Edge-Face Chromatic Number of Plane Graphs with High Maximum Degree*

Wang Weifan Zhang Kemin

Department of Mathematics, Nanjing University,
Nanjing, 210093 P.R.China

Abstract

The edge-face chromatic number $\chi_{ef}(G)$ of a plane graph G is the smallest number of colors assigned to the edges and faces of G so that any two adjacent or incident elements have different colors. Borodin(1994) proved that $\Delta(G) \leq \chi_{ef}(G) \leq \Delta(G)+1$ for each plane graph G with $\Delta(G) \geq 10$ and the bounds are sharp. The main result of this paper is to give a sufficient and necessary condition for $\chi_{ef}(G) = \Delta(G)+1$ if $\Delta(G) \geq |G|-2$.

1 Introduction

Throughout this paper, all graphs are finite simple plane graphs. Let G be a plane graph, whose vertex set, edge set, face set, vertex number, edge number, maximum degree and minimum degree of vertices are denoted by $V(G)$, $E(G)$, $F(G)$, $p(G)$, $q(G)$, $\Delta(G)$ and $\delta(G)$ respectively. Let $G[S]$ denote the induced subgraph of G on $S \subseteq V(G)$, and $N_G(u)$ the neighboor set of a vertex u in G. Moreover set $N_G^c(u) = V(G) - (N_G(u) \cup \{u\})$. A vertex (or face) of degree k is said to be a k-vertex (or k-face) of G. A n-face f whose boundary, denoted by $b(f)$, contains the vertices $u_1, u_2, \cdots, u_n$ in some order is written as $f = u_1u_2\cdots u_n$. Let $V_k(G)$ $(k = 0, 1, \cdots, \Delta = \Delta(G))$ denote the set of k-vertices of G. If C_k is a cycle of length k in a connected plane graph G, then let $V_{int}(C_k)$ and $V_{ext}(C_k)$ denote the sets of vertices in G contained in the interior and exterior of C_k respectively. We say that C_k is a k-separating cycle of G if $V_{int}(C_k) \neq \emptyset$ and $V_{ext}(C_k) \neq \emptyset$. In particular, C_3 is called a separating triangle. A graph G is called an h_k-graph if $\Delta(G) = p(G) - k$, $k = 1, 2, \cdots$.

A plane graph G is k-edge-face colorable if the elements of $E(G) \cup F(G)$ can be colored with k colors so that any two distinct adjacent or incident elements receive different colors. The edge-face chromatic number $\chi_{ef}(G)$ is defined as the minimum number k for which G is k-edge-face colorable.

*The project supported by NSFC(No.19471037) and NSFJC.

Clearly, $\chi_{ef}(G) \geq \Delta(G)$. On the other hand, Melnikov [4] conjectured that $\chi_{ef}(G) \leq \Delta(G) + 3$. Without using the Four-Color Theorem, this conjecture was proved for $\Delta \leq 3$ [3, 5] and for $\Delta = 4$ [6]. Borodin [2] showed that $\chi_{ef}(G) \leq \Delta(G)+1$ for $\Delta(G) \geq 10$ and the bound is sharp. Recently, using the Four-Color Theorem and Vizing's Theorem, Waller [7] proved the conjecture to be true for all plane graphs. Thus the main problem in this area is to determine the precise bounds of $\chi_{ef}(G)$ for $3 \leq \Delta(G) \leq 9$ or to give a complete classification of plane graphs according to their edge-face chromatic numbers. In this paper, we present a necessary and sufficient condition for $\chi_{ef}(G) = \Delta(G) + 1$ if $\Delta(G) \geq |V(G)| - 2$ and $p(G) \geq 7$.

