On the list chromatic index of nearly bipartite multigraphs

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Abstract

Galvin ([7]) proved that every k-edge-colorable bipartite multigraph is k-edge-choosable. Slivnik ([11]) gave a streamlined proof of Galvin's result. A multigraph G is said to be *nearly bipartite* if it contains a *special* vertex v_s such that $G - v_s$ is a bipartite multigraph. We use the technique in Slivnik's proof to obtain a list coloring analog of Vizing's theorem ([12]) for nearly bipartite multigraphs, and to obtain an extension (suggested by Woodall ([13])) of Galvin's result to multigraphs whose underlying simple graph is bipartite 'plus one edge'. We also prove that for any nearly bipartite multigraph G with special vertex v_s of degree at most six, if G is k-edge-colorable then G is k-edge-choosable.

1 Introduction

We refer the reader to ([1]) or ([8]) for all terminology and notation that is not defined in this paper.

Let G be a multigraph with vertex set $V(G)$ and edge set $E(G)$. A *proper edge coloring* of G is an assignment of colors to the edges of G in such a way that no two adjacent edges are assigned the same color. Multigraph G is said to be *k-edge-colorable* if there exists a proper edge coloring of G in k colors. Given a family of sets of colors $\mathcal{C} = \{C(e)\colon e \in E(G)\}$, G is said to be *$\mathcal{C}$-list-colorable* if there exists a proper edge coloring of G such that for each edge $e \in E(G)$, e is assigned a color from $C(e)$. Multigraph G is said to be *k-edge-choosable* if G is $\mathcal{C}$-list-colorable for **any** family of sets of colors $\mathcal{C} = \{C(e)\colon e \in E(G)\}$ satisfying $|C(e)| \geq k$ for every $e \in E(G)$. The *chromatic index* of G (denoted by $\chi'(G)$) is the minimum k for which G is k-edge-colorable. The *list chromatic index* of G (denoted by $\chi'_l(G)$) is the minimum k for which G is k-edge-choosable.

For a multigraph G, we denote the degree of vertex $v \in V(G)$ by $\deg_G(v)$, the maximum degree of G by $\Delta(G)$, and, the maximum edge-multiplicity of G by $\mu(G)$ respectively. The following theorems of König ([9]), Vizing ([12]), and Shannon ([10])

are well-known.

Theorem 1 (König [9]). *For any bipartite multigraph G, $\chi'(G) = \Delta(G)$.*

Theorem 2 (Vizing [12]). *For any multigraph G, $\chi'(G) \leq \Delta(G) + \mu(G)$.*

Theorem 3 (Shannon [10]). *For any multigraph G, $\chi'(G) \leq \frac{3}{2}\Delta(G)$.*

Dinitz ([6]) asked the following question.

Dinitz's Question. *Given an $n \times n$ array of n-sets, is it always possible to choose one element from each set, keeping the chosen elements distinct in every row, and distinct in every column?*

It is clear that Dinitz's question asks whether it is true that $\chi'_l(K_{n,n}) = n$, where $K_{n,n}$ is the complete bipartite graph with n vertices in each of its partite sets. Galvin ([7]) obtained the following list-coloring analog of König's Theorem 1, and settled Dinitz's question in the affirmative by proving that (more generally) the list chromatic index of any bipartite multigraph is equal to its maximum degree.

Theorem 4 (Galvin [7]). *For any bipartite multigraph G, $\chi'_l(G) = \Delta(G)$.*

Borodin, Kostochka, and, Woodall ([2]) strengthened Galvin's Theorem 4 as follows to Theorem 5, and, used this strengthening to obtain a list-coloring analog of Shannon's Theorem 3 as follows in Theorem 6.

Theorem 5 (Borodin, Kostochka, Woodall [2]). *For any bipartite multigraph G, if $\mathcal{C} = \{C(e) \colon e \in E(G)\}$ is any family of sets of colors such that for each $e = (u, v) \in E(G)$, we have that $|C(e)| \geq \max\{\deg_G(u), \deg_G(v)\}$ then, G is $\mathcal{C}$-list-colorable.*

Theorem 6 (Borodin, Kostochka, Woodall [2]). *For any multigraph G, $\chi'_l(G) \leq \frac{3}{2}\Delta(G)$.*

Clearly, we have that $\chi'_l(G) \geq \chi'(G)$ for any multigraph G. Note that since by Theorem 1, $\chi'(G) = \Delta(G)$ for any bipartite multigraph G, Galvin's theorem implies that $\chi'_l(G) = \chi'(G)$ for any bipartite multigraph G. It has been conjectured (see ([4]) for a history of this conjecture and results leading up to Galvin's Theorem) that $\chi'_l(G) = \chi'(G)$ for any multigraph G. This conjecture has become known as the *List Chromatic Conjecture* (LCC).

Conjecture 1 (List Chromatic Conjecture (LCC)). *For any multigraph G, $\chi'_l(G) = \chi'(G)$.*

For any multigraph G, it is clear that $\chi'(G)$ is the minimum number of matchings of G that are required to cover $E(G)$, and that the maximum degree of G is a lower bound for $\chi'(G)$. Another lower bound for $\chi'(G)$ can be derived as follows. We first note that if H is a multigraph of odd order at least 3 then, $\chi'(H) \geq \frac{|E(H)|}{\frac{1}{2}(|V(H)|-1)}$ since any matching in H contains at most $\frac{1}{2}(\mid V(H) \mid -1)$ edges. We denote this lower bound on $\chi'(H)$ by $t(H)$. Now, for $S \subseteq V(G)$, denote by $\langle S \rangle$ the subgraph of G induced by the vertices in S.

Define $\Gamma(G)$ by

$$\Gamma(G) = \max\{t(\langle S \rangle) : S \subseteq V(G), \mid S \mid \geq 3, \mid S \mid \text{odd}\}.$$

Clearly, $\lceil \Gamma(G) \rceil$ provides another lower bound for $\chi'(G)$.

