Isomorphisms and Normality of Cayley Digraphs of A_5

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Abstract

It is proved that each 2-element generating set of A_5 is a CI-subset and that the corresponding Cayley digraph is normal. It is furthermore proved that for each 3-element generating set of A_5 the corresponding Cayley digraph is normal.

1 Introduction

Let G be a finite group and S a subset of G not containing the identity element 1. We define the *Cayley digraph* $X = \mathrm{Cay}(G, S)$ of G with respect to S by

$$\begin{aligned} V(X) &= G, \\ E(X) &= \{(g, sg) \mid g \in G, s \in S\}. \end{aligned}$$

If $S = S^{-1}$, then the adjacency relation is symmetric and $\mathrm{Cay}(G, S)$ is called the *undirected* Cayley graph of G with respect to S. The group G acting by right multiplication (that is, $g_R : x \mapsto xg$) is a subgroup of automorphisms of $\mathrm{Cay}(G, S)$ and acts transitively on vertices. We call $G_R = \{g_R \mid g \in G\}$ the *right regular representation* of G. Let $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Obviously $\mathrm{Aut}(X) \geq G_R\mathrm{Aut}(G, S)$. If $G_R = \mathrm{Aut}(X)$, then X is called a digraphical regular representation (DRR) of G and a DRR of a group G is a normal Cayley graph of G.

Let $A = \mathrm{Aut}(X)$. We have

Lemma 1.1 ([2, Proposition 1.3])

(1) $N_A(G_R) = G_R\mathrm{Aut}(G, S)$;

(2) $A = G_R\mathrm{Aut}(G, S)$ *is equivalent to* $G_R \lhd A$.

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Definition 1.2 *The* Cayley *(di)graph* $X = \mathrm{Cay}(G, S)$ *is called* normal *if* G_R*, the right regular representation of* G*, is a normal subgroup of* $\mathrm{Aut}(X)$.

So, normal Cayley digraphs are just those which have the smallest possible full automorphism group. The following obvious result is a direct consequence of the above definition and Lemma 1.1.

Lemma 1.3 *Let* $X = \mathrm{Cay}(G, S)$ *be the Cayley digraph of* G *with respect to* S*, and let* $A = \mathrm{Aut}(X)$*. Let* A_1 *be the stabilizer of the identity element 1 in* A*. Then* X *is normal if and only if every element of* A_1 *is an automorphism of the group* G.

Let $X = \mathrm{Cay}(G, S)$ be the Cayley digraph of G with respect to S. Let $\alpha \in \mathrm{Aut}(G)$. Then it is easy to see that α is a graph isomorphism from $\mathrm{Cay}(G, S)$ to $\mathrm{Cay}(G, S^{\alpha})$. We call this kind of isomorphism between Cayley digraphs of G a *trivial* automorphism. The subset S is said to be a *CI-subset of* G, if for any graph isomorphism $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, there exists an $\alpha \in \mathrm{Aut}(G)$ such that $S^{\alpha} = T$. In other words, that S is CI means that there are only trivial isomorphisms between $\mathrm{Cay}(G, S)$ and other Cayley digraphs of G.

The motivation for this paper comes from a survey of Xu [2] and an unpublished result of Li [7] which states that some 2-element generating sets of A_5 are CI-subsets, and the corresponding Cayley digraph is normal. In order to make this paper self-contained, we prove Li's result in section 2 while in section 3 we give a further extension. The main results of this paper are the following two theorems.

Theorem 1.4 (See [2]) *Each 2-element generating set of* A_5 *is a* CI*-subset and the corresponding Cayley digraph is normal.*

This result was originally proved by X. Li. However, the proof of Theorem 1.4 in section 2 is independent of Li's.

Theorem 1.5 *Let* $G = A_5$ *and* $S = \{a, b, c\}$ *be a 3-element generating set of* G *not containing the identity 1. Then* $X = \mathrm{Cay}(G, S)$ *is a normal Cayley digraph.*

In this paper the symbol G will always denote the group A_5 and 1 will denote its identity. For $x \in G$ we let $o(x)$ denote the order of x. The symbol X will always denote a simple graph. By $V(X)$, $E(X)$ and $A(X) = A$ we denote the vertex set, the edge set and the automorphism group of X, respectively. By $A_v(X) = A_v$ we denote the stabilizer of the vertex $v \in V(X)$. For every set T, 1_T denotes the identity permutation on T.

