{m}{2} \\ 2a - m & \text{if } a > \frac{m}{2} \end{cases},$$

so τ_m is a permutation.

It is not difficult to see that each orbit of σ_m is a subset of an orbit of τ_m. The importance of τ_m is that if an m-trail has a subcycle of length a then the lengths of the odd subcycles in the 2-step iterations are precisely the elements of the orbit of τ_m containing a.

Theorem 2.3. *Let m and k be odd numbers. Then a 2-perfect trail system with a subcycle of size k is possible if and only if the orbit of τ_m containing k contains no number of the form $\pm(2^r \pm 1) \pmod m$.*

Example 2.4. The smallest prime for which the premises of the theorem hold is 127 (since, as was shown in [2] for all smaller primes, even the orbits of σ_m contain numbers of the above form). Here τ has an orbit $11, 69, 29, 49, 39, 83, 105$ and we have a trail

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 1, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 1

whose 2-step iterates

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119, 121, 123, 125, 127, 2, 4, 6, 8, 10, 1, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 1;

1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77, 81, 85, 89, 93, 97, 101, 105, 109, 113, 117, 121, 125, 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98, 102, 106, 110, 114, 118, 122, 126, 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71, 75, 79, 83, 87, 91, 95, 99, 103, 107, 111, 115, 119, 123, 127, 4, 8, 1, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100, 104, 108, 112, 116, 120, 124, 1;

1, 9, 17, 25, 33, 41, 49, 57, 65, 73, 81, 89, 97, 105, 113, 121, 2, 10, 18, 26, 34, 42, 50, 58, 66, 74, 82, 90, 98, 106, 114, 122, 3, 11, 19, 27, 35, 43, 51, 59, 67, 75, 83, 91, 99, 107, 115, 123, 4, 1, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100, 108, 116, 124, 5, 13, 21, 29, 37, 45, 53, 61, 69, 77, 85, 93, 101, 109, 117, 125, 6, 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, 110, 118, 126, 7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87, 95, 103, 111, 119, 127, 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, 112, 120, 1;

1, 17, 33, 49, 65, 81, 97, 113, 2, 18, 34, 50, 66, 82, 98, 114, 3, 19, 35, 51, 67, 83, 99, 115, 4, 20, 36, 52, 68, 84, 100, 116, 5, 21, 37, 53, 69, 85, 101, 117, 6, 22, 38, 54, 70, 86, 102, 118, 7, 23, 39, 55, 71, 87, 103, 119, 8, 24, 40, 56, 72, 88, 104, 120, 9, 25, 41, 57, 73, 89, 105, 121, 10, 26, 42, 58, 74, 90, 106, 122, 11, 27, 43, 59, 75, 91, 107, 123, 1, 28, 44, 60, 76, 92, 108, 124, 13, 29, 45, 61, 77, 93, 109, 125, 14, 30, 46, 62, 78, 94, 110, 126, 15, 31, 47, 63, 79, 95, 111, 127, 16, 32, 48, 64, 80, 96, 112, 1;

1, 33, 65, 97, 2, 34, 66, 98, 3, 35, 67, 99, 4, 36, 68, 100, 5, 37, 69, 101, 6, 38, 70, 102, 7, 39, 71, 103, 8, 40, 72, 104, 9, 41, 73, 105, 10, 42, 74, 106, 11, 43, 75, 107, 1, 44, 76, 108, 13, 45, 77, 109, 14, 46, 78, 110, 15, 47, 79, 111, 16, 48, 80, 112, 17, 49, 81, 113, 18, 50, 82, 114, 19, 51, 83, 115, 20, 52, 84, 116, 21, 53, 85, 117, 22, 54, 86, 118, 23, 55, 87, 119, 24, 56, 88, 120, 25, 57, 89, 121, 26, 58, 90, 122, 27, 59, 91, 123, 28, 60, 92, 124, 29, 61, 93, 125, 30, 62, 94, 126, 31, 63, 95, 127, 32, 64, 96, 1;

1, 65, 2, 66, 3, 67, 4, 68, 5, 69, 6, 70, 7, 71, 8, 72, 9, 73, 10, 74, 11, 75, 1, 76, 13, 77, 14, 78, 15, 79, 16, 80, 17, 81, 18, 82, 19, 83, 20, 84, 21, 85, 22, 86, 23, 87, 24, 88, 25, 89, 26, 90, 27, 91, 28, 92, 29, 93, 30, 94, 31, 95, 32, 96, 33, 97, 34, 98, 35, 99, 36, 100, 37, 101, 38, 102, 39, 103, 40, 104, 41, 105, 42, 106, 43, 107, 44, 108, 45, 109, 46, 110, 47, 111, 48, 112, 49, 113, 50, 114, 51, 115, 52, 116, 53, 117, 54, 118, 55, 119, 56, 120, 57, 121, 58, 122, 59, 123, 60, 124, 61, 125, 62, 126, 63, 127, 64, 1;

are edge-disjoint from it and from one another.