Combining the two lower bounds, $\Delta(G)$ and $\lceil \Gamma(G) \rceil$ for $\chi'(G)$ we get an improved lower bound, $\phi(G) = \max\{\Delta(G), \lceil \Gamma(G) \rceil\}$ for $\chi'(G)$. We have that,

$$\chi'(G) \geq \phi(G).$$

A multigraph G is said to be *nearly bipartite* if it contains a *special* vertex v_s such that $G - v_s$ is a bipartite multigraph. Eggan and Plantholt ([5]) extended Theorem 1 to nearly bipartite multigraphs in the following way.

Theorem 7 (Eggan and Plantholt [5]) *For any nearly bipartite multigraph G, $\chi'(G) = \phi(G)$.*

In this paper we study the list chromatic index of nearly bipartite multigraphs. Slivnik ([11]) has given a streamlined proof of Theorem 4. In Section 2 we give a sketch of Slivnik's proof of Theorem 4, and use his proof technique to obtain the following list coloring analog of Theorem 2 for nearly bipartite multigraphs.

Theorem 8 *For any nearly bipartite multigraph G, $\chi'_l(G) \leq \Delta(G) + \mu(G)$.*

In Section 3 we show that if G is a nearly bipartite multigraph with special vertex v_s, $\mathcal{C} = \{C(e) : e \in E(G)\}$ is a family of sets of colors satisfying $|C(e)| \geq \chi'(G)$ for each $e \in E(G)$, and the colors in $\bigcup\{C(e)$: e is incident to $v_s\}$ can be partitioned in a certain way, then G is $\mathcal{C}$-list-colorable. In Section 3 we also obtain an extension (suggested by Woodall ([13])) of Galvin's result to multigraphs whose underlying simple graph is bipartite 'plus one edge'. Although we have not been able to prove the LCC for nearly bipartite multigraphs in general, in Section 3, we obtain the following theorem for nearly bipartite multigraphs for which the degree of the special vertex is at most six.

Theorem 9 *Let G be any nearly bipartite multigraph with special vertex v_s. If $\deg_G(v_s) \leq 6$, and if G is k-edge-colorable, then G is k-edge-choosable.*

2 Slivnik's proof and bounds on the list chromatic index of nearly bipartite multigraphs

In this section we give an outline of Slivnik's proof of Theorem 4 by Galvin, and then use the proof technique to obtain a list coloring analog (Theorem 8 in the Introduction) of Vizing's theorem (Theorem 2 in the Introduction) for nearly bipartite multigraphs.

Let G be a bipartite multigraph G with vertex partition $V(G) = L \bigcup R$, and let $\lambda : E(G) \to \mathcal{N}$ be a proper edge coloring of G. For $e \in E(G)$ we denote by L_e and R_e the end vertices of e in L and R respectively. A *dominating matching in G with respect to λ* is a matching $M \subseteq E(G)$ such that, for every edge $e \in E(G) \setminus M$, there exists an edge $f \in M$ with either $L_e = L_f$ and $\lambda(f) > \lambda(e)$ or $R_e = R_f$ and $\lambda(f) < \lambda(e)$. Slivnik ([11]) proved that there exists a dominating matching in any bipartite multigraph with respect to any proper edge coloring. This result is also implicit in [7].

Theorem 10 (Slivnik [11], Galvin [7]). *For any bipartite multigraph G, and any proper edge coloring, $\lambda : E(G) \to \mathcal{N}$, there exists a dominating matching in G with respect to λ.*

We refer the reader to (Slivnik [11]) for details of the proof of this theorem.

Given a multigraph G and a function $\theta : E(G) \to \mathcal{N}$, G is said to be *θ-edge-choosable* if G is $\mathcal{C}$-list-colorable for every family of sets of colors $\mathcal{C} = \{C(e) \colon (e \in E(G))\}$ satisfying $|C(e)| \geq \theta(e)$, for every $e \in E(G)$. Note that G is k-edge choosable if G is θ-edge-choosable for the constant function $\theta(e) = k$ for each edge $e \in E(G)$. Given a bipartite multigraph G and a proper edge coloring, $\lambda : E(G) \to \mathcal{N}$, Slivnik defined the following function $T_{(G,\lambda)} : E(G) \to \mathcal{N}$.

$$\begin{aligned} T_{(G,\lambda)}(e) &= |\{f \in E(G) : L_e = L_f, \lambda(f) > \lambda(e)\}| \\ &\quad + |\{f \in E(G) : R_e = R_f, \lambda(f) < \lambda(e)\}| \end{aligned}$$

Slivnik then used Theorem 10 to show constructively that any bipartite multigraph with proper edge coloring, $\lambda : E(G) \to \mathcal{N}$, is $(T_{(G,\lambda)} + 1)$-edge-choosable. For completeness, we give an outline of Slivnik's proof of this result below.

Theorem 11 (Slivnik [11]). *Any bipartite multigraph G with any proper edge coloring, $\lambda : E(G) \to \mathcal{N}$, is $(T_{(G,\lambda)} + 1)$-edge-choosable, where the function $T_{(G,\lambda)}$ is as defined above.*

Proof. Suppose that we are given any family of sets of colors $\mathcal{C} = \{C(e) \colon e \in E(G)\}$ satisfying $|C(e)| \geq T_{(G,\lambda)}(e) + 1$, for every $e \in E(G)$. Slivnik proves constructively that G is $\mathcal{C}$-list-colorable by iterating the following procedure, $\texttt{SLIVNIK}(G, C, \lambda)$. When edge e is assigned a color from its list $C(e)$, we will say that edge e is *list-colored.*

$\texttt{SLIVNIK}(G, C, \lambda)$:

(1) Select some color $c \in \bigcup\{C(e) : e \in E(G)\}$, and let $E_c = \{e \in E(G) : c \in C(e)\}$.