In what follows, a k-edge-face coloring of a plane graph G is abbreviated to a k-EF coloring. Let $\sigma(x)$ denote the color assigned to the element $x \in E(G) \cup F(G)$ under a given coloring σ, and for $u \in V(G)$, let $C_\sigma(u)$ denote the set of colors which are colored on the edges incident with u under σ. For $S \subseteq \cup E(G) \cup F(G)$, we write $S \to \alpha$ to express that all the elements of S are simultaneously colored with the color α. And $S[m]$ denotes that at most m colors can not be used when coloring all the elements of S with the same color. In particular, $y[m] = S[m]$ if $S = \{y\}$. Other terms and notations not defined in this paper can be found in [1]

2 Preliminary

Lemma 2.1 *If G is an h_k-graph with $p(G) \geq 3k+3$ ($k \geq 1$), then $|V_\Delta(G)| \leq 2$.*

Proof By contradiction. Suppose that $|V_\Delta(G)| \geq 3$. Then there are $u_1, u_2, u_3 \in V_\Delta(G)$ such that $d_G(u_i) = p(G) - k$, $i = 1, 2, 3$. Thus

$$|N_G^c(u_i)| = p(G) - 1 - d_G(u_i) = k - 1.$$

Then we have

$$|N_G^c(u_1)| + |N_G^c(u_2)| + |N_G^c(u_3)| = 3k - 3.$$

However, by $p(G) \geq 3k+3$, we deduce

$$|V(G) - ((\bigcup_{i=1}^{3} N_G^c(u_i)) \cup \{u_1, u_2, u_3\})|$$

$$\geq |V(G)| - |(\bigcup_{i=1}^{3} N_G^c(u_i)) \cup \{u_1, u_2, u_3\}|$$

$$\geq p(G) - (3k-3) - 3 \geq 3k + 3 - 3k = 3.$$

This implies that u_1, u_2 and u_3 are simultaneously adjacent to at least three vertices, say v_1, v_2 and v_3, of G. It follows that

$$K_{3,3} \subseteq G[\{u_1, u_2, u_3, v_1, v_2, v_3\}],$$

which contradicts the planarity of G.□

Corollary 2.2 *Let G be an h_k-graph of order p. Then*

(1) $|V_{p-1}(G)| \le 2$ *if* $k=1$ *and* $p(G) \ge 6$.

(2) $|V_{p-2}(G)| \le 2$ *if* $k=2$ *and* $p(G) \ge 9$.

Lemma 2.3 *Let G be an h_1-graph with $p(G) \ge 3$ which contains two Δ-vertices w_1 and w_2. Then*

(1) $2 \le d_G(u) \le 4$ *for each* $u \in V(G) \setminus \{w_1, w_2\}$.

(2) $3 \le d_G(f) \le 4$ *for each* $f \in F(G)$.

Proof Obvious.

For $i \ge 1$, an h_i-graph is said to be an h_i^*-graph if there are a vertex $u \in V_\Delta(G)$ and a face $f \in F(G)$ such that all the edges incident to u lie on the boundary of f. Let x be a vertex of a connected graph G, and let the components of $G-x$ have vertex sets $V_1, V_2, \cdots, V_n$ $(n \ge 1)$. Then the induced subgraphs $G_i = G[V_i \cup \{x\}]$, $i=1,2,\cdots,m$, are called the x-components of G.

Lemma 2.4 *Let G be an h_1-graph with $p(G) \ge 2$ and let w be a Δ-vertex of G. Then G is an h_1^*-graph iff (1) each w-component of G is either K_3 or K_2; and (2) G does not contain a separating triangle.*

Lemma 2.5 *Let G be an h_2-graph with $p(G) \ge 5$ and a unique Δ-vertex w and let $N_G^c(w) = \{x\}$. Then G is an h_2^*-graph iff (1) G contains no separating cycle through x and w, and (2) $G-x$ is an h_1^*-graph.*

It is not difficult to prove the above two lemmas. In fact, an h_1^*-graph is an outerplane graph and an h_2^*-graph is a 1-outerplane graph (i.e. after removing at most one vertex it becomes an outerplane graph).