Proof of Theorem 2.3. The lengths of the odd subcycles in the iterated 2-step trails are precisely the elements of the orbit of τ_m containing k.

Now suppose this orbit contains a number ℓ of the form $2^r \pm 1 \pmod m$ Let us write the corresponding trail as $1, 2, 3, \ldots, \ell, 1, \ell+2, \ldots, m, 1$. After $r-1$ 2-step iterations we shall obtain the trail starting $1, 2^r+1, \ldots$. But 2^r+1 is either ℓ or

$\ell + 2$, so this trail has an edge in common with the original trail. (Note that this cannot be the original trail, which is reached after a number of steps equal to s where $2^s \equiv \pm 1 \pmod m$.) Thus we cannot get an edge-disjoint decomposition in this case.

Now suppose this orbit contains a number ℓ of the form $-(2^r \pm 1) \pmod m$. Let us write the corresponding trail as $1, 2, 3, \ldots, \ell, 1, \ell + 2, \ldots, m, 1$. After $r - 1$ 2-step iterations we shall obtain the trail ending $1 - 2^r, 1 \ldots$. But $1 - 2^r$ is either ℓ or $\ell + 2$, so this trail also has an edge in common with the original trail. Thus we cannot get an edge-disjoint decomposition in this case.

Conversely, suppose two of the 2-step iterates of a trail of length m with a cycle of length k have an edge in common. Let us number the vertices of the first trail as $1, 2, \ldots, n, 1, n+2, \ldots, m, 1$. Then the vertices of the second will be of the form $1, 2^r + 1, \ldots, n + 1 - 2^r, 1, n + 1 + 2^r, \ldots, m + 1 - 2^r$. The only possibilities for coincidences are edges involving 1. Note first that if $2^r \equiv \pm 1 \pmod m$ then the two trails coincide, so we can eliminate any coincidences that would imply this. We are left with $n + 1 - 2^r \equiv 2 \pmod m$, $n + 1 - 2^r \equiv m \pmod m$, $n + 1 + 2^r \equiv 2 \pmod m$ and $n + 1 + 2^r \equiv m \pmod m$. Thus n is of the form $\pm(2^r \pm 1) \pmod m$, as stated.

In fact, we have found no example where it is not sufficient to check the orbit of σ_m, rather than that of τ_m for this property. It would be nice to be able to replace τ by σ in the theorem. Certainly, the non-existence of a number of the form $2^r \pm 1$ in an orbit of σ is a necessary condition for the existence of a strong trail system. As the following example shows, for Mersenne and Fermat primes, the lengths of the orbits of σ containing numbers of this form are very short. Thus for reasonable size m of this form there will be orbits not containing numbers of this form.

Examples 2.5.

(1) Let $m = 2^r - 1$ and $k = 2^s - 1$, where $s < r$. Then the orbit of σ_m containing k is $(k, 2^{r-s} - 1)$.
(2) Let $m = 2^r - 1$ and $k = 2^s + 1$, where $s < r - 1$. Then the orbit of σ_m containing k is $(k, 2^{r-1} - 2^{s-1} - 1, 2^{r-s} + 1, 2^{r-1} - 2^{r-s-1} - 1)$.
(3) Let $m = 2^r + 1$ and $k = 2^s - 1$, where $s < r$. Then the orbit of σ_m containing k is $(k, 2^{r-1} - 2^{s-1} + 1, 2^{r-s} + 1)$.
(4) Let $m = 2^r + 1$ and $k = 2^s + 1$, where $s < r - 1$. Then the orbit of σ_m containing k is $(k, 2^{r-s} - 1, 2^{r-1} - 2^{r-s-1} + 1)$.