(2) Find a dominating matching M (guaranteed to exist by Theorem 10) in the bipartite multigraph $G_c = (V(G), E_c)$ with respect to the proper edge coloring

λ. List color the edges in M by the color c.

(3) Replace G by $G \setminus M$, $C(e)$ by $C(e) \setminus \{c\}$, for each edge $e \in E(G)$, and return to (1).

The justification for being able to iterate the above procedure SLIVNIK(G, C, λ) is that since the set of edges in M that get list colored in step (2) form a dominating matching in G with respect to λ, if in step (3), $|C(e)|$ reduces by one for some $e \in E(G)$, then $T_{(G,\lambda)}(e)$ also reduces by one in step (3), and hence the inequality $|C(e)| \geq T_{(G,\lambda)}(e)+1$ for every $e \in E(G)$, continues to hold for the updated bipartite multigraph G, the associated function $T_{(G,\lambda)} : E(G) \to \mathcal{N}$, and the updated family of sets of colors $\{C(e) : e \in E(G)\}$.

Theorem 11 implies Galvin's Theorem 4 as follows. Let $\lambda : E(G) \to \{1, 2, 3, \ldots, \Delta(G)\}$ be a proper edge coloring (guaranteed by Theorem 1) of a bipartite multigraph G. Then, it is easy to see that $T_{(G,\lambda)}(e) + 1 \leq \Delta(G)$, for each $e \in E(G)$. Now, Theorem 11 implies that G is $(T_{(G,\lambda)} + 1)$-edge-choosable, and hence G is $\Delta(G)$-edge-choosable.

We will now use the technique in Slivnik's proof of Galvin's theorem to obtain list coloring analogs of Vizing's theorem ([12]) and Shannon's theorem ([10]) for nearly bipartite multigraphs. Let G be a nearly bipartite multigraph with special vertex v_s, and, let the bipartition of $G - v_s$ be given by $L \bigcup R$. The edges of G naturally partition into $E(G) = E_l \bigcup E_r \bigcup E_b$, where, $E_l = \{(u, v_s) \in E(G) : u \in L\}$, $E_r = \{(u, v_s) \in E(G) : u \in R\}$, and $E_b = \{(u, v) \in E(G) : u \in L, v \in R\}$. We denote by G_l the bipartite subgraph of G with edge set $E(G_l) = E_l \bigcup E_b$, and vertex bipartition $V(G_l) = L \bigcup (R \bigcup \{v_s\})$; we denote by G_r the bipartite subgraph of G with edge set $E(G_r) = E_r \bigcup E_b$, and vertex bipartition $V(G_r) = (L \bigcup \{v_s\}) \bigcup R$; finally we denote by G_b the bipartite multigraph $G - v_s$. We denote by $\mu_G(u, v)$ the number of parallel edges between vertices u and v in the multigraph G. We now prove the following analog of Vizing's theorem ([12]) for nearly bipartite multigraphs. Note that this theorem is stronger than Theorem 8 promised in the Introduction.

Theorem 12 *Let G be a nearly bipartite multigraph with special vertex v_s, and let $L \bigcup R$ be the vertex partition of the bipartite multigraph $G - v_s$. Define m_1 and m_2 by $m_1 = \max\{\mu_G(u, v_s) : u \in L\}$, and $m_2 = \max\{\mu_G(u, v_s) : u \in R\}$. Then, $\chi'_l(G) \leq \Delta(G) + \min\{m_1, m_2\}$.*

Proof. We first show that $\chi'_l(G) \leq \Delta(G) + m_1$, and then note that a similar method of proof will give that $\chi'_l(G) \leq \Delta(G) + m_2$, thus proving the theorem. Suppose that we are given a family of sets of colors $\mathcal{C} = \{C(e) : e \in E(G)\}$ satisfying $|C(e)| \geq \Delta(G) + m_1$ for each edge $e \in E(G)$. We will show that G is $\mathcal{C}$-list-colorable by successively applying Slivnik's procedure to two subgraphs of G.

Since G_r is a bipartite multigraph with $\Delta(G_r) \leq \Delta(G)$, Theorem 1 implies that the edges of G_r can be properly colored in $\Delta(G)$ colors. We can assume (by renaming the colors if necessary) that in this proper edge coloring of G_r, the edges in E_r are assigned colors from the set $\{1, 2, \ldots, |E_r|\}$. Now there is a proper edge coloring,

$\lambda : E(G) \to \{1, 2, \ldots, |E_r|, |E_r| + 1, \ldots, \Delta(G), \Delta(G) + 1, \ldots, \Delta(G) + |E_l|\}$, of G, where for $e \in E_r$, $\lambda(e) \in \{1, 2, \ldots, |E_r|\}$, for $e \in E_b$, $\lambda(e) \in \{1, 2, \ldots, \Delta(G)\}$, and for $e \in E_l$, $\lambda(e) \in \{\Delta(G) + 1, \Delta(G) + 2, \ldots, \Delta(G) + |E_l|\}$. We now intend to invoke the procedure SLIVNIK(G_l, C, λ) with some modifications. Clearly, $T_{(G_r,\lambda)}(e) \leq \Delta(G) - 1$ for each edge $e \in E_b$, so that $T_{(G_l,\lambda)}(e) \leq \Delta(G) - 1 + m_1$ for each edge $e \in E_b$. Also, $T_{(G_r,\lambda)}(e) \leq |E_r| - 1$ for each edge $e \in E_r$. We now run $|E_l|$ iterations of the procedure SLIVNIK(G_l, C, λ) with two modifications: (a) In step (1) we select $c \in \bigcup\{C(e) : e \in E_l,\ e$ not yet list-colored$\}$, and, (b) In step (3) we replace $C(e)$ by $C(e) \setminus \{c\}$ for each edge $e \in E(G_l) \bigcup E_r$. Note that since $\lambda(e) \leq \Delta(G)$ for each $e \in E_b$, and $\lambda(e) \geq \Delta(G) + 1$ for each $e \in E_l$, the dominating matching M found each time step (2) of SLIVNIK(G_l, C, λ) is executed, must contain precisely one edge from E_l. Hence, each edge $e \in E_l$ will be list-colored after precisely $|E_l|$ iterations of SLIVNIK(G_l, C, λ).

