The diameter of lifted digraphs

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Abstract

The theory of voltage assignments enables one to construct large graphs (directed as well as undirected) as covering spaces of smaller base graphs. All properties of the large graph, called the lift, are determined by the structure of the base graph and by an assignment of voltages (elements of some group) to its arcs. In this paper we prove several upper bounds on the diameter of the lift in terms of some properties of the base graph and the voltage group.

1. Introduction

When designing a large communication network one usually has to consider several constraints. Among the most frequently appearing restrictions the following two seem to play an important role. The number of connections attached to any node as well as the communication distance between any two nodes should be relatively small. Since a network is often modelled by a directed graph (*digraph*), this draws our attention to the well-known

Degree/Diameter Problem. Construct digraphs with the largest possible number of vertices $n(d,k)$ for given maximum (in- and out-) degree d and diameter k.

A natural upper bound for $n(d,k)$ is the *Moore bound*

$$M_{d,k} = 1 + d + d^2 + \cdots + d^k.$$

The equality $n(d,k) = M_{d,k}$ can be attained only for $d = 1$ or $k = 1$ [5,9]. In the case $d \geq 2$ and $k = 2$ we have $n(d,k) = M_{d,k} - 1$ [2,3].

In [8] it was shown that if $d = 2$ and $k \geq 3$ then $n(d,k) \leq M_{d,k} - 3$. According to [4] for $d = 3$ and $k \geq 3$ the inequality $n(d,k) \leq M_{d,k} - 2$ holds.

However, it is not known whether the value $M_{d,k} - 1$ can be attained for $d \geq 4$, $k \geq 3$. It is quite remarkable that all these upper bounds for $n(d,k)$ differ from $M_{d,k}$ just by a small constant.

As for lower bounds for $n(d,k)$, the Kautz digraphs $K(d,k)$ [7] imply that $n(d,k) \geq d^k + d^{k-1}$. A mild improvement on this bound can be obtained for $d = 2$.

Since $n(2,4) = 25$ (the corresponding digraph was found by Alegre; its construction using voltage assignments is in [1]), by repeating the line-digraph-construction we get $n(2,j) \geq 25.2^{j-4} = 2^j + 2^{j-1} + 2^{j-4}$ for $j \geq 4$.

The authors of [1] approached the problem by means of voltage digraphs. This theory enables one to construct "large" digraphs – the *lifts* – from a given digraph G with help of a mapping (called *voltage assignment*) from its arc set $D(G)$ to some group Γ. The structure of the lift is completely determined by the structure of the base digraph G and the voltage assignment. In what follows it is sufficient to examine the "small" digraph G in order to find the properties (e.g. the diameter) of the lift. Lifts appear to be an appropriate tool for attacking the Degree/Diameter Problem because (as shown in [1]) many of the currently largest known digraphs of given degree and diameter are lifts.

The paper consists of five sections. In Section 2 we introduce basic concepts and notation. The next two Sections present a number of upper bounds on the diameter of the lift (assuming certain conditions on the structure of the base digraph) when Γ is an arbitrary group (Section 3) and when Γ is an abelian group (Section 4). Finally, Section 5 concludes the paper with an outline of a recursive construction which could yield digraphs with numbers of vertices asymptotically close to the Moore bound.

2. Voltage assignments and Cayley digraphs

Let G be a digraph and $D(G)$ the set of its *arcs* (i.e. directed edges) – we allow loops as well as multiple arcs. A $u \to v$ *walk* of length k in G is a sequence $P = e_1 e_2 \dots e_k$ of arcs of G such that u is the initial vertex of e_1, for $2 \leq i \leq k$ the terminal vertex of e_{i-1} coincides with the initial vertex of e_i and v is the terminal vertex of e_k. Let $d_G(u,v)$ denote the distance from a vertex u to a vertex v in G, i.e. the length of the shortest $u \to v$ walk in G. As usual, the *indegree (outdegree)* of a vertex u is the number of arcs terminating at (emanating from) u.