Lemma 2.6 *Let G be a h_2-graph with $p(G) \ge 8$ and a unique Δ-vertex w. Let $N_G^c(w) = \{x\}$ with $d_G(x) \ge 2$. Then at least one of the following cases is true for G:*

(1) *There is a 1-vertex u adjacent to w.*

(2) *There is a 2-verex u on a 3-face uwy.*

(3) *There is a 3-vertex u with $N_G(u) = \{w, v_1, v_2\}$ such that $uwv_1, uwv_2 \in F(G)$.*

Proof Let G be an h_2-graph satisfying the conditions of the lemma. Suppose that the vertices of $N_G(w)$ are put in the order $u_1, u_2, \cdots, u_m$, where $m = d_G(w) = \Delta(G) = p-2$. By $wx \notin E(G)$, we have $N_G(x) \subseteq N_G(w)$. Since G has a unique Δ-vertex, it follows that $N_G(x) \ne N_G(w)$ and hence $N_G(w) \setminus N_G(x) \ne \emptyset$. Then $N_G(x)$ partitions $N_G(w) \setminus N_G(x)$ into n nonempty maximal subsets $S_1, S_2, \cdots, S_n$, where $1 \le n \le d_G(w) - d_G(x)$. Since $S_1 \ne \emptyset$ and $S_1 \subseteq N_G(w) \setminus N_G(x)$, we let that $S_1 = \{u_{j+1}, u_{j+2}, \cdots, u_{j+t}\}$, where $t = |S_1| \ge 1$ and the suffixes are taken modulo m. From the maximality of S_i, it follows $u_j, u_{j+t+1} \in N_G(x)$. This implies that the interior of the 4-cycle $xu_jwu_{j+t+1}x$ does not contain the vertices in $N_G(x)$ and the edges incident to x. If there is no separating triangle inside $xu_jwu_{j+t+1}x$, then (1) holds when some $u_k \in S_1$ has degree one, (2) holds when some $u_k \in S_1$ has degree two and (3) occurs when all the vertices of S_1 have degree three. Otherwise let $C = wu_{j+s}u_{j+l}w$ be a separating triangle inside $xu_jwu_{j+t+1}x$ with as few vertices in

$V_{int}(C)$ as possible, where $u_{j+s}, u_{j+l} \in \{u_j, u_{j+1}, \cdots, u_{j+t+1}\}$, $2 \le l-s \le t$. Observing the internal vertices $u_{j+s+1}, u_{j+s+2}, \cdots, u_{j+l-1}$ of C, we can similarly get (1), (2) or (3). The lemma is proved.□

Lemma 2.7 *Let G be an h_1-graph with $p(G) \ge 2$ and let w be a Δ-vertex of G. Then at least one of the following cases is true for G:*

(1) $\delta(G) = 1$.

(2) *There is a 2-vertex u on a 3-face uwy.*

(3) *There is a 3-vertex u with $N_G(u) = \{w, v_1, v_2\}$ such that $uwv_1, uwv_2 \in F(G)$.*

The proof is similar to that of Lemma 2.6. In order to prove the following theorem, we introduce two notations. Let G be an h_1-graph with a unique Δ-vertex w. We denote by $E_{in}^w(G)$ the set of inner edges in G incident to w and let $m_w(G) = |E_{in}^w(G)|$. An edge is called an inner edge if it does not lie on the boundary of the unbounded face of G. Obviously, $E_{in}^w(G) \subseteq E_{in}(G)$, and G is not an h_1^*-graph iff $m_w(G) \ge 1$.

Lemma 2.8 *Let G be an h_1-graph with $p(G) \ge 7$ and w a Δ-vertex of G. If G is not an h_1^*-graph, then at least one of the following cases is true for G.*

(1) *There is a 1-vertex u adjacent to w such that $H_1 = G - u$ is not an h_1^*-graph.*

(2) *There is a 2-vertex u on a 3-face uwy such that $H_2 = G - u$ is not an h_1^*-graph.*

(3) *There is a 3-vertex u with $N_G(u) = \{w, v_1, v_2\}$ and $uwv_1, uwv_2 \in F(G)$ such that H_3 is not an h_1^*-graph, where $H_3 = G - u$ if $v_1v_2 \in E(G)$ and $H_3 = G - u + v_1v_2$ otherwise.*

Proof Let G be an h_1-graph with $p(G) \ge 7$ and not an h_1^*-graph. By Corollary 2.2, $1 \le |V_\Delta(G)| \le 2$. We consider two cases below:

Case 1 $|V_\Delta(G)| = 2$. Suppose that $V_\Delta(G) = \{w_1, w_2\}$. Then $w_1w_2 \in E(G)$ and $uw_1, uw_2 \in E(G)$ for each $u \in V(G) \setminus \{w_1, w_2\}$. Let v_1, v_2, $\cdots$, v_k denote the vertices in $V(G) \setminus \{w_1, w_2\}$ which are arranged on one side of the edge w_1w_2 such that the 3-cycle $w_1w_2v_jw_1$ is contained in all 3-cycles $w_1w_2v_sw_1$, $j+1 \le s \le k$, $j = 1, 2, \cdots, k-1$, and symmetrically y_1, y_2, $\cdots$, y_m on the other side of w_1w_2 such that the 3-cycle $w_1w_2y_iw_1$ is contained in all 3-cycles $w_1w_2y_lw_1$, $i+1 \le l \le m$, $i = 1, 2, \cdots, m-1$. Thus $k + m = p(G) - 2 \ge 5$ and $k, m \ge 0$. Assume that $k \ge m$. Hence $k \ge \lceil \frac{1}{2}(p(G) - 2) \rceil \ge 3$. Since $2 \le d_G(v_1) \le 3$, we can form $H_2 = G - v_1$ if $d_G(v_1) = 2$ and $H_3 = G - v_1$ if $d_G(v_1) = 3$. Then H_j $(j = 2, 3)$ is an h_1-graph with two Δ-vertices w_1 and w_2 and $v_2w_i \in E_{in}^{w_i}(H_j)$ $(i = 1, 2)$. Hence H_2 or H_3 is not an h_1^*-graph.

Case 2 $|V_\Delta(G)| = 1$. Let $V_\Delta(G) = \{w\}$. Since G is not an h_1^*-graph, $m_w(G) \ge 1$. There are two subcases:

2.1 $m_w(G) \ge 2$. By Lemma 2.7, we consider three possibilities:

(i) There is a 1-vertex u such that $uw \in E(G)$. We form $H_1 = G - u$.

(ii) There is a 2-vertex u on a 3-face uwy. We form $H_2 = G - u$.

(iii) There is a 3-vertex u with $N_G(u) = \{w, v_1, v_2\}$ such that $uwv_1, uwv_2 \in F(G)$. In this case, we put $H_3 = G - u$ if $v_1v_2 \in E(G)$ and $H_3 = G - u + v_1v_2$ if $v_1v_2 \notin E(G)$.

Obviously H_i $(i = 1, 2, 3)$ is an h_1-graph with $p(H_i) \ge 6$ and $w \in V_\Delta(H_i)$. By Corollary 2.2, $1 \le |V_\Delta(H_i)| \le 2$. First suppose that $|V_\Delta(H_i)| = 1$. Note that

$E^w_{in}(H_i) \subseteq E^w_{in}(G)$ and $m_w(H_i) \geq m_w(G) - 1 \geq 1$. It follows that H_i is not an h_1^*-graph. Next let $|V_\Delta(H_i)| = 2$. Referring to the proof of Case 1, we deduce that for each $x \in V_\Delta(H_i)$, $E^x_{in}(H_i) \neq \emptyset$. Thus H_i is not an h_1^*-graph.

2.2 $m_w(G) = 1$. Let $e^* = wx \in E^w_{in}(G)$ and let $G_1, G_2, \cdots, G_k$ be the w-components of G. We claim that $k \geq 2$. In fact, if $k = 1$, then w is not a cut vertex of G and so G is 2-connected. Thus at most two edges incident to w are not inner edges of G. It follows that $m_w(G) \geq d_G(w) - 3 = p(G) - 1 - 3 \geq 3$, a contradiction. Note that each G_i is an h_1-graph with w as a Δ-vertex. In particular, when $|V(G_i)| \geq 3$, G_i is 2-connected. If there is some G_j such that $|V(G_j)| \geq 5$, then by $E^w_{in}(G_j) \subseteq E^w_{in}(G)$, it follows that $m_w(G) \geq m_w(G_j) \geq |V(G_j)| - 3 \geq 2$, also a contradiction. Thus $|V(G_i)| \leq 4$ for all i. In addition, there exists at most one w-component of G having four vertices. In fact, if there are two such w-components, say G_i and G_j, then $m_w(G) \geq m_w(G_i) + m_w(G_j) \geq 1 + 1 = 2$, a contradiction. Hence we may suppose that $2 \leq |V(G_1)| \leq 4$, $2 \leq |V(G_i)| \leq 3$, $i = 2, 3, \cdots, k$. Now the discussion can be divided into two cases.