Let us denote by E'_b the set of edges in E_b that remain to be list-colored, and by $C'(e)$, the updated list of colors for edge e after the $|E_l|$ iterations of the procedure SLIVNIK(G_l, C, λ) performed above. Now consider the bipartite multigraph G' with vertex set $V(G)$, edge set $E'_b \bigcup E_r$, vertex bipartition $(L \bigcup\{v_s\}) \bigcup R$, and proper edge coloring λ. We now intend to invoke the procedure SLIVNIK(G', C', λ) to obtain a list coloring of the remaining edges of G. We first verify that $T_{(G',\lambda)}(e) \leq |C'(e)| - 1$ for each edge $e \in E(G')$. For $e \in E'_b$, it is clear that $T_{(G',\lambda)}(e) \leq |C'(e)| - 1$, because the matching M found each time step (2) was executed during the $|E_l|$ iterations of SLIVNIK(G_l, C, λ) was a dominating matching in G_l with respect to the proper edge coloring λ. For $e \in E_r$, since in the beginning we had that $|C(e)| \geq \Delta(G) + m_1$, we have that $|C'(e)| > \Delta(G) - |E_l| + m_1 - 1 \geq |E_r| + m_1 - 1 \geq |E_r| - 1 \geq T_{(G',\lambda)}(e)$. Thus, we can now invoke SLIVNIK(G', C', λ) to obtain a $\mathcal{C}$-list-coloring of the remaining edges of G. This proves that $\chi'_l(G) \leq \Delta(G) + m_1$. We note that the above proof with the roles of E_l and E_r interchanged gives that $\chi'_l(G) \leq \Delta(G) + m_2$, thus proving the theorem.

We mention here that the referee has pointed out that the list-coloring analog of Vizing's Theorem for nearly bipartite multigraphs (Theorem 8 in the Introduction) can be derived easily from Theorem 5 in the Introduction as follows: first list color the edges incident with v_s, and then apply Theorem 5 to the bipartite multigraph $G - v_s$ with color lists reduced as needed. In fact, this argument proves a weak form of Theorem 12 above, with $\min\{m_1, m_2\}$ replaced by $\max\{m_1, m_2\}$. Theorem 12 itself, however, does not seem to follow from Theorem 5.

We also point out here that the list coloring analog of Shannon's theorem (Theorem 3 in the Introduction) for nearly bipartite multigraphs can be independently (without recourse to Theorem 6 in the Introduction) obtained as a direct consequence of Theorem 12 since we have that $\min\{m_1, m_2\} \leq \frac{1}{2}\Delta(G)$ for any nearly bipartite graph G, where m_1, and m_2 are as defined in the statement of Theorem 12.

3 The list chromatic index of nearly bipartite multigraphs with special vertex of degree at most six

Let G be a nearly bipartite multigraph that is k-edge-colorable with special vertex v_s of degree $d \le k$, with the vertex bipartition of $G - v_s$ given by $L \bigcup R$, and with E_l, E_r, and E_b as defined before. We prove that if $\mathcal{C} = \{C(e) : e \in E(G)\}$ is a family of sets of colors such that $\bigcup\{C(e) : e \in E_l \bigcup E_r\}$ can be partitioned in a certain way, then G is $\mathcal{C}$-list-colorable.

Theorem 13 *Let G be a nearly bipartite multigraph that is k-edge-colorable, with special vertex v_s of degree $d \le k$, and let $\mathcal{C} = \{C(e) \colon e \in E(G)\}$ be any family of sets of colors satisfying $|C(e)| \ge d$ for each $e \in E_l \bigcup E_r$, and $|C(e)| \ge k$ for each $e \in E_b$. Let $A = \bigcup\{C(e) : e \in E_l \bigcup E_r\}$. If A can be partitioned into $A = A_l \bigcup A_r$ such that $|A_l \bigcap C(e)| \ge |E_l|$ for each $e \in E_l$, and $|A_r \bigcap C(e)| \ge |E_r|$ for each $e \in E_r$, then G is $\mathcal{C}$-list-colorable.*

Proof. We split the special vertex v_s into two vertices v_l and v_r and construct a bipartite multigraph G' with vertex set $V(G') = V(G_b) \bigcup \{v_l, v_r\}$ and edge set $E(G') = E_b \bigcup E'_l \bigcup E'_r$, where $E'_l = \{(u, v_r) : u \in L, (u, v_s) \in E_l\}$, and, $E'_r = \{(u, v_l) : u \in R, (u, v_s) \in E_r\}$. Consider a proper k-edge-coloring $\lambda : E(G) \to \{1, 2, \ldots, k\}$ of G, and assume (by renaming the colors if necessary) that for $e \in E_r$, $\lambda(e) \in \{1, 2, \ldots, |E_r|\}$, and that for $e \in E_l$, $\lambda(e) \in \{k - |E_l| + 1, k - |E_l| + 2, \ldots, k - 1, k\}$.