Let Γ be an arbitrary group and let $\alpha : D(G) \to \Gamma$ be a mapping. Then α is called *voltage assignment* and G (with α) a *voltage digraph.* The *lift* G^α is the digraph defined as follows. The vertex and arc sets of G^α are $V(G^\alpha) = V(G) \times \Gamma$, $D(G^\alpha) = D(G) \times \Gamma$, and there is an arc (x,f) emanating from (u,g) and terminating at (v,h) if and only if $f = g$, x is an arc from u to v (in G) and $h = g\alpha(x)$. Since we deal with finite digraphs only, Γ will always be a finite group.

Voltage assignments on undirected graphs were for the first time closely examined in [6]. However, the theory can be extended to directed graphs easily. Let us now mention some basic facts.

Obviously the indegree (outdegree) of a vertex (u,g) is the same as the one of the vertex u in G. In particular, the maximum (in- and out-) degree of the digraph G^α is equal to the maximum (in- and out-) degree of the starting digraph G. This is extremely important, since, for our purposes, we need to keep the degrees of vertices of the lift G^α small. If $P = e_1 e_2 \dots e_k$ is a walk in G then let $\alpha(P) = \alpha(e_1)\alpha(e_2)\dots\alpha(e_k)$ denote its *net voltage* (or simply *voltage*). The *trivial* walk of length 0 has voltage 1, the neutral element of Γ. The distance of the vertices (u,g) and (v,h) in G^α is then equal to the length of the shortest $u \to v$

walk with voltage $g^{-1}h$ (in G). Therefore $diam(G^\alpha) \le k$ holds if and only if for any ordered pair of vertices $u, v \in V(G)$ (including the case $u = v$) and for any $g \in \Gamma$ there exists (in G) a $u \to v$ walk with voltage g and length at most k. We will use this fact massively throughout the paper.

Let Γ be an arbitrary group and let X be its subset. The *Cayley digraph* $C(\Gamma, X)$ has Γ as its vertex set and it contains an arc $\overrightarrow{gh}$ (for $g, h \in \Gamma$) if and only if there is some $x \in X$ such that $gx = h$.

The digraph $C(\Gamma, X)$ is vertex-transitive of (in- and out-) degree $|X|$. In general, $C(\Gamma, X)$ is (weakly) connected if and only if X is a generating set for Γ. However, the group Γ is finite and therefore $C(\Gamma, X)$ is strongly connected if and only if it is weakly connected. In what follows we briefly refer to *connected* Cayley digraphs.

If $1 \in \Gamma$ is an element of X then $C(\Gamma, X)$ differs from $C(\Gamma, X \setminus \{1\})$ only by having a loop at every vertex. Since we are only interested in the diameter of Cayley digraphs, let us (for the sake of simplicity) exclude the case $1 \in X$.

Let G be a digraph and let $w \in V(G)$. Define r_w^+ to be the largest distance from w to a vertex in G and let r_w^- be the largest distance from a vertex in G to w. Let $r(G) = \min_{w \in V(G)} \{r_w^+ + r_w^-\}$; a vertex w for which $r_w^+ + r_w^- = r(G)$ will be called *central*. Let $\delta(G)$ be the minimum of outdegrees of the vertices of G.

The following result was proved in [1].

Theorem 0. *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph and let G be a strongly connected digraph such that $\delta(G) \ge |X| + 1$. Then there exists a voltage assignment $\alpha : D(G) \to \Gamma$ such that*

$$diam(G^\alpha) \le r(G) + diam(H).$$

The assumptions of *Theorem 0* are rather general, which has some drawbacks as well. One of them is the impossibility to use it in the case $\delta(G) = 2$ (which is quite interesting) because the diameter of a Cayley digraph with a generating set X having only one element is too large ($|\Gamma| - 1$). Therefore it is reasonable to try to weaken the condition $\delta(G) \ge |X| + 1$ even if this requires adding some extra assumptions.

3. Voltage assignments in general groups

In our considerations a special type of spanning trees will play an important role. A spanning tree of the digraph G will be called *(inward) radial* if there exists a central vertex w of G such that for every $u \in V(G)$ we have $d_T(u, w) \le r_w^-$. Such a spanning tree has its "root" w and is directed inward to w. Therefore in T there is no arc emanating from w and exactly one arc emanating from any vertex $u \ne w$. Let us start with a simple observation.