The group and graph-theoretic notation and terminology used here are generally standard, and the reader can refer to [3] and [6] when necessary.

2 The Proof of Theorem 1.4

Lemma 2.1 *Let* $S = \{a, b\}$ *and* $T = \{a', b'\}$ *be two 2-element generating subsets of* $G = A_5$*. If* $X = \mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T) = X'$ *and if min* $\{o(a), o(b)\} \leq 3$*, then there exists* $\alpha \in \mathrm{Aut}(G)$ *such that* $S^{\alpha} = T$.

Proof Let α be a graph isomorphism. Without loss of generality we may assume that $1^\alpha = 1$, since Cayley digraphs are vertex-transitive. Hence $S^\alpha = T$. By renaming the elements of T if necessary, we may also assume that

$$a^\alpha = a', \ b^\alpha = b'.$$

We use induction on n to show that $(x_1x_2\cdots x_n)^\alpha = x_1'x_2'\cdots x_n'$, where $x_i = a$ or b for $i = 1, \ldots, n$. (This implies that $\alpha \in \mathrm{Aut}(G)$, as required.) We distinguish two cases.

Case 1: $\min\{o(a), o(b)\} = 2$.

Without loss of generality, we may assume that $o(a) = 2$. Then $o(b) \neq 2$ since $\langle a, b\rangle \cong A_5$. Thus $(a, 1) \in E(X)$ and $(b, 1) \notin E(X)$ by definition of the Cayley digraph X and so $(a', 1) \in E(X')$. By definition of the Cayley digraph X', $(a', b'a') \in E(X')$ and $(a', (a')^2) \in E(X')$. If $b'a' = 1$, this would contradict $\langle a', b'\rangle \cong A_5$, so it follows that $(a')^2 = 1$ and thus $(ba)^\alpha = b'a'$ and $o(b') \neq 2$. Using the fact that the graphs have indegrees and outdegrees equal to 2, we have that $(ab)^\alpha = a'b'$ and $(b^2)^\alpha = (b')^2$. So $(x_1x_2)^\alpha = x_1'x_2'$.

Suppose $n > 2$. Set $x = x_3x_4\cdots x_n$ and $x' = x_3'x_4'\cdots x_n'$. From the inductive assumption we can suppose that $x^\alpha = x'$, $(ax)^\alpha = a'x'$ and $(bx)^\alpha = b'x'$. An argument similar to the one above shows that $(x_1x_2x)^\alpha = x_1'x_2'x'$. Thus we have that for any positive integer n, $(x_1x_2\cdots x_n)^\alpha = x_1'x_2'\cdots x_n'$.

Case 2: $\min\{o(a), o(b)\} = 3$.

Let $o(a) = 3$. As $1 \mapsto a \mapsto a^2 \mapsto 1$ is a directed circuit of length 3 in X we have that $1 \mapsto a' \mapsto (a')^2 \mapsto (a')^3$ must be a directed circuit of length 3 in X' because $a'b'a' \neq 1$, $(b')^2a' \neq 1$ and $b'(a')^2 \neq 1$ (otherwise $G = \langle a', b'\rangle$ is abelian). Thus $(a')^3 = 1$ and $(a^2)^\alpha = (a')^2$, $(ba)^\alpha = b'a'$.

If $o(b) = 3$, similarly we have $(b^2)^\alpha = (b')^2$ and $(ab)^\alpha = a'b'$. If $o(b) \neq 3$ then we also have $(b^2)^\alpha = (b')^2$ and $(ab)^\alpha = a'b'$ because there is a unique directed circuit of length 3 through b and b' in X and X' respectively.

Thus for $n = 1$ or 2, $(x_1x_2\cdots x_n)^\alpha = x_1'x_2'\cdots x_n'$. Using the same method as in (1) completes the inductive step.

It follows that $\alpha \in \mathrm{Aut}(G)$. □

We now prove Theorem 1.4.

First we show that each 2-element generating subset S of G is CI.