Now consider the proper k-edge-coloring $\lambda' : E(G') \to \{1, 2, \ldots, k\}$ of G', in which, for $e \in E_b$, $\lambda'(e) = \lambda(e)$; for $e' = (u, v_r) \in E'_l$, $\lambda'(e') = \lambda(e)$, where $e = (u, v_s) \in E_l$, and finally for $e' = (u, v_l) \in E'_r$, $\lambda'(e') = \lambda(e)$, where $e = (u, v_s) \in E_r$. Suppose that we are given any family of sets of colors $\{C(e) \colon e \in E(G)\}$ satisfying $|C(e)| \ge d$ for each $e \in E_l \bigcup E_r$, and $|C(e)| \ge k$ for each $e \in E_b$. Let $A = \bigcup\{C(e) : e \in E_l \bigcup E_r\}$. Also suppose that as in the statement of the theorem, A can be partitioned into $A = A_l \bigcup A_r$ such that $|A_l \bigcap C(e)| \ge |E_l|$ for each $e \in E_l$, and $|A_r \bigcap C(e)| \ge |E_r|$ for each $e \in E_r$. Define a family of sets of colors $\mathcal{C}' = \{C'(e) \colon e \in E(G')\}$ as follows: for $e \in E_b$, $C'(e) = C(e)$, for $e' = (u, v_r) \in E'_l$, $C'(e') = C(e) \bigcap A_l$, where $e = (u, v_s) \in E_l$, and for $e' = (u, v_l) \in E'_r$, $C'(e') = C(e) \bigcap A_r$, where $e = (u, v_s) \in E_r$. Note that since $C'(e_1) \bigcap C'(e_2) = \emptyset$ for each $e_1 \in E'_l$ and each $e_2 \in E'_r$, a $\mathcal{C}'$-list-coloring of the edges of G' naturally translates into a $\mathcal{C}$-list-coloring of the edges of G.

We complete the proof by showing that G' is $\mathcal{C}'$-list-colorable. We first verify that $T_{(G',\lambda')}(e) \le |C'(e)| - 1$ for each $e \in E(G')$. For each $e \in E_b$, we have $T_{(G',\lambda')}(e) \le k - 1 \le |C(e)| - 1 = |C'(e)| - 1$. For each $e' \in E'_l$, we have $T_{(G',\lambda')}(e') \le |E'_l| - 1 \le |C'(e)| - 1$. For each $e' \in E'_r$, we have $T_{(G',\lambda')}(e') \le |E'_r| - 1 \le |C'(e)| - 1$. We can now invoke the procedure $\texttt{SLIVNIK}(G', \mathcal{C}', \lambda')$ to obtain a $\mathcal{C}'$-list-coloring of the edges of G' which as noted earlier naturally translates into a $\mathcal{C}$-list-coloring of the edges of G.

We note here that the following example given by Bryant ([3]) shows that for $A = \bigcup\{C(e) : e \in E_l \bigcup E_r\}$, A cannot always be partitioned into $A = A_l \bigcup A_r$ such that $|A_l \bigcap C(e)| \geq |E_l|$ for each $e \in E_l$, and $|A_r \bigcap C(e)| \geq |E_r|$ for each $e \in E_r$. Let $d = 6$ with $E_l = \{e_1, e_2, e_3\}$, $E_r = \{e_4, e_5, e_6\}$, and $C(e_1) = C(e_4) = \{1,2,3,4,5,6\}$, $C(e_2) = C(e_5) = \{1,2,3,7,8,9\}$, $C(e_3) = C(e_6) = \{4,5,6,7,8,9\}$. We have $A = \{1,2,3,4,5,6,7,8,9\}$, and it is easy to see that in order to satisfy $|A_l \bigcap C(e)| \geq 3$ for each $e \in E_l$, A_l must contain at least 5 elements from A. If A_l contains at least 5 elements of A then A_r must contain at most 4 elements from A, and the condition $|A_r \bigcap C(e)| \geq |E_r|$ for each $e \in E_r$ cannot be satisfied.

We now prove Theorem 9 promised in the introduction. We will follow the terminology for nearly bipartite multigraphs established immediately before Theorem 12 in section 2. We first prove a Lemma that will be repeatedly used in the proof of Theorem 9.

Lemma 1 *Let G be any k-edge-colorable nearly bipartite multigraph with special vertex v_s of degree d, and E_l, E_r, E_b, and G_l as defined in section 2. Let $\lambda : E(G) \to \{1,2,\ldots,k\}$ be a proper k-edge-coloring of G, and assume without loss of generality that for $e \in E_r, \lambda(e) \in \{1,2,\ldots,|E_r|-1,|E_r|\}$, and for $e \in E_l, \lambda(e) \in \{k-|E_l|+1, k-|E_l|+2,\ldots,k-1,k\}$. Let $\mathcal{C}_0 = \{C_0(e) : e \in E(G)\}$ be any family of sets of colors satisfying $|C_0(e)| \geq d$ for $e \in E_l \bigcup E_r$, and $|C_0(e)| \geq k (\geq d)$ for $e \in E_b$. Let $C_u \subseteq \bigcup\{C_0(e) : e \in E(G_l)\}$ be the set of colors used in list coloring some subset of the edges in E_l by running an appropriate number of iterations of* SLIVNIK*(G_l, C, λ), where initially, $C(e) = C_0(e)$ for each $e \in E(G)$. For each $e \in E_r$, replace $C(e)$ by $C(e) \setminus C_u$.*

(i) *If $e \in E_l$ is an edge that is not yet list colored, then the updated set of colors $C(e)$ for edge e must satisfy $|C(e)| \geq d - |E_l| + 1$.*

(ii) *If all edges in E_l have been list colored, and, if the updated sets of colors $\{C(e) : e \in E_r\}$ satisfy $|C(e)| \geq |E_r|$ for each $e \in E_r$, then G is $\mathcal{C}_0$-list-colorable.*

Proof.

(i) This follows immediately because we have that initially $|C(e)| \geq d$, and, $T_{(G_l,\lambda)}(e) \leq |E_l| - 1$ for each $e \in E_l$.