Lemma 1. *Every strongly connected digraph has a radial spannning tree.*

Proof. A radial spanning tree T can be obtained as follows. Let w be any central vertex of G. For each vertex $u \ne w$ let $D(T)$ contain exactly one of the arcs

$\overrightarrow{uv} \in D(G)$ for which $d_G(v,w) = d_G(u,w) - 1$. The resulting spanning tree is clearly radial. $\square$

As mentioned before, the diameter of a Cayley digraph $C(\Gamma, X)$ with $|X| \le \delta(G) - 1$ might be too large if $\delta(G)$ was small (e.g. $\delta(G) = 2$). Thus, it would be of advantage to extend the set X and obtain a Cayley digraph with much smaller diameter.

The next proposition is a generalisation of *Theorem 0*, which gives a better upper bound on the diameter of the lift G^α in such cases.

Theorem 2. *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph and let G be a strongly connected digraph such that $\delta(G) \ge \left\lceil \frac{|X|}{m} \right\rceil + 1$. Let T be a radial spanning tree of G with root w and let w_1 be a vertex of G such that $\overrightarrow{ww_1} \in D(G)$ and $d_T(w_1, w) \equiv m-1 \pmod m$. Then there exists a voltage assignment $\alpha : D(G) \to \Gamma$ such that*

$$diam(G^\alpha) \le r(G) + m\ diam(H).$$

Remark. A special radial spanning tree T as in *Theorem 2* exists for example if there is an arc $\overrightarrow{ww_1} \in D(G)$ such that w is central and $d_G(w_1, w) \equiv m - 1 \pmod m$. In this case we can take for T the spanning tree constructed in the proof of *Lemma 1*, for it is obvious that $d_T(w_1, w) = d_G(w_1, w)$.

This condition is trivially satisfied if $m = 1$ and then the statement is equivalent to *Theorem 0.*

Proof. Let T' be a digraph obtained from T by adding the arc $\overrightarrow{ww_1}$. From every vertex of T' there emanates exactly one arc and it is easy to verify that if $\overrightarrow{u_1u_2} \in D(T')$ then $d_T(u_1, w) - d_T(u_2, w) \equiv 1 \pmod m$. Therefore if W is a $u_1 \to u_2$ walk in T' of length s then $d_T(u_1, w) - d_T(u_2, w) \equiv s \pmod m$.

Let us now define the voltage assignment α. Set $\alpha(e) = 1 \in \Gamma$ for $e \in D(T')$. Let $\{X_0, \dots, X_{m-1}\}$ be a decomposition of X, where $|X_i| \le \left\lceil \frac{|X|}{m} \right\rceil$, $\quad i = 0, \dots, m-1$. From any vertex $v \in V(G)$ there emanate at least $\left\lceil \frac{|X|}{m} \right\rceil$ arcs not in T'. Assign them the voltages from X_k, where $k \equiv d_T(v, w) \pmod m$, so that every element of X_k is assigned to at least one of them. Do this for every $v \in V(G)$.

Let us, for arbitrary $v_1, v_2 \in V(G)$ and $g \in \Gamma$, find a $v_1 \to v_2$ walk with voltage g. Obviously there exists a $w \to v_2$ walk P of length at most r_w^+. Let its voltage be $\alpha(P) = h$ and let $gh^{-1} = x_1x_2 \dots x_k$, where $x_i \in X$ and $k \le diam(H)$. Let $x_1 \in X_j$ and $d_T(v_1, w) - j \equiv s \pmod m$ (where $s < m$). Take the unique walk Q_1 in T' emanating from v_1 of length s (denote by u its terminal vertex). Clearly $\alpha(Q_1) = 1$ and $d_T(u, w) \equiv j \pmod m$. Thus from u there emanates an arc $e_1 = \overrightarrow{uu_1}$ with voltage x_1. The length of the walk $P_1 = Q_1e_1$ is at most m and its voltage is x_1.

Using the same algorithm we can find a $u_1 \to u_2$ walk P_2 with length at most m and voltage x_2, etc. Finally denote by P_{k+1} the $u_k \to w$ walk in T of length at most r_w^- (T is radial) and voltage 1. Now, joining the walks $P_1, P_2, \dots, P_k, P_{k+1}, P$ we obtain a $v_1 \to v_2$ walk of length at most $km + r_w^- + r_w^+ \le r(G) + m\ diam(H)$ and with voltage $x_1x_2 \dots x_k 1 h = gh^{-1}h = g$. $\square$

In the case $m = 2$ the additional condition can be ommited, however, this increases the upper bound by 1. Let us call *simple* an arc $e \in D(G)$ with initial and terminal vertices u and v, respectively, if $u \neq v$ and there is no other arc $\overrightarrow{uv} \in D(G)$ (i.e. e is neither a loop nor a multiple arc).