3. Strongly 2-perfect p-cycle systems which are not equationally defined

In this section we give some general conditions under which the quasigroups arising from strongly 2-perfect p-cycle systems do not form the finite members of a variety.

Definition 3.1. A graph G with vertex set V and edge set E is said to have a *derangement* if there exists a permutation Ω of V such that

(1) Ω has no fixed points,

(2) $(\Omega(a) = b) \to (\{a, b\} \in E)$.

Definition 3.2. A strongly 2-perfect p-trail (or cycle) (S, C) system is *t-derangeable* if the graph G formed by the edges in

$$\{\{a, b\} \mid (\forall (x_1, x_2, \ldots, x_p) \in C)\{a, b\} \notin \Lambda_t(x_1, x_2, \ldots, x_p)\}$$

has a derangement.

To see that it is reasonable to expect derangeable systems to exist, take a 127-trail with a repeat at distance 11 and then form a graph G with $e = 7 \times 127$ edges by including the distance $2, 4, 8, 16, 32, 64$ graphs of the 127-trail. (Example 2.4 shows that this is a simple graph). Now, for some x it is "possible" that there exists a cyclic decomposition of K_v, with $v = 2ex + 1$, into edge disjoint copies of G. Notice that the number of copies of G in a decomposition of K_v is $v(v-1)/(2e) = x(2ex + 1) = xv$. Hence we have x starters mod v. It seems plausible that for sufficently large v a cyclic decomposition of this type exists for any graph G.

Suppose such a cyclic decomposition does exist for some v. This decomposition gives us a strongly 2-perfect 127-cycle system of K_v and we claim that it is 11-derangeable.

There are $e = 7 \times 127$ pairs of vertices occurring at distance 1 in G and so there are xe differences covered in the starter trail(s) (this is all $(v-1)/2$ differences mod v). However, there are at most $x(e-1)$ pairs of vertices at distance 11 in the starter cycles because there is a repeated pair at distance 11 in each starter. Hence there are at most $x(e-1)$ differences covered at distance 11 and so there are differences which do not occur at distance 11 in the starter cycles. Let d be a difference which is not covered at distance 11 in the starter cycle(s). Then the edges $\{0, d\}, \{1, 1 + d\}, \{2, 2 + d\}, \ldots, \{v - 1, v - 1 + d\}$ do not appear at distance 11 in any trail of our strongly 2-perfect 127-trail system. Clearly, these edges form a graph which has a derangement—the permutation is $(0, d, 2d, \ldots, -d)$.

The above argument can be applied in the case of any m-trail T for which $r > 1$ is the smallest integer such that $\Lambda_r(T) = T$ and the union of the distance $1, 2, 4, \ldots, 2^{r-1}$ graphs of T form a simple graph.

Theorem 3.3. *If p is a prime for which a strongly 2-perfect derangeable p-trail system exists, then the quasigroups arising from strongly 2-perfect p-cycle systems do not form the finite members of a variety.*

Theorem 3.3 is an immediate consequence of Theorem 3.4 and Lemma 3.5.

Theorem 3.4. *Suppose there exists an $n^2 \times p$ array, based on the set $\{1, 2, \ldots, n\}$, with the following properties:*

1. *columns i and $i + 1$ (including p and 1) are orthogonal. (That is, they include all n^2 ordered pairs in corresponding rows.)*
2. *if $x_1, x_2, \ldots, x_p$ is a row then so also is $\Lambda_2(x_1, x_2, \ldots, x_p)$.*

(3) *Let $s \in \{1, 2, \ldots, p\}$ be such that its orbit O under τ_p contains no element of the form $\pm 2^r \pm 1 \pmod p$. Then for $i = 1, 2, \ldots, n^2$ and for each $t \in O$, the entries in row i of column 1 and column $t+1$ are distinct.*

(4) *There exists a strongly 2-perfect p-trail system (V, T) of order v in which there exists a p-trail of the form*

$$(\underbrace{x_1, x_2, \ldots, x_t, x_1}_{\textit{distance } t}, x_{t+2}, \ldots, x_p)$$

where $x_1, x_2, \ldots, x_t, x_{t+2}, \ldots, x_p$ are distinct, $t \in O$ (as in (3)) and every proper trail is of this form (that is, there exists a repeat at distance t, and any repeat is at distance t).