In Lemma 2.1, we checked that all 2-element generating subsets of G are CI-subsets except the case $o(a) = o(b) = 5$. Thus to prove our statement it suffices to prove that given two 2-element generating subsets $S = \{a, b \mid o(a) = o(b) = 5\}$ and $T = \{a', b'\}$ such that $X = \mathrm{Cay}(G, S)$ and $X' = \mathrm{Cay}(G, T)$ are isomorphic, there exists an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$.

Suppose that $\alpha : X = \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, T)$ is a graph isomorphism. Since Cayley digraphs are vertex-transitive, without loss of generality we may assume that $1^\alpha = 1$. Hence $S^\alpha = T$. By renaming the elements of T if necessary, we may also assume that

$$a^\alpha = a', \ b^\alpha = b'.$$

As $1 \mapsto a \mapsto a^2 \mapsto a^3 \mapsto a^4 \mapsto 1$ is a directed circuit of length 5 in X we have that $1 \mapsto a' \mapsto (a^2)' \mapsto (a^3)' \mapsto (a^4)' \mapsto (a^5)'$ must be a directed circuit of length 5 in X'. We now distinguish three cases:

(i) $(b')^i(a')^j = 1$, for $0 < i \leq 4$ and $i + j = 5$.

Then $(b')^{-i} \in \langle a' \rangle$ and so $(b') \in \langle a' \rangle$ which is a contradiction. So (i) cannot happen.

(ii) $(b'a')^2a' = 1$ or $(a'b')^2a' = 1$.

If $(b'a')^2a' = 1$ then $a' \in \langle b'a' \rangle$ and thus $b' \in \langle b'a' \rangle$ which contradicts $A_5 \cong \langle a', b' \rangle \leq \langle a'b' \rangle$. Similarly $(a'b')^2a' \neq 1$ and so (ii) cannot happen.

(iii) $a'(b')^2(a')^2 = 1$ or $a'(b')^3a' = 1$.

If $a'(b')^2(a')^2 = 1$, then it is easy to check $(b')^2 \neq 1$ and thus $(b')^2 \in \langle a' \rangle$. So $b' \in \langle a' \rangle$, a contradiction. Similarly $a'(b')^2 \neq 1$.

By (i), (ii), (iii), we have $(a')^5 = (a^5)' = 1$ and thus $1 \mapsto a' \mapsto (a')^2 \mapsto (a')^3 \mapsto (a')^4 \mapsto (a')^5$ is a directed circuit of length 5 in X'. So $(a')^i = (a^i)'$ for i=1,2,3,4,5 and thus $b'a' = (ba)'$. Similariy we have $(b')^2 = (b^2)'$ and $a'b' = (ab)'$.

Thus for $n = 1$ or 2, $(x_1x_2 \cdots x_n)^\alpha = x_1'x_2' \cdots x_n'$. Using the same inductive method as in the proof of Lemma 2.1, we can show that $\alpha \in \mathrm{Aut}(G)$.

Now we show that $X = \mathrm{Cay}(G, S)$ is a normal Cayley digraph of G.

Let $S = \{a, b\}$ be a 2-element generating subset of G and $A = \mathrm{Aut}(X)$. Then $A = A_1G_R$. For each $\phi \in A_1$,

$$\phi : X = \mathrm{Cay}(G, S) \longrightarrow X' = \mathrm{Cay}(G, S^\phi)$$

is graph isomorphism. As shown above, S is CI, so $\phi \in \mathrm{Aut}(G)$ and this implies $\phi \in \mathrm{Aut}(G, S)$. It follows that $A_1 = \mathrm{Aut}(G, S)$. By Lemma 1.1, $\mathrm{Cay}(G, S)$ is normal as required. □

3 The Proof of Theorem 1.5

The proof is organized into twelve Lemmas.