Theorem 3. *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph and let G be a strongly connected digraph such that $\delta(G) \geq \left\lceil \frac{|X|}{2} \right\rceil + 1$ and all arcs terminating at some central vertex w are simple. Then there exists a voltage assignment $\alpha : D(G) \to \Gamma$ such that*

$$diam(G^\alpha) \leq r(G) + 2\,diam(H) + 1.$$

Proof. Take the central vertex w and construct a radial spanning tree T in the same way as in the proof of *Lemma 1*. The special way of construction and the fact that all arcs terminating at w are simple implies that T contains all arcs terminating at w. Construct T' by adding an arbitrary arc $\overrightarrow{ww_1}$ ($w_1 \neq w$) into T. If $d_T(w_1, w)$ is odd then T, w, w_1 satisfy the conditions of the previous *Theorem* and the result follows.

If $d_T(w_1, w)$ is even then let $\{X_0, X_1\}$ be a decomposition of X such that $|X_i| \leq \left\lceil \frac{|X|}{2} \right\rceil$, $i = 0, 1$. Define α in the same way as in the proof of the previous *Theorem* and use the same algorithm when looking for a $v_1 \to v_2$ walk with voltage g. Problems can occur when u_i – one of the vertices $u_0\,(= v_1), u_1, \ldots, u_{k-1}$ – is w and x_{i+1} is an element of X_1. In this case there is no arc of voltage x_{i+1} emanting from u_i. But such an arc does not emanate from w_1 either; it emanates from the vertex w_2, where $\overrightarrow{w_1w_2} \in D(T')$. Thus the walk P_{i+1} would have to have length 3 instead of 2 $(= m)$. However, this case can occur only once – when $w = u_0\,(= v_1)$ – because the last arcs in the walks $P_1, \ldots, P_k$ are not from T' (those arcs have voltages from X) and therefore can not terminate at w. Thus we have $u_i \neq w$ $(i = 1, \ldots, k)$ and the inequality holds. □

Remark. It seems paradoxical that one has to increase the bound from the previous *Theorem* just because of the case $v_1 = w$, where w is a central vertex of G.

4. Voltage assignments in abelian groups

In this section we consider Γ to be an abelian group. However, we will keep the multiplicative notation with 1 to be the neutral element of Γ. We focus on the case $\delta(G) \geq |X|$. This bound differs from the assumption in *Theorem 0* only by one, but this fact may be important for some recursive constructions which we will describe at the end of this paper.

Let us define $min_2(X)$ as the second smallest of the orders (in the group Γ) of elements of X.

Theorem 4. *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph, where Γ is an abelian group and $|X| \geq 2$, and let G be a strongly connected digraph such that $\delta(G) \geq |X|$. Then there exists a voltage assignment $\alpha : D(G) \to \Gamma$ such that*

$$diam(G^\alpha) \leq r(G) + diam(H) + min_2(X) - 1.$$

Proof. Let x and y have smallest orders of all elements of X. Take any radial spanning tree with the root w and set $\alpha(e) = 1$ for $e \in D(T)$. There are at least $|X|$ arcs emanating from w and not belonging to T. Assign them voltages from X so that every element of X is assigned to at least one arc. From any $u \neq w$ there emanate at least $|X| - 1$ arcs not belonging to T. If $d_T(u, w)$ is even (odd), assign them voltages from $X \setminus \{x\}$ ($X \setminus \{y\}$) so that every element of $X \setminus \{x\}$ ($X \setminus \{y\}$) is assigned to at least one arc emanating from u.

It is obvious that if there is no arc with voltage x (y) emanating from some vertex u then $u \neq w$ and an arc with voltage x (y) emanates from the one vertex v for which $\overrightarrow{uv} \in D(T)$.