(ii) Denote by E'_b the set of edges in E_b that remain to be list-colored after an appropriate number of iterations of SLIVNIK(G_l, C, λ) that result in a list coloring of the edges in E_l. Now consider the bipartite multigraph G' with vertex set $V(G)$, edge set $E'_b \bigcup E_r$, vertex bipartition $(L \bigcup \{v_s\}) \bigcup R$, and proper edge coloring λ. For $e \in E'_b$, it is clear that $T_{(G',\lambda)}(e) \leq |C(e)| - 1$, because the matching M found each time step (2) is executed during SLIVNIK(G_l, C, λ) is a dominating matching in G_l with respect to the proper edge coloring λ. For $e \in E_r$, we have that $T_{(G',\lambda)}(e) \leq |E_r| - 1 \leq |C(e)| - 1$. Hence, $T_{(G',\lambda)}(e) \leq |C(e)| - 1$ for each $e \in E(G')$. Now, invoking the procedure SLIVNIK(G', C, λ) yields a $\mathcal{C}_0$-list-coloring of the remaining edges of G.

Before proving Theorem 9, we mention here that the referee has pointed out a recent result by Woodall ([13]) that proves the LCC (Conjecture 1 in the Introduction) for 'multicircuits' (multigraphs whose underlying simple graphs are cycles). Woodall ([13]) also observes in his paper that the underlying simple graphs of multicircuits have the form 'bipartite plus one edge', and asks for an extension of his result to multigraphs having this form. Lemma 1 above immediately yields a proof of the LCC for multigraphs whose underlying simple graphs have the form 'bipartite plus one edge' as follows in Theorem 14 below.

Theorem 14 *Let G be a multigraph that contains two vertices, u and v, such that deleting all edges between u and v results in a bipartite multigraph. If G is k-edge-colorable, then G is k-edge-choosable.*

Proof. We can view G as a nearly bipartite graph with special vertex $v_s = v$, edge set E_l consisting of all edges between vertices u and v, and, edge sets E_r and E_b given by $E_r = \{e : e = (v, w) \in E(G), w \neq u\}$, and, $E_b = E(G) \backslash (E_l \bigcup E_r)$. As in the statement of Lemma 1, let $\lambda : E(G) \to \{1, 2, \ldots, k\}$ be a proper k-edge-coloring of G, and assume without loss of generality that for $e \in E_r, \lambda(e) \in \{1, 2, \ldots, |E_r|-1, |E_r|\}$, and for $e \in E_l, \lambda(e) \in \{k-|E_l|+1, k-|E_l|+2, \ldots, k-1, k\}$. Let $\mathcal{C}_0 = \{C_0(e) : e \in E(G)\}$ be any family of sets of colors satisfying $|C_0(e)| \geq k$ for each $e \in E(G)$. We run $|E_l|$ iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (where initially, $C(e) = C_0(e)$ for each $e \in E(G)$) with two modifications: (a) In step (1) we select $c \in \bigcup\{C(e) : e \in E_l,\ e$ not yet list-colored$\}$, and, (b) In step (3) we replace $C(e)$ by $C(e) \setminus \{c\}$ for each edge $e \in E(G_l) \bigcup E_r$. It is clear that after these $|E_l|$ iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$, each edge $e \in E_l$ is list colored, and that the updated sets of colors $\{C(e) : e \in E_r\}$ satisfy $|C(e)| \geq k - |E_l| \geq |E_r|$. Hence, part (ii) of Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable.

Theorem 9 *Let G be any nearly bipartite multigraph with special vertex v_s of degree $d \leq 6$. If G is k-edge-colorable, then G is k-edge-choosable.*

Proof. We will prove the stronger result that if $\mathcal{C}_0 = \{C_0(e) : e \in E(G)\}$ is any family of sets of colors satisfying $|C_0(e)| \geq d$ for $e \in E_l \bigcup E_r$, and $|C_0(e)| \geq k (\geq d)$ for $e \in E_b$, then G is $\mathcal{C}_0$-list-colorable. In what follows, we give details of the proof in the case when $d = 6$, and point out that the cases when $d < 6$ can be similarly (but more easily) handled.

Suppose that $d = 6$. Let $\mathcal{C}_0 = \{C_0(e) : e \in E(G)\}$ be any family of sets of colors satisfying $|C_0(e)| \geq 6$ for $e \in E_l \bigcup E_r$, and $|C_0(e)| \geq k (\geq 6)$ for $e \in E_b$. Label the edges of G so that $E_r = \{e_i : i = 1, 2, \ldots, |E_r| - 1, |E_r|\}$, and, $E_l = \{e_i : i = k - |E_l| + 1, k - |E_l| + 2, \ldots, k - 1, k\}$. Let $\lambda : E(G) \to \{1, 2, \ldots, k - 1, k\}$ be a proper k-edge-coloring of G, and assume (by renaming the colors if necessary) that $\lambda(e_i) = i$ for each $e_i \in E_l \bigcup E_r$. Since the roles of E_l and E_r are interchangeable, the following cases exhaust all possibilities. In each case below, we will run an appropriate number of iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with $C(e) = C_0(e)$ initially for each edge $e \in E(G)$, and, careful choices of the color c in step (1)) to obtain a list coloring of the edges in E_l. We then let $C_u \subseteq \{C(e) : e \in E(G_l)\}$ be the set of colors used in list coloring the edges in E_l by this process, and for each $e \in E_r$, we replace $C(e)$ by $C(e) \setminus C_u$. In each case below, we will ensure that the choices of the color c in step (1) of SLIVNIK(G_l, C, λ) are such that the updated set of colors $C(e)$ satisfy $|C(e)| \geq 6 - |E_l|$ for each $e \in E_r$. Then, Lemma 1 will imply that G is $\mathcal{C}_0$-list-colorable.

Case (a): $|E_l| = 0, |E_r| = 6$. In this case, G is bipartite and Galvin's Theorem 4 and Theorem 1 imply that $\chi_l'(G) = \chi'(G) = \Delta(G) \leq k$.