(5) *There exists a strongly 2-perfect p-trail system (S, C) of order n.*

Then there exists a strongly 2-perfect p-cycle system of order nv such that the quasigroup corresponding to this cycle system has a homomorphism onto a quasigroup which does not correspond to a strongly 2-perfect p-cycle system.

Proof. For each trail $(x_1, x_2, \ldots, x_p)$, where x_i is the repeated vertex, and for each row $y_1, y_2, \ldots, y_p$ of the array, let

$$((x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)) \in R.$$

Also, for each $x \in V$ and each p-cycle $(z_1, z_2, \ldots, z_p) \in C$, let

$$((x, z_1), (x, z_2), \ldots, (x, z_p)) \in R.$$

Then $(V \times S, R)$ is the required strongly 2-perfect p-cycle system. Its quasigroup has a homomorphism, ϕ, onto the quasigroup of (V, T), namely $\phi(x, y) = x$ for all $(x, y) \in V \times S$.

Lemma 3.5. *If there exists a strongly 2-perfect p-trail system of order n which is t-derangeable, where $t \in \text{orbit}(s)$ and orbit(s) contains no $\pm(2^r \pm 1) \pmod p$, then there exists an array satisfying the conditions of Theorem 5.1.*

Proof. First we note that

$$\begin{aligned} &\{\{a, b\} \mid (\forall (x_1, x_2, \ldots, x_p) \in C)\{a, b\} \notin \Lambda_{t_i}(x_1, x_2, \ldots, x_p)\} \\ &= \{\{a, b\} \mid (\forall (x_1, x_2, \ldots, x_p) \in C)\{a, b\} \notin \Lambda_{t_j}(x_1, x_2, \ldots, x_p)\} \end{aligned}$$

for any $t_j \in \text{orbit}(t_i)$.

Form an $n^2 \times p$ array, A^*, as follows:

For each $(x_1, x_2, \ldots, x_p) \in C$, let the $2p$ rows

$$\begin{array}{l} x_1, x_2, \ldots, x_p \\ x_2, x_3, \ldots, x_1 \\ \vdots \\ x_p, x_1, x_2, \ldots, x_{p-1} \\ x_1, x_p, x_{p-1} \cdot \ldots, x_2 \\ x_2, x_1, x_p, x_{p-1}, \ldots, x_2 \\ \vdots \\ x_p, x_{p-1}, \ldots, x_1 \end{array}$$

be in the array A^*. Also let $(x, x, \ldots, x)$be a row for each $x \in S$. Now apply the derangement Ω to the first column of A^* to obtain the required array A.

We now check in turn the five conditions of the Theorem.

(1) Since each unordered pair $\{a, b\}$ with $a \neq b$ is adjacent in a unique trail $(x_1, x_2, \ldots,\ x_p) \in C$, it is clear that for every ordered pair (a, b) and every $i \in 1, 2, \ldots, p$, there is a unique row of A^* with a in column i and b in column $i+1$. Since Ω is a permutation acting on a column, A also has this property.

(2) Suppose $x_1, x_2, \ldots, x_p$ is a row of A^*. Then $(x_1, x_2, \ldots, x_p) \in C$ or $x_1 = x_2 = \cdots = x_p$. If $x_1 = x_2 = \cdots = x_p$, then $\Lambda_2(x_1, x_2, \ldots, x_p) = (x_1, x_2, \ldots, x_p)$ is also a row. If $(x_1, x_2, \ldots, x_p) \in C$, then, since$(S, C)$ is strongly 2-perfect, $\Lambda_2(x_1, x_2, \ldots, x_p) \in C$ and so $\Lambda_2(x_1, x_2, \ldots, x_p)$ is also a row of A^*. Hence A^* has property (2). Now, if $(x_1, x_2, \ldots, x_p)$ is a row of A, then $(\Omega^{-1}(x_1), x_2, \ldots, x_p)$ is a row of A^*, so $\Lambda_2(\Omega^{-1}(x_1), x_2, \ldots, x_p)$ is a row of A^*. It follows that $\Lambda_2(x_1, x_2, \ldots, x_p)$ is a row of A, and so A has property (2).