Let us now take arbitrary $v_1, v_2 \in V(G)$ and $g \in \Gamma$. Let P be any $w \to v_2$ walk of length at most r_w^+ and with voltage h. Let $gh^{-1} = z_1 z_2 \ldots z_k x^m y^n$, where $z_i \in X \setminus \{x, y\}$, $k + m + n \leq diam(H)$ and $0 \leq m <$ order of x, $0 \leq n <$ order of y (such $z_1, \ldots, z_k, m, n$ do exist thanks to the abelianity of Γ). Obviously $\max(m, n) \leq min_2(X) - 1$.

Clearly there is an arc $\overrightarrow{v_1 u_1}$ with voltage z_1 (for some u_1), then there is an arc $\overrightarrow{u_1 u_2}$ with voltage z_2 (for some u_2), etc. This way we can construct some $v_1 \to u_k$ walk of length k and with voltage $z_1 \ldots z_k x^0 y^0$.

From u_k there emanates an arc $\overrightarrow{u_k u_{k+1}}$ with voltage either x or y, thus we can extend our walk to u_{k+1} – it will have voltage either $z_1 \ldots z_k x^1 y^0$ or $z_1 \ldots z_k x^0 y^1$. From u_{k+1} emanates an arc with voltage either x or y as well, so we can extend the walk the same way again. We may continue till we get a $v_1 \to u_l$ walk Q of length l and with voltage either $z_1 \ldots z_k x^m y^{n_1}$ (where $0 \leq n_1 \leq n$ and $l = k + m + n_1$) or $z_1 \ldots z_k x^{m_1} y^n$ (where $0 \leq m_1 \leq m$ and $l = k + m_1 + n$). This is possible because Γ is abelian. Without loss of generality, let us consider the case $\alpha(Q) = z_1 \ldots z_k x^m y^{n_1}$.

If there is an arc with voltage y emanating from u_l then we can extend Q by adding this arc. If not, then such an arc emanates from the unique vertex u_{l+1} for which $\overrightarrow{u_l u_{l+1}} \in D(T)$ and we can extend Q by adding these two arcs. In both cases the length of our walk increases by 2 at most and its voltage will be $z_1 \ldots z_k x^m y^{n_1+1}$. We may continue this way until we get a $v_1 \to u_s$ walk P_1 of length $s \leq k + m + n_1 + 2(n - n_1) \leq diam(H) + n - n_1 \leq diam(H) + min_2(X) - 1$ and with voltage $z_1 \ldots z_k x^m y^n = gh^{-1}$.

Obviously, there is a $u_s \to w$ walk P_2 of length at most r_w^- and with voltage 1 in T. The union of P_1, P_2 and P is then a $v_1 \to v_2$ walk of length at most $r_w^- + r_w^+ + diam(H) + min_2(X) - 1 = r(G) + diam(H) + min_2(X) - 1$ and with voltage $gh^{-1}1h = g$. □

If certain conditions are fulfilled, it is possible to improve the last result. First, we need a voltage assignment with special properties.

Lemma 5. *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph, where Γ is an abelian group. Let G be a strongly connected digraph and T its radial spanning tree with the root w. Let the voltage assignment $\alpha : D(G) \to \Gamma$ have the following properties:*

(1) $\alpha(e) = 1$ *for* $e \in D(T)$,
(2) *for any* $x \in X$ *there is an arc with voltage* x *emanating from* w,

(3) *if $u \neq w$ then there is at most one $x \in X$ such that there is no arc with voltage x emanating from u and in this case an arc with voltage x emanates from the unique vertex v such that $\overrightarrow{uv} \in D(T)$,*

(4) *if from a vertex u there emanates at least one arc with voltage $x \in X$ then there is an arc $\overrightarrow{uv} \in D(G)$ with voltage x such that from the vertex v also emanates an arc with voltage x.*

Then

$$diam(G^{\alpha}) \leq r(G) + diam(H) + 1.$$

Proof. Let us again find a $v_1 \to v_2$ walk with voltage g. Let P be a $w \to v_2$ walk of length at most r_w^+ and with voltage h and let $gh^{-1} = x_1^{m_1} \dots x_k^{m_k}$, where $X = \{x_1, \dots, x_k\}$, $m_i \geq 0$ $(i = 1, \dots, k)$ and $m_1 + \dots + m_k \leq diam(H)$.