Lemma 3.1 *Let $G = A_5$ and $S = \{a, b, c\}$ be a 3-element generating subset of G. Set $X = \mathrm{Cay}(G, S)$ and $A = \mathrm{Aut}(X)$. Then $X = \mathrm{Cay}(G, S)$ is normal if the following conditions hold:*

(1) *for each $\phi \in A_1$, $\phi|_S = 1_S$ implies $\phi|_{S^2} = 1_{S^2}$,*

(2) *for each $\phi \in A_1$, $\phi^2|_S = 1_S$.*

Proof Let H_S denote the subgroup of A which fixes 1, a, b, and c. First we show that H_S is trivial. Note that since A is transitive on $V(X) = G$, condition (1) applies to every vertex v, that is, for any $\phi \in A_v(X)$, $\phi|_{Sv} = 1_{Sv}$ implies $\phi|_{S^2v} = 1_{S^2v}$. Now let $\phi \in H_S$, then $\phi \in A$, and $\phi|_S = 1_S$ so $\phi|_{S^2} = 1_{S^2}$. Let $x \in S$, then $x^\phi = x$, so $\phi \in A_x(X)$. Also $Sx \subseteq S^2$ so $\phi|_{Sx} = 1|_{Sx}$. Hence $\phi|_{S^2x} = 1_{S^2}$. Since this holds for all $x \in S$, we have $\phi|_{S^3} = 1_{S^3}$. By induction, $\phi|_{S^t} = 1_{S^t}$ holds for any positive integer t. Since $G = \langle S \rangle$ and G is finite, we have that if $\phi_S = 1_S$ then $\phi = 1_G$. This implies that if $\phi \in H_S$, then $\phi = 1_G$ and so $H_S = 1$ as required.

Since condition (2) says that for each $\phi \in A_1$, $\phi^2 \in H_S$, we have $\phi^2 = 1_G$ for all $\phi \in A_1$, so A_1 is 2-group. Since $A_1 \subseteq S_3 H_S = S_3$, $|A_1| \leq 2$. As $A = A_1 G_R$ we have $G_R \lhd A$. Thus $X = \mathrm{Cay}(G, S)$ is normal. □

Using this lemma we can analyze the normality of a Cayley digraph of A_5 in terms of its generating set.

Let S be a generating subset of G of cardinality 3 and let l, m, n be integers ≥ 2. We call S an *(l, m, n)-generating set* of G if $a, b, c \in S$, $a^l = b^m = c^n = 1$ and $G = \langle a, b, c \rangle$.

Lemma 3.2 *Let S be a (2,3,5)-generating set of G. Then $X = \mathrm{Cay}(G, S)$ is a normal Cayley digraph. Moreover X is a DRR of G.*

Proof Now A_5 is by definition the group given by the presentation $\langle a, b, c \mid a^2 = b^3 = c^5 = 1, ab = c \rangle$ (or $\langle a, b, c \mid a^2 = b^3 = c^5 = 1, ac = b \rangle$). Thus if $S = \{a, b, c\}$ as above, then $C_1 = (1, a)$, $C_2 = (1, b, b^2)$ and $C_3 = (1, c, c^2, c^3, c^4)$ are unique cycles of lengths 2, 3 and 5 at the point 1. So for each $\phi \in A_1$, we have $C_1^\phi = C_1$, $C_2^\phi = C_2$ and $C_3^\phi = C_3$. Since C_1, C_2 and C_3 are directed dicircuits, therefore ϕ fixes C_1, C_2 and C_3 pointwise, and so $b^\phi = b$, $c^\phi = c$ and $a^\phi = a$. It follows that $A_1 = A_a = A_b = A_c$. Finally because a, b, c generate the group G, therefore $A_1 = A_g$ for any $g \in G$ and so $A_1 = 1$ and $A = G_R$ as required. □

In the remaining part of this section we shall discuss the case $o(a) = o(b) \neq o(c)$. If ϕ is a non-trivial graph automorphism which fixes the point 1, it must have the form

$$\phi\,|_S = (a, b), \qquad c^\phi = c.$$

This shows that $\phi^2\,|_S = 1_S$. Applying Lemma 3.1 in this case, to prove that $\mathrm{Cay}(G, S)$ is a normal Cayley digraph of G, we need only check that the condition (1) of Lemma 3.1 holds, that is, we focus on testing which digraphs meet the condition: for $\phi \in A_1$,

$$\text{if } \phi\,|_S = 1_S, \text{ then } \phi\,|_{S^2} = 1_{S^2}. \tag{3.1}$$

Lemma 3.3 *Let S be a (3,5,5)-generating set of G or a (3,3,5)-generating set of G. Then* $\mathrm{Cay}(G, S)$ *is a normal digraph.*