2.2.1 $|V(G_1)| = 4$. Since G_1 is a 2-connected h_1-graph with w as a Δ-vertex, $m_w(G_1) \geq |V(G_1)| - 3 \geq 1$. On the other hand, $m_w(G_1) \leq m_w(G)$ is obvious. Therefore $m_w(G_1) = m_w(G) = 1$. This implies that $e^* \in E^w_{in}(G_1)$ and so $x \in V(G_1)$. We claim that $V(G_1)$ can not be contained in any separating cycle of G. Suppose that the assertion is false, then for every $u \in V(G_1) \setminus \{w\}$, $uw \in E^w_{in}(G)$. So $m_w(G) \geq |V(G_1) \setminus \{w\}| \geq 3$, a contradiction. Now, by $k \geq 2$, we may choose a w-component of G, say $G_t (2 \leq t \leq k)$, which is not contained any separating cycle of G. Applying Lemma 2.7 for G_t, we can form either H_1 or H_2 with $e^* \in E(H_i)$, $i = 1$ or 2. Thus H_i is not an h_1^*-graph, the lemma is shown in this case.

2.2.2 $|V(G_1)| \leq 3$. Now each w-component of G is either K_3 or K_2. Thus $1 \leq d_G(u) \leq 2$ for each $u \in V(G) \setminus \{w\}$. If $d_G(x) = 2$, let $y \in N_G(x) \setminus \{w\}$. Then both wy and wx must simultaneously be inner edges of G, thus $m_w(G) \geq 2$, a contradiction. Hence we must have $d_G(x) = 1$. It follows that there is $s \in \{1, 2, \cdots, k\}$ such that $G_s = G[\{e^*\}]$. Since $e^* \in E^w_{in}(G)$, G_s must be contained in some 3-cycle C of G. Clearly, $C = G_{j_0}$ $(1 \leq j_0 \leq k)$. We claim that G_{j_0} can not be contained in other w-components of G, since otherwise we can similarly deduce that $m_w(G) \geq 3$. By $p(G) \geq 7$, we may select a w-component G_t $(t \neq s, j_0)$ which is not contained in any separating cycle of G. Then the problem can be reduced to 2.2.1. The proof is completed.□

Lemma 2.9 *Let G be an h_2-graph with $p(G) \geq 8$ that is not an h_2^*-graph. If G contains a unique Δ-vertex w and $N^c_G(w) = \{x\}$ with $d_G(x) \geq 2$, then at least one of the following cases is true for G:*

(1) There is a 1-vertex u adjacent to w such that $H_1 = G - u$ is not an h_2^*-graph.

(2) There is a 2-vertex u on a 3-face uwy such that $H_2 = G - u$ is not an h_2^*-graph.

(3) There is a 3-vertex u with $N_G(u) = \{w, v_1, v_2\}$ and $uwv_1, uwv_2 \in F(G)$ such that H_3 is not an h_2^*-graph, where $H_3 = G - u$ if $v_1v_2 \in E(G)$ and $H_3 = G - u + v_1v_2$ otherwise.

By Lemmas 2.5 and 2.6, and using a method similar to that of the proof of Lemma

2.8, we can establish this lemma.

Lemma 2.10 *Let G be an h_2-graph with $p(G) \geq 9$ and two adjacent Δ-vertices w_1 and w_2. Then at least one of the following cases holds for G:*

(1) There is a 2-vertex $u \in N_G(w_1) \cap N_G(w_2)$ such that $uw_1w_2 \in F(G)$.