Case (b): $|E_l| = 1, |E_r| = 5$. We run one iteration of SLIVNIK(G_l, C, λ) with the modification that in step (1) we select $c \in C(e_k)$. Because $\lambda(e_k) = k$, it is clear that e_k will be in the dominating matching found in Step (2) of SLIVNIK(G_l, C, λ). Clearly, the updated set of colors $C(e)$ satisfies $|C(e)| \geq 5$ for each edge $e \in E_r$, and, Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable.

Case (c): $|E_l| = 2, |E_r| = 4$. Suppose first that $C_0(e_k) \bigcap C_0(e_{k-1}) \neq \emptyset$. We run one iteration of SLIVNIK(G_l, C, λ) with the modification that in step (1) we select $c \in C(e_k) \bigcap C(e_{k-1})$. Since $\lambda(e_k) = k$, and $\lambda(e_{k-1}) = k - 1$, it is clear that exactly one of the edges e_k and e_{k-1} must be in the dominating matching, M found in step (2) of this first iteration of SLIVNIK(G_l, C, λ).

If $e_{k-1} \in M$, we now run another (second) iteration of SLIVNIK(G_l, C, λ) (with updated G_l and C) with the modification that in step (1) we select $c \in C(e_k)$. It is clear that the edge e_k must be in the dominating matching found in step (2) of this second iteration of SLIVNIK(G_l, C, λ). Clearly, the updated set of colors, $C(e)$ satisfies $|C(e)| \geq 4$ for each $e \in E_r$, and, Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable.

If e_k is in the dominating matching M found in the first iteration of SLIVNIK(G_l, C, λ), then there must be an edge $e^* \in M$, with $\lambda(e^*) = k$, and $L_{e_{k-1}} = L_{e^*}$. We now run another (second) iteration of SLIVNIK(G_l, C, λ) (with updated G_l and C) with the modification that in step (1) we select $c \in C(e_{k-1})$. Note that since e^* was in the matching M found in the first iteration of SLIVNIK(G_l, C, λ), and since $\lambda(e^*) = k$, after the first iteration of SLIVNIK(G_l, C, λ), there does not exist $e \in E(G_l)$ with $L_e = L_{e_{k-1}}$ and $\lambda(e) = k$. Hence, $\lambda(e_{k-1}) = k - 1$ implies that the dominating matching found in step (2) of this second iteration of SLIVNIK(G_l, C, λ) must contain the edge e_{k-1}. As before, Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable.

Now, suppose that $C_0(e_k) \bigcap C_0(e_{k-1}) = \emptyset$. We run one iteration of SLIVNIK(G_l, C, λ) with the modifications that in step (1) we select $c = c_1 \in C(e_k)$. It is clear that e_k must be in the dominating matching M found in step (2) of this first iteration of SLIVNIK(G_l, C, λ). We now run another (second) iteration of SLIVNIK(G_l, C, λ) with the modification that in step (1) we select $c = c_2 \in C(e_{k-1})$. If e_{k-1} is in the dominating matching M found in step (2) of this second iteration of SLIVNIK(G_l, C, λ), then Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable. So, suppose now that $e_{k-1} \notin M$. Since $\lambda(e_{k-1}) = k - 1$, there must exist $e^* \in M$ with $L_{e_{k-1}} = L_{e^*}$, and $\lambda(e^*) = k$. We now run another (third) iteration of SLIVNIK(G_l, C, λ) (with updated G_l and C) with the modification that in step (1) we select $c = c_3 \in C(e_{k-1})$. It is clear that e_{k-1}

must be in the dominating matching M found in step (2) of this third iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$. Let G' be the bipartite multigraph with edge set $(E(G_l) \bigcup E_r) \backslash M$ (with G_l updated). At this point, we have used up the set of colors $C_u = \{c_1, c_2, c_3\}$ with $c_1 \in C_0(e_k), c_2 \in C_0(e_{k-1})$, and $c_3 \in C_0(e_{k-1})$ in order to list color the edges e_k and e_{k-1}. Since $|C_0(e_k)| \geq 6$, and $|C_0(e_{k-1})| \geq 6$, the number of ways in which C_u can be chosen is at least $6\binom{6}{2} = 90$. A 6-set which has k elements in common with $C_0(e_k)$ has at most $6 - k$ elements in common with $C_0(e_{k-1})$, and, hence the number of ways of choosing C_u such that the updated set of colors $C(e)$ satisfies $|C(e)| \leq 3$ for some $e \in E_r$ is at most $\max_i 4i\binom{6-i}{2} = 48$. Hence any of the other 42 ways of choosing C_u gives that $|C(e)| \geq 4$ for each $e \in E_r$, and Lemma 1 implies that $\mathcal{C}_0$-list-colorable.

Case (d): $|E_l| = 3, |E_r| = 3$.

Subcase (d1): $C_0(e_k) \bigcap C_0(e_{k-1}) \bigcap C_0(e_{k-2}) \neq \emptyset$. We run one iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with $C(e) = C_0(e)$ for each $e \in E(G)$ initially) with the modification that in step (1) we select $c \in C(e_k) \bigcap C(e_{k-1}) \bigcap C(e_{k-2})$. Since $\lambda(e_k) = k, \lambda(e_{k-1}) = k - 1$, and $\lambda(e_{k-2}) = k - 2$, it is clear that exactly one of the edges e_k, e_{k-1} and e_{k-2} must be in the dominating matching found in step (2) of $\texttt{SLIVNIK}(G_l, C, \lambda)$. Now, a proof technique similar to the one in Case (c) above proves that G is $\mathcal{C}_0$-list-colorable.