Proof Suppose that $S = \{b, c_1, c_2\}$ and $G = \langle b, c_1, c_2 \rangle$ or $S = \{b_1, b_2, c\}$ and $G = \langle b_1, b_2, c \rangle$, where b, b_1, b_2 are elements of order 3 and c_1, c_2, c are elements of order 5 in G. Let $S_1 = \{b, c_1\}$ and $S_2 = \{b_1, c\}$. Since a subgoup of A_5 which contains an element of order 3 and an element of order 5 is A_5, so $G = \langle S_i \rangle$ for $i = 1, 2$. If $\phi\,|_S = 1_S$ for $\phi \in A_1$ then ϕ induces an action on S_i for $i = 1, 2$. By Theorem 1.4 $\mathrm{Cay}(G, S_i)$ is a normal Cayley digraph. Hence $\phi \in \mathrm{Aut}(G, S_i)$, for $i = 1, 2$. In either case ϕ is an automorphism of G. Therefore ϕ fixes every element of G and hence $\phi = 1$ (certainly we have $\phi\,|_{S^2} = 1_{S^2}$). So $G \lhd A$ as required. □

Lemma 3.4 *Let S be a (2,5,5)-generating set of G. Then $X = \mathrm{Cay}(G,S)$ is a normal Cayley digraph.*

Proof Let $S = \{a, c_1, c_2\}$ be a (2,5,5)-generating set of G, where a is an involution and c_1, c_2 are elements of order 5.

If $G = \langle a, c_i\rangle$, $i = 1$ or $i = 2$, the lemma can be proved similarly to Lemma 3.3.

So we suppose that $\langle a, c_i\rangle \neq G$ for $i = 1, 2$. Since $G = \langle a, c_1, c_2\rangle$, we have $\langle c_1\rangle \neq \langle c_2\rangle$. Thus $G = \langle c_1, c_2\rangle$ and hence for the digraph $\mathrm{Cay}(G, \{c_1, c_2\})$, the set $\{c_1, c_2\}$ is CI by Theorem 1.4, and so $\phi \in \mathrm{Aut}(G)$ for $\phi \in A_1$. Therefore if $\phi\,|_S = 1_S$, then $\phi\,|_{S^2} = 1_{S^2}$ and so (3.1) shows that $X = \mathrm{Cay}(G,S)$ is normal. □

Lemma 3.5 *Let S be a (2,3,3)-generating set of G. Then $X = \mathrm{Cay}(G,S)$ is a normal Cayley digraph.*

Proof Assume that $S = \{a, b_1, b_2 \mid a^2 = b_1^3 = b_2^3 = 1\}$. If $\langle a, b_i\rangle = G$ or $\langle b_1, b_2\rangle = G$, for $i = 1, 2$, then the lemma can be proved similarly to Lemma 3.4. So it suffices to prove the result in the case:

(i) $\langle a, b_i\rangle \neq G$, $i = 1, 2$; and

(ii) $\langle b_1, b_2\rangle \neq G$. (3.5)

Since A_5 has only one conjugacy class of involutions, without loss of generality we may assume that $a = (12)(34)$. First we claim that **if** $\langle a, b_1\rangle \cong A_4$ **then** $\langle a, b_2\rangle \cong S_3$ by condition (3.5). Indeed, if $\langle a, b_1\rangle \cong A_4$ then we may suppose that $b_1 = (123)$ and $b_2 = (i, j, k)$. Since $G = \langle a, b_1, b_2\rangle$, $\{i, j, k\}$ must contain 5; say $k = 5$. Thus $\{i, j\}$ cannot be one of $\{1,3\}$, $\{2,4\}$, $\{2,3\}$ or $\{1,4\}$ otherwise $\langle a, b_2\rangle \cong A_5$. So $\{i, j\}$ must be either $\{1,2\}$ or $\{3,4\}$. In either case we have $\langle a, b_2\rangle \cong S_3$. But if $b_2 = (345)$ or (354) then $\langle b_1, b_2\rangle \cong A_5$. From this it follows that $b_2 = (125)$ or (152). Similarly, suppose that $a = (12)(34)$, $b_2 = (125)$. If $\langle a, b_1\rangle \cong S_3$, then b_1 must be (345) or (435); in either case we have $\langle b_1, b_2\rangle = G$ which contradicts condition (ii). Since $\langle a, b_1\rangle \neq G$, it follows that $\langle a, b_1\rangle \cong A_4$.