If at least two m_i are positive then there is an arc $\overrightarrow{v_1u_1}$ (for some u_1) with voltage x_p such that $m_p > 0$. Again, if at least two of the numbers $m_1, \dots, m_{p-1}, m_p - 1, m_{p+1}, \dots, m_k$ are positive then there is an arc $\overrightarrow{u_1u_2}$ (for some u_2) with voltage x_q such that the created $v_1 \to u_2$ walk of length 2 will have voltage $x_px_q = x_1^{n_1} \dots x_k^{n_k}$, where $n_i \leq m_i$ $(i = 1, \dots, k)$. We may continue this way until we get a $v_1 \to u_l$ walk Q of length $l = n_1 + \dots + n_k$ and with voltage $x_1^{n_1} \dots x_k^{n_k}$ such that $n_{i_0} \leq m_{i_0}$ for some i_0 and $n_i = m_i$ for all $i \neq i_0$. Without loss of generality let us suppose $i_0 = 1$.

If from u_l there emanates an arc with voltage x_1 then there is an arc $\overrightarrow{u_lu_{l+1}}$ with voltage x_1 such that from u_{l+1} also emanates an arc with voltage x_1 (because of (4)). Then there is an arc $\overrightarrow{u_{l+1}u_{l+2}}$ with voltage x_1 such that from u_{l+2} also emanates an arc with voltage x_1, etc. This way we can construct a $u_l \to u_s$ walk (for some u_s) of length $s - l = m_1 - n_1$ and with voltage $x_1^{m_1-n_1}$. After joining with Q we get a $v_1 \to u_s$ walk P_1 of length $l + m_1 - n_1 = m_1 + \dots + m_k \leq diam(H)$ and with voltage $x_1^{m_1} \dots x_k^{m_k} = gh^{-1}$.

If there is no arc with voltage x_1 emanating from u_l then $u_l \neq w$ and such an arc emanates from the unique vertex v such that $\overrightarrow{u_lv} \in D(T)$ (and therefore $\alpha(\overrightarrow{u_lv}) = 1$). Using the same algorithm as before, we construct a $v_1 \to u_s$ walk P_1 of length $l + 1 + m_1 - n_1 = m_1 + \dots + m_k + 1 \leq diam(H) + 1$ and with voltage gh^{-1}.

Obviously there is a $u_s \to w$ walk P_2 in T of length at most r_w^- and with voltage 1. The union of P_1, P_2 and P is then a $v_1 \to v_2$ walk of length at most $r(G) + diam(H) + 1$ and with voltage g. □

The next proposition gives us a sufficient condition for the existence of a voltage assignment described in *Lemma 5*. Recall that $1 \notin X$.

Lemma 6. *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph, where Γ is an abelian group. Let G be a strongly connected digraph such that $\delta(G) \geq |X| \geq 2$ and let T be its radial spanning tree with the root w. Let $f : V(G) \to X \cup \{1\}$ be a function with these properties:*

(1) *$f(u) = 1$ if and only if $u = w$,*

(2) *if $\overrightarrow{uv} \in D(T)$ then $f(u) \neq f(v)$,*

(3) *if* $\overrightarrow{uv_1}, \dots, \overrightarrow{uv_k}$ *are all arcs emanating from a vertex* u *and not belonging to* T *then either the values* $f(v_1), \dots, f(v_k)$ *are not all equal to each other or* $f(v_1) = \dots = f(v_k) \in \{f(u), 1\}$.

Then there exists a voltage assignment $\alpha : D(G) \to \Gamma$ *from* Lemma 5.

Proof. Let us define voltage assignment β as follows. If $e \in D(T)$ then set $\beta(e) = 1$. If $u \in V(G)$ and $u \neq w$ ($u = w$) then there are at least $|X|-1$ ($|X|$) arcs emanating from u and not belonging to T. Assign them voltages from $X \setminus \{f(u)\}$ so that every element of this set is assigned to at least one of them. Note that $X \setminus \{f(u)\}$ does contain at least one element because $|X| \geq 2$. Thus for every $u \in V(G)$ and $x \in X$, $x \neq f(u)$ ($x = f(u)$) there is an arc (no arc) with voltage x emanating from u.