(3) Suppose that in row i of A, the entry in column 1 is identical to the entry in column $t+1$ (say it is x). Then this row cannot correspond to a row in A^* of the form $(x, x, \ldots, x)$, since Ω has no fixed points. Thus $\Omega^{-1}(x)$ and x occur at distance t in a trail of C. That is, $\{\Omega^{-1}(x), x\} \in \Lambda_t(x_1, x_2, \ldots, x_p)$ for some $(x_1, x_2, \ldots, x_p) \in C$. Hence $\{\Omega^{-1}(x), x\} \notin E(G)$ (where G is defined as in Definition 2.6). But $\Omega(\Omega^{-1}(x)) = x$ implies that $\{\Omega^{-1}(x), x\} \in E(G)$, a contradiction.

(4) Since $t \in$orbit(s), and orbit(s) contains no $\pm(2^r \pm 1) \pmod p$, we have (4) for sufficiently large v.

(5) This follows from the conditions of the lemma.

4. Strongly 2-perfect 9-cycle systems

In this section we prove the following theorem:

Theorem 4.1. *The quasigroups arising from finite strongly 2-perfect 9-cycle systems do not form the finite members of a variety.*

Proof. First note that if G is a 9-cycle, then $G \cup \Lambda_2(G) \cup \Lambda_4(G) \cong K_{3,3,3}$ (the complete tripartite graph with three vertices in each part) and $\Lambda_8(G) = G$. Now consider the infinite graph $K_{\infty,\infty,\infty}$ with vertex set

$$\{u_0, u_1, u_2, \dots\} \cup \{v_0, v_1, v_2, \dots\} \cup \{w_0, w_1, w_2, \dots\}$$

and edge set

$$\{\{u_i, v_j\} \mid i = 0, 1, 2 \dots, j = 0, 1, 2, \dots\} \cup \{\{u_i, w_j\} \mid i = 0, 1, 2 \dots, j = 0, 1, 2, \dots\} \\ \cup \{\{v_i, w_j\} \mid i = 0, 1, 2 \dots, j = 0, 1, 2, \dots\}.$$

For all $x \equiv 0 \pmod 3$ and all $y \equiv 0 \pmod 3$, let $G_{x,y} \cong K_{3,3,3}$ be the complete tripartite graph with vertex set

$$\{u_x, u_{x+1}, u_{x+2}\} \cup \{v_y, v_{y+1}, v_{y+2}\} \cup \{w_{x*y}, w_{(x*y)+1}, w_{(x*y)+2}\},$$

where $(\{0, 3, 6, \dots\}, *)$ is any infinite quasigroup, and the obvious edge set. Then the set $\{G_{x,y} \mid x, y \in \{0, 3, 6, \dots\}\}$ forms a decomposition of $K_{\infty,\infty,\infty}$ in edge disjoint copies of $K_{3,3,3}$ and hence yields a strongly 2-perfect 9-cycle system of $K_{\infty,\infty,\infty}$. The union of the 9-cycles in this strongly 2-perfect 9-cycle system with those in the three infinite strongly 2-perfect 9-cycle systems having underlying sets $\{u_0, u_1, u_2, \dots,\}$, $\{v_0, v_1, v_2, \dots\}$ and $\{w_0, w_1, w_2, \dots\}$ form an infinite strongly 2-perfect 9-cycle system with underlying set $\{u_0, u_1, u_2, \dots,\} \cup \{v_0, v_1, v_2, \dots\} \cup \{w_0, w_1, w_2, \dots\}$. Furthermore, the mapping $u_i \mapsto u$, $v_i \mapsto v$, $w_i \mapsto w$ for $i = 0, 1, 2, \dots$ is a homomorphism from the quasigroup Q corresponding to this system onto a Steiner quasigroup of order 3 (with underlying set $\{u, v, w\}$).

It remains to show that Q is in the variety V generated by the class of quasigroups corresponding to finite strongly 2-perfect 9-cycle systems. If $Q \notin V$, then there exists an identity I which holds in V but not in Q. Since I does not hold in Q, there is a finite collection of 9-cycles, in the 9-cycle system corresponding to Q, which define a finite partial quasigroup in which I fails. This collection of 9-cycles can be embedded in a finite strongly 2-perfect 9-cycle system. To see this, note that the collection of 9-cycles is contained in a collection of copies of $K_{3,3,3}$ and that by Wilson's theorem, for sufficiently large n, there is a decomposition of K_n into copies of the graph formed by the union of these copies of $K_{3,3,3}$. Hence, I fails in the finite quasigroup corresponding to the constructed strongly 2-perfect 9-cycle system. This is a contradiction and so $Q \in V$.

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