Finally it remains to show the result in the case that $\langle a, b_1\rangle \cong A_4$ and $\langle a, b_2\rangle \cong S_3$, where $a = (12)(34)$, $b_1 = (123)$, and $b_2 = (125)$.

Indeed, we claim that $\phi\,|_S = 1_S$ for **each** $\phi \in A_1$ in this case. If $\phi\,|_S \neq 1_S$ then ϕ fixes a and interchanges b_1 and b_2 and so ϕ induces a graph automorphism between $\mathrm{Cay}(\langle a, b_1\rangle, \{a, b_1\})$ and $\mathrm{Cay}(\langle a, b_2\rangle, \{a, b_2\})$. This contradicts the assumption $\langle a, b_1\rangle = A_4$ and $\langle a, b_2\rangle = S_3$. Thus $\phi\,|_S = 1_S$. So ϕ fixes a, b_1, b_2 and hence $A_1 = A_a = A_{b_1} = A_{b_2}$. It follows that $A_1 = 1$ and the lemma is proved. □

Lemma 3.6 *Let S be a (2,2,3)-generating set of G. Then $X = \mathrm{Cay}(G,S)$ is a normal Cayley digraph.*

Proof In this case G is generated by a pair of involutions a_1, a_2 and an element b of order 3. Without loss of generality we assume $b = (123)$. Let $\phi \in A_1$. If $\phi\,|_S \neq 1_S$, as before ϕ fixes b and interchanges a_1 and a_2 and so it induces a graph automorphism from $\mathrm{Cay}(\langle a_1, b\rangle, \{a_1, b\})$ to $\mathrm{Cay}(\langle a_2, b\rangle, \{a_2, b\})$. So $|\langle a_1, b\rangle| = |\langle a_2, b\rangle|$ and it follows that $\langle a_1^{\phi}, b\rangle = \langle a_2, b\rangle$.

Now we consider three cases.

(1) $\langle a_1, b\rangle \cong \langle a_2, b\rangle \cong A_5$.

Since $\mathrm{Cay}(\langle a_i, b\rangle, S_i)$ is a normal Cayley digraph by Theorem 1.4 (where $S_i = \{a_i, b\}$ and $i = 1, 2$), we have $\phi \in \mathrm{Aut}(G, \{a_i, b\}) \leq \mathrm{Aut}(G)$ and so the result is proved by Lemma 1.3.

(2) $\langle a_1, b\rangle \cong \langle a_2, b\rangle \cong A_4$.

It is obvious that $\langle a_1, b\rangle \neq \langle a_2, b\rangle$. Let $\phi \in A_1$. If $\phi\,|_S = 1_S$ then ϕ induces an automorphism of the subgraph $X_i = \mathrm{Cay}(\langle a_i, b\rangle, S_i)$, where $S_i = \{a_i, b\}$ and $i = 1, 2$. Since $C_1 = (1, a_1)$ is the unique circuit of length 2 and $C_2 = (1, b, b^2)$ is the unique circuit of length 3 at the point 1, it follows that ϕ fixes C_1 and C_2 pointwise. Since ϕ fixes b, ϕ fixes the neighbourhood of b in X_1 and thus ϕ fixes $a_1 b$. Since ϕ fixes a_1, ϕ fixes the neighbourhood of a_1 in X_1 and thus ϕ fixes ba_1. Since ϕ fixes a_1, ϕ fixes the neighbourhood of a_1 in X and thus ϕ fixes $a_2 a_1$. Similarly we can check that ϕ fixes the points $a_2 b, a_2^2, ba_2$ and $a_1 a_2$. This shows that $\phi\,|_{S^2} = 1_{S^2}$. By Lemma 3.1, $\mathrm{Cay}(G, S)$ is normal as required.

(3) $\langle a_1, b\rangle \cong \langle a_2, b\rangle \cong S_3$.

This case can not happen, since for each element of order 3 in the unique subgroup which is isomorphic to S_3, if $\langle a_1, b\rangle \cong \langle a_2, b\rangle \cong S_3$ then $\langle a_1, b\rangle = \langle a_2, b\rangle \cong S_3$ which contradicts $\langle a_1, a_2, b\rangle = G$.