Subcase (d2):
$C_0(e) \bigcap C_0(f) = \emptyset$ for each $e, f \in E_l, e \neq f$, or $C_0(e) \bigcap C_0(f) = \emptyset$ for each $e, f \in E_r, e \neq f$. Without loss of generality assume that $C_0(e) \bigcap C_0(f) = \emptyset$ for each $e, f \in E_r, e \neq f$. We run three iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with $C(e) = C_0(e)$ for each edge $e \in E(G)$ initially) with the modification that at each iteration in step (1) we choose c to be in the current set of colors of the edge in E_l that has the highest index among those edges in E_l that have not yet been list colored. It is easy to verify that after these three iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$, at most one edge in E_l remains to be list colored; if all the edges in E_l get list colored, then Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable. Suppose that edge $e^* \in E_l$ is not yet list colored after these three iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$. At this point, Lemma 1 implies that the updated set of colors $C(e^*)$ has cardinality at least four. We now run an appropriate number $p (\leq 3)$ of iterations (till e^* gets list colored) of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with updated G_l and C) with the modification that we choose $c \in C(e^*)$ each time. Note that in this final iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$, e^* will be in the dominating matching found in step (2), and that we have four choices for the color c in step (1). Since $C_0(e) \bigcap C_0(f) = \emptyset$ for each $e, f \in E_r, e \neq f$, in each of these last p iterations at least one choice of $c \in C(e^*)$ is such that the updated set of colors $C(e)$ satisfies $|C(e)| \geq 3$ for each $e \in E_r$. Now, Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable.

Subcase (d3):
Neither Subcase (d1) nor Subcase (d2) holds, but $|C_0(e) \bigcap C_0(f)| \geq 2$ for some $e, f \in E_l, e \neq f$, or $|C_0(e) \bigcap C_0(f)| \geq 2$ for some $e, f \in E_r, e \neq f$. Assume without

loss of generality that $a, b \in C_0(e_{k-2}) \bigcap C_0(e_{k-1})$, and, $a \neq b$. Note that since we are not in Subcase (d1), $a \notin C_0(e_k)$, and, $b \notin C_0(e_k)$. We run one iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with $C(e) = C_0(e)$ for each $e \in E(G)$ initially) with the modification that in step (1) we choose $c = a$. We then run a second iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with updated G_l and C) with the modification that in step (1) we choose $c = b$. If both edges e_{k-2} and e_{k-1} get list colored in these two iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$, then a third iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) we choose $c \in C(e_k)$ will result in e_k being list colored, and then, Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable. So, suppose that one of the edges e_{k-2} and e_{k-1}, call it e^*, is not yet list colored after the first two iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$. Note that the updated set of colors for each edge in E_r has cardinality at least four. If the updated sets of colors $C(e)$ satisfy $|C(e)| \geq 5$ for each $e \in E_r$, then a third iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) we choose $c \in C(e^*)$ will result in e^* being list colored, and then, a fourth iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) we choose $c \in C(e_k)$ will result in e_k being list colored. Lemma 1 then implies that G is $\mathcal{C}_0$-list-colorable. Hence, suppose that $|C(e)| = 4$ for some $e \in E_r$. Note that since we are not in Subcase (d1), there exist at most two edges in E_r whose updated color sets have cardinality four; suppose without loss of generality that these edges are e_1 and e_2.

After the first two iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$, suppose that there exists $c^* \in C(e^*)$ such that $|C(e_i) \setminus c^*| \geq 4$ for $i = 1$, or $i = 2$. In this case, we run a third iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) we choose $c = c^*$, and this results in edge e^* being list colored. Note that at this point, the updated set of colors $C(e_k)$ has cardinality at least five, and hence, there exists $c' \in C(e_k)$ such that the updated sets of colors $C(e)$ satisfy $|C(e) \setminus c'| \geq 3$ for each $e \in E_r$. We run a fourth iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) we choose $c = c'$. This results in edge e_k being list colored, and, Lemma 1 implies that G is $\mathcal{C}_0$-list-colorable.

On the other hand, suppose that after the first two iterations of $\texttt{SLIVNIK}(G_l, C, \lambda)$, for each $c \in C(e^*)$, we have that $|C(e_i) \setminus c| = 3$ for $i = 1$ and $i = 2$. This implies that $C_0(e^*) \supseteq C_0(e_1) = C_0(e_2)$, and hence that there exists $c'' \in C_0(e_k)$ such that $c'' \notin C_0(e_i)$, for $i = 1, 2$. Now, a third iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) $c \in C(e^*)$, and a fourth iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ with the modification that in step (1) $c = c''$ results in edge e_k being list colored. Lemma 1 now implies that G is $\mathcal{C}_0$-list-colorable.

Subcase (d4):

None of Subcases (d1), (d2), and (d3) holds. Since we are not in Subcase (d2), we can assume that there exists $a \in C_0(e_{k-1}) \bigcap C_0(e_k)$. We run one iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ (with $C(e) = C_0(e)$ initially for each $e \in E(G)$) with the modification that we choose $c = a$ in step (1), and a second iteration of $\texttt{SLIVNIK}(G_l, C, \lambda)$ choosing a color that is not in $C(e_{k-2})$ but is in the updated list of colors of the one edge from $\{e_{k-1}, e_k\}$ that is not yet list colored. After these two iterations, both e_{k-1} and e_k are list colored, and, $C(e_{k-2})$ still has cardinality 6.

We now need to perform at most three more iterations (iterations 3, 4, and, 5) of $\texttt{SLIVNIK}(G_l, C, \lambda)$ in order to list color e_{k-2} if we choose $c \in C(e_{k-2})$ at each iteration. We wish to do so, and still keep $|C(f)| \geq 3$ for each $f \in E_r$, because then the result will follow from Lemma 1, part (ii). To see that this is possible, note that because of the minimal set overlap that results from our not being in Subcase (d1) or (d3), after iteration 3 or 4, the cardinality of at most one of the sets $C(e_1), C(e_2)$, and, $C(e_3)$ can be reduced to three. If one of these sets of colors is reduced to cardinality 3, we simply avoid picking those 3 colors when we choose c at iterations 4 and 5. The result now follows.

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