(2) There is a 3-cycle yw_1w_2 such that its interior (or exterior) contains only a vertex u and three edges uy, uw_1 and uw_2 and $d_G(y) \leq 6$.

(3) There are three vertices $u_1, u_2, u_3 \in N_G(w_1) \cap N_G(w_2)$ such that $d_G(u_1) \leq 5$, $d_G(u_2) \leq 4$, $d_G(u_3) \leq 5$ and the interior (or exterior) of the 4-cycle $u_1w_1u_3w_2u_1$ contains only u_2 and the edges incident to u_2.

Proof By the definition of h_2-graph, we suppose that $N_G^c(w_i) = \{x_i\}$, $i = 1, 2$. Consider the graph $H = G - x_1 - x_2$. If $x_1 \neq x_2$, then $p(H) = p(G) - 2$ and $\Delta(H) = \Delta(G) - 1 = p(G) - 2 - 1 = p(H) - 1$. If $x_1 = x_2$, then $p(H) = p(G) - 1$ and $\Delta(H) = \Delta(G) = p(G) - 2 = p(H) - 1$. This means that H always is an h_1-graph with $p(H) \geq 7$. Obviously w_1 and w_2 are two Δ-vertices of H. Let $v_1, v_2, \cdots, v_k$ denote the vertices in $V(H) \setminus \{w_1, w_2\}$ located on one side of the edge w_1w_2 such that the 3-cycle $w_1w_2v_jw_1$ is contained in all 3-cycles $w_1w_2v_sw_1$, $j + 1 \leq s \leq k$, $j = 1, 2, \cdots, k-1$, and $y_1, y_2, \cdots, y_m$ on the other side of w_1w_2 such that the 3-cycle $w_1w_2y_iw_1$ is contained in all 3-cycles $w_1w_2y_lw_1$, $i + 1 \leq l \leq m$, $i = 1, 2, \cdots, m-1$. Thus $k + m + 1 = \Delta(H) = p(H) - 1 \geq 6$ and $k, m \geq 0$. By virtue of Lemma 2.3, $2 \leq d_H(u) \leq 4$ for each $u \in V(H) \setminus \{w_1, w_2\}$. Thus $d_G(u) \leq d_H(u) + 2 \leq 6$, and $d_G(u) = 6$ iff $d_H(u) = 4$ and $ux_1, ux_2 \in E(G)$. Furthermore, each face f of H has degree either 3 or 4 and $b(f)$ contains at most two vertices in $V(H) \setminus \{w_1, w_2\}$. Noting that in G x_i $(i = 1, 2)$ must lie inside some face f_i of H, we deduce that $d_G(x_i) \leq 3$, $|V_6(G)| \leq 2$, and $uv \in E(G)$ and $\{f_1, f_2\} = \{uvw_1, uvw_2\}$ if there do exist two 6-vertices u and v in G.

Now suppose that (1) and (2) of the lemma are both false, we prove that (3) must hold. Let $m \leq k$. We distinguish three cases.

Case 1 $m = 0$. Then $k = \Delta(H) - 1 \geq 5$. Since both (1) and (2) do not hold, it follows that one of x_1 or x_2 lies inside the 3-cycle $w_1w_2v_2w_1$ and the other inside some face in H with v_k as a boundary vertex. Taking $u_1 = v_2$, $u_2 = v_3$ and $u_3 = v_4$, we obtain (3).

Case 2 $m = 1$. In this case, $k = \Delta(H) - 2 \geq 4$. With the same reason, one of x_1 or x_2 must lie inside the 3-cycle $w_1w_2v_2w_1$ and the other inside a face in H with y_1 as a boundary vertex. Again taking $u_1 = v_2$, $u_2 = v_3$ and $u_3 = v_4$, we deduce (3).