It is easy to verify that β has properties (1),(2) and (3) from *Lemma 5*. Let us define $N(\beta) = \{(u,x) : u \in V(G),\ x \in X$ and property (4) from *Lemma 5* does not hold for the couple $(u,x)\}$. If $|N(\beta)| = 0$ then β is the voltage assignment we look for.

Let $|N(\beta)| > 0$ and $(u,x) \in N(\beta)$. Therefore if $\overrightarrow{uv_1}, \dots, \overrightarrow{uv_l}$ are all arcs emanating from u and having voltage x (obviously $\overrightarrow{uv_i} \notin D(T)$ because $1 \notin X$) then $l \geq 1$ and from the vertices $v_1, \dots, v_l$ do not emanate arcs with voltages x, i.e. $f(v_1) = \dots = f(v_l) = x$. However, $f(u) \neq x$ (because the arc $\overrightarrow{uv_1}$ has voltage x), in what follows (by (3)) there exists an arc $\overrightarrow{uv} \in D(G) \setminus D(T)$ such that $f(v) \neq x$. Let its voltage be $y \in X$, $y \neq x$.

Let us create from β a new voltage assignment γ by exchanging the voltages of arcs $\overrightarrow{uv_1}$ and $\overrightarrow{uv}$. Clearly γ will retain the properties (1),(2) and (3) from *Lemma 5* as well as the property that for every $u \in V(G)$ and $x \in X$, $x \neq f(u)$ ($x = f(u)$) there is an arc (no arc) with voltage x emanating from u. Moreover, $N(\gamma) = N(\beta) \setminus \{(u,x),(u,y)\}$ because the arc $\overrightarrow{uv}$ has new voltage x and there already is an arc with voltage x emanating from v (recall that $f(v) \neq x$) and the arc $\overrightarrow{uv_1}$ has new voltage y and there already is an arc with voltage y emanting from v_1 too ($f(v_1) = x \neq y$). Hence property (4) from *Lemma 5* does hold for pairs (u,x) and (u,y). It is a matter of routine to show that the exchange of voltages has no other impact on $N(\beta)$.

Since $(u,x) \in N(\beta)$, we have $|N(\gamma)| < |N(\beta)|$. Thus, by repeating the algorithm above we finally come to a voltage assignment α such that $|N(\alpha)| = 0$, which completes the proof. □

Now we will show that the function f exists if the number of vertices of G is not very large.

Theorem 7. *Let* $H = C(\Gamma, X)$ *be an arbitrary connected Cayley digraph, where* Γ *is an abelian group. Let* G *be a strongly connected digraph without multiple arcs such that* $\delta(G) \geq |X| \geq 3$ *and* $|V(G)| \leq (|X|-1)^{\delta(G)-2}$. *Then there exists a voltage assignment* $\alpha : D(G) \to \Gamma$ *such that*

$$diam(G^\alpha) \leq r(G) + diam(H) + 1.$$

Proof. Let T (with the root w) be any radial spanning tree of G. Define a *branch* of T to be any component of (weak) connectivity of the digraph $T - w$. We will show that there exists a function as in *Lemma 6*.

Let $k = |X|$, $l = \delta(G)$, m = number of branches in T and $n = |V(G)|$. Let us find the number of functions $f : V(G) \to X \cup \{1\}$ which have properties (1) and (2) from *Lemma 6*.

If f satisfies (1) then (to satisfy (2) too) it is neccessary and sufficient that $f(u) \neq f(v)$ for any pair of vertices u, v such that $\overrightarrow{uv}$ is an arc in $T - w$ (i.e. in one of the branches of T). This means that in any branch we have k possibilities to define f on some vertex u ($f(u)$ can be any element of X) but only $k - 1$ possibilities for any of its neighbours (because of (2)), $k - 1$ possibilities for any of their neighbours, etc. Therefore there are $k(k-1)^{p-1}$ possibilities to define f on a branch with p vertices, which implies that there are exactly $k^m(k-1)^{n-m-1}$ functions f having properties (1) and (2) from *Lemma 6*.