By (1), (2) and (3), the lemma is proved. □

Lemma 3.7 *Let S be a (2,2,5)-generating set of G. Then $X = \mathrm{Cay}(G, S)$ is a normal Cayley digraph.*

Proof Suppose that G is generated by a pair of involutions a_1, a_2 and an element c of order 5. If $\langle a_1, c\rangle \cong D_{10}$ then $\langle a_2, c\rangle \not\cong D_{10}$ by the fact that c is in a unique subgroup $H \cong D_{10}$, and so $\langle a_2, c\rangle \cong A_5$. Now we distinguish two cases.

(1) $\langle a_1, c\rangle \cong D_{10}$ and $\langle a_2, c\rangle \cong A_5$;

(2) $\langle a_1, c\rangle \cong \langle a_2, c\rangle \cong A_5$.

The lemma follows in either case in the same way as the proof of Lemma 3.5. □

Lemma 3.8 *Let S be a (5,5,5)-generating set of G. Then $X = \mathrm{Cay}(G, S)$ is a normal Cayley digraph.*

Proof Let $S = \{c_1, c_2, c_3 \mid c_1^5 = c_2^5 = c_3^5 = 1\}$. Clearly if $\langle c_1\rangle \neq \langle c_2\rangle$, then $\langle c_1, c_2\rangle = A_5$. Since $G = \langle c_1, c_2, c_3\rangle$, we may assume that $\langle c_1, c_2\rangle = A_5$. If $\langle c_2, c_3\rangle \cong Z_5$, then $\phi\,|_S = (c_1, c_2, c_3)$ is not a graph automorphism from $\mathrm{Cay}(G, \{c_1, c_2\})$ to $\mathrm{Cay}(G^{\phi}, \{c_1, c_2\}^{\phi})$ since $\langle c_1, c_2\rangle \cong A_5$ and $\langle c_1, c_2\rangle^{\phi} = \langle c_2, c_3\rangle \cong Z_5$.

So we assume that $\langle c_1, c_2\rangle = \langle c_2, c_3\rangle = \langle c_3, c_1\rangle = G$. Let A_{1,c_1} denote the subgroup of A which fixes 1 and c_1. Then $|A_1| \leq 3|A_{1,c_1}|$. Let $\phi \in A_{1,c_1}$, then ϕ fixes c_1 and stabilizes the set $\{c_2, c_3\}$. Set $S_1 = \{c_2, c_3\}$. Thus ϕ induces a graph automorphism of $\mathrm{Cay}(G, S_1)$. Since $\mathrm{Cay}(G, S_1)$ is a normal Cayley graph by Theorem 1.4, it follows that $\phi \in \mathrm{Aut}(G)$. But $\mathrm{Aut}(G)$ has no non identity automorphism that interchanges a pair of elements (c_2 and c_3) of order 5 and fixes another element c_1 of order 5, so we have $\phi = 1$ and thus $A_{1,c_1} = 1$ and $|A_1| \leq 3$.

If $|A_1| = 3$, then $|A| = 180$ and A is not simple group. So A contains a nontrivial normal subgroup N, that is $N \neq 1$ and $N \neq A$. If $G \cap N \neq 1$ then $G \cap N$ is a nontrivial normal subgroup of G which contradicts $G \cong A_5$. It follows that $G \cap N = 1$ and thus $A = N \cdot G$ and $A_1 = N$. Since A_1 is the stabilizer of the point 1, we get $N = 1$, a contradiction. It follows that $|A_1| \leq 2$ and $\mathrm{Cay}(G, S)$ is a normal Cayley digraph as claimed. □

Lemma 3.9 ([4, Lemma 2.2]) *Let $X = \mathrm{Cay}(G, S)$. Then X is a normal Cayley digraph of G if the following conditions hold:*

(i) *for each $\phi \in A_1$ there exists $\sigma \in \mathrm{Aut}(G)$ such that $\phi|_S = \sigma|_S$;*

(ii) *for each $\phi \in A_1$, $\phi|_S = 1_S$ implies $\phi|_{S^2} = 1_{S^2}$.*

Proof (1) Condition (ii) implies that if $\phi \in A_1$ and $\phi|_S = 1_S$, then $\phi = 1_G$.