Now we will find the upper bound on the number of f satisfying (1) and (2) but not (3). If $\overrightarrow{wv_1}, \ldots, \overrightarrow{wv_s}$ are all arcs emanating from w (obviously $s \geq l$) then (3) does not hold for the vertex w if and only if $f(v_1) = \cdots = f(v_s) \neq 1$ (hence $v_i \neq w$). So we have at most k possibilities of defining f on $v_1, \ldots, v_s$. If a branch of T with p vertices contains $q > 0$ vertices of $v_1, \ldots, v_s$ then (using the same argumentation as above) the number of possibilities of defining f on the remaining vertices in that branch is at most $(k-1)^{p-q}$ and in the case $q = 0$ it is $k(k-1)^{p-1}$. Therefore if $j \geq 1$ is the number of branches containing at least one of the vertices $v_1, \ldots, v_s$ then the number of f satisfying (1) and (2) but for the vertex $u = w$ not satisfying (3) is at most

$$k\, k^{m-j}(k-1)^{n-s-(m-j)-1} \leq k^m(k-1)^{n-l-m}$$

(note that $v_1, \ldots, v_s$ are different vertices because G does not contain multiple arcs).

If $u \neq w$ and $\overrightarrow{uv_1}, \ldots, \overrightarrow{uv_s}$ are all arcs emanating from u and not in T ($s \geq l - 1$) then the fact that (3) does not hold for the vertex u implies $f(v_1) = \cdots = f(v_s) \neq 1$. Again we have at most k possibilities to define f on $v_1, \ldots, v_s$. Using the same argumentation as above we come to the conclusion that the number of f which have properties (1) and (2) but for the vertex $u \neq w$ do not satisfy (3) is at most

$$k\, k^{m-j}(k-1)^{n-s-(m-j)-1} \leq k^m(k-1)^{n-(l-1)-m}.$$

Since f does not satisfy (3) if and only if it does not satisfy (3) for some vertex u, the number of f which satisfy (1) and (2) but not (3) is at most

$$k^m(k-1)^{n-l-m} + (n-1)k^m(k-1)^{n-(l-1)-m} = k^m(k-1)^{n-l-m}(1+(n-1)(k-1))$$

which is (thanks to $k \geq 3$) strictly smaller than

$$k^m(k-1)^{n-l-m}n(k-1) \leq k^m(k-1)^{n-l-m+1+l-2} = k^m(k-1)^{n-m-1},$$

what is the number of f satisfying (1) and (2). Thus, there is at least one function f with all the three properties from *Lemma 6*. The result now follows from the last two *Lemmas*. □

5. Concluding Remarks

All theorems in this paper are fairly general. We may therefore expect that for some special class of digraphs $\mathcal{G}$ and for some special group Γ it is possible to improve our results. Let us now outline a possible way to construct digraphs of order "close" to the Moore bound provided that there was a suitably powerful bound on the diameter of a lift, at least for some very special classes of digraphs.

Suppose that there were Cayley digraph $H = C(X, \Gamma)$ and digraph $G_0 \in \mathcal{G}$, both having indegree and outdegree of every vertex equal to $d = |X|$ such that the numbers of their vertices were close to the Moore bound, say $|V(H)| \geq d^{diam(H)}$ and $|V(G_0)| \geq d^{diam(G_0)}$. Suppose in addition that we could prove that for any $G \in \mathcal{G}$ with $\delta(G) \geq |X|$ there was a voltage assignment $\alpha : D(G) \to \Gamma$ such that

$$diam(G^\alpha) \leq diam(G) + diam(H)$$

and $G^\alpha \in \mathcal{G}$.

Then we could apply this proposition to G_0 and construct $G_1 = G_0^\alpha \in \mathcal{G}$ with diameter at most $diam(G_0) + diam(H)$. From G_1 we could construct $G_2 = G_1^\alpha \in \mathcal{G}$ with diameter at most $diam(G_0) + 2diam(H)$, etc. After n iterations we would obtain a digraph G_n with indegree and outdegree of every vertex equal to d, with diameter at most $diam(G_0) + n\, diam(H)$ and with $|V(G_0)||V(H)|^n \geq d^{diam(G_n)}$ vertices. Thus the number of vertices of the "large" digraph G_n would be at least asymptotically close to the Moore bound as well.

The above outline could provide a further motivation to look for improvements of the results of [1] and this paper for suitable digraphs and groups.

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