(2) We show that $A_1 \leq \mathrm{Aut}(G, S)$. By the hypothesis (i), for each $\phi \in A_1$, we may take $\sigma \in \mathrm{Aut}(G)$ such that $\phi|_S = \sigma|_S$. Then $\phi\sigma^{-1}|_S = 1_S$. By the proof above we have $\phi\sigma^{-1} = 1_G$ and $\phi = \sigma \in \mathrm{Aut}(G)$. Thus $A_1 \leq \mathrm{Aut}(G, S)$.

(1) and (2) imply that X is a normal graph of the group G. □

Lemma 3.10 *Let S be a (3,3,3)-generating set of G. Then $X = \mathrm{Cay}(G, S)$ is a normal Cayley digraph.*

Proof First suppose that G is generated by three elements b_1, b_2 and b_3 of order 3. We now distinguish two cases:

(i) There exist $b_i, b_j \in S$ such that $\langle b_i, b_j \rangle = G$. In this case there must exist b_i or b_j such that $\langle b_i, b_k \rangle \neq G$ or $\langle b_j, b_k \rangle \neq G$. As in Lemma 3.8 we can prove $\mathrm{Cay}(G, S)$ is a normal digraph of G.

(ii) There are no $b_i, b_j \in S$ such that $\langle b_i, b_j \rangle = G$. In this case there exist b_i, b_j such that $\langle b_i, b_j \rangle = A_4$. It is clear that the 3-cycle b_i and b_j are in a same subgroup of G which is isomorphic to A_4 if and only if they have two symbols that are the same. So without loss of generality, we assume that $b_1 = (123)$ and $b_2 = (124)$. Since $\langle b_1, b_2, b_3 \rangle = A_5$, thus $b_3 = (i, j, 5)$, and in addition $\langle b_i, b_3 \rangle \cong A_4$, for $i = 1, 2$ (If $\langle b_i, b_3 \rangle \cong Z_3$, then $b_3 \in \langle b_i, b_j \rangle \cong A_4$, a contradiction.) Hence we have $b_3 = (125)$ or (152).

In this case for each $\phi \in A_1$, if $\phi|_S = (b_1, b_2, b_3)$, then there exists $\sigma \in \mathrm{Aut}(G, S)$ such that $\phi|_S = \sigma|_S$; in fact by the assumption, $\sigma = (345)$. If $\phi|_S = 1_S$, then ϕ induces a graph automorphism of $\mathrm{Cay}(\langle b_i, b_j \rangle, \{b_i, b_j\})$ for $i, j \in \{1, 2, 3\}$. Set $S_1 = \{b_i, b_j\}$. It is easy to check that the Cayley graph $\mathrm{Cay}(\langle b_i, b_j \rangle, S_1)$ satisfies the conditions of Lemma 3.9. So $\mathrm{Cay}(\langle b_i, b_j \rangle, S_1)$ is normal and hence $\phi|_{S_1} \in \mathrm{Aut}(\langle b_i, b_j \rangle)$. It follows that ϕ fixes b_i and b_j and hence fixes b_i^2, b_j^2, b_ib_j and b_jb_i. Similarly ϕ fixes the other elements of S^2. So if $\phi|_S = 1_S$ then $\phi|_{S^2} = 1_{S^2}$. By Lemma 3.9 our statement follows.

By (i) and (ii), the lemma is proved. □

Now consider the case when G is generated by three involutions a_1, a_2 and a_3 and $X = \mathrm{Cay}(G, S)$ is an undirected graph. We will use the following result.

Lemma 3.11 ([5, Theorem 1.3]) *Suppose that G is a nonabelian simple group. Then G is a 3-CI-group if and only if $G = A_5$.*

Lemma 3.12 *Let S be a (2,2,2)-generating set of G. Then $X = \mathrm{Cay}(G, S)$ is a normal Cayley graph.*

Proof Since $S = \{a_1, a_2, a_3\}$ is 3-CI, for each $\phi \in A_1$ we have $\phi \in \mathrm{Aut}(G)$ by Lemma 3.11 and the lemma follows. □

PROOF OF THEOREM 1.5 This is given by Lemmas 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.10 and 3.12. □

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