Case 3 $k \geq m \geq 2$. Similarly, one of x_1 or x_2 must lie inside the 3-cycle $w_1w_2v_2w_1$ and the other inside the 3-cycle $w_1w_2y_2w_1$. Thus three consecutive vertices in $\{v_2, v_3, \cdots, v_k, y_m, y_{m-1}, \cdots, y_2\}$ satisfy the requirement of (3). The lemma is proved.□

3 Main results

Lemma 3.1 *If G is either an h_1^*-graph with $p(G) \geq 5$ or an h_2^*-graph with $p(G) \geq 6$, then $\chi_{ef}(G) = \Delta(G) + 1$.*

Theorem 3.2 *If G is an h_1-graph with $p(G) \geq 6$, then $\Delta(G) \leq \chi_{ef}(G) \leq \Delta(G)+1$; and $\chi_{ef}(G) = \Delta(G) + 1$ iff G is an h_1^*-graph.*

Proof We use induction on $p(G)$. By enumeration, we can prove the theorem holds for $p(G) = 6$. Assume that it is true for all h_1-graphs with fewer than p vertices, and let G be an h_1-graph of order p (≥ 8). If G is an h_1^*-graph, it follows from Lemma 3.1 that $\chi_{ef}(G) = \Delta(G) + 1$. If G is not an h_1^*-graph, we shall prove $\chi_{ef}(G) = \Delta(G)$. Let w denote a Δ-vertex of G and then consider three cases by Lemma 2.8.

Case 1 There is a 1-vertex u adjacent to w such that $H = G - u$ is not an h_1^*-graph. Then $\Delta(H) = \Delta(G) - 1$, and H is a h_1-graph with $p - 1$ vertices. By the induction assumption, $\chi_{ef}(H) = \Delta(H)$. Thus we first give a $(\Delta(G) - 1)$-EF coloring λ of H with a color set C. Then we assign a new color $\beta \notin C$ to the edge uw in G. A $\Delta(G)$-EF coloring σ of G is constructed.

Case 2 There is a 2-vertex u on a 3-face uwy such that $H = G - u$ is not an h_1^*-graph. A similar discussion yields a $(\Delta(G) - 1)$-EF coloring λ of H with a color set C. Based on λ, we color the edge uw in G with a new color $\beta \notin C$. If $d_G(y) \leq \Delta(G) - 2$, then the edge uy can be properly colored because it has at most $\Delta(G) - 1$ color restrictions. Otherwise, since $\Delta(G) = p(G) - 1 \geq 7$, there must exist a vertex $v \in N_H(y) \setminus \{w\}$ such that $\lambda(vy)$ differs from $\lambda(f_0)$, where f_0 is the face of H with yw as a boundary edge, which is subdivided into the union of uyw and a face in G. In this case, we recolor the edge vy with β and color uy with $\lambda(vy)$. Afterward we put $uwy[5]$.

Case 3 There is a 3-vertex u with $N_G(u) = \{w, v_1, v_2\}$ and $uwv_1, uwv_2 \in F(G)$ such that H is not an h_1^*-graph, where $H = G - u$ if $v_1v_2 \in E(G)$ and $H = G - u + v_1v_2$ otherwise. It follows from Corollary 2.2 that $min\{d_G(v_1), d_G(v_2)\} \leq \Delta(G) - 1$, say $d_G(v_1) \leq \Delta(G) - 1$. Similarly, H has a $(\Delta(G) - 1)$-EF coloring λ with a color set C. We form a $\Delta(G)$-EF coloring σ of G by considering two subcases:

3.1 $v_1v_2 \in E(G)$. Based on λ, we color both uw and v_1v_2 with a new color $\beta \notin C$ and then put: $uv_2[\Delta - 1]$, $uv_1[\Delta - 1]$, $uwv_1[4]$, $uwv_2[5]$, $uv_1v_2[6]$.

3.2 $v_1v_2 \notin E(G)$. Let f_0 denote the face of G with uv_1 and uv_2 as two boundary edges. If $d_G(v_1) \leq \Delta(G) - 2$, based on λ, we can color both the edge uw and the face f_0 with a new color $\beta \notin C$. Then we put: $uv_2[\Delta - 1]$, $uv_1[\Delta - 1]$, $uwv_1[5]$, $uwv_2[6]$. If $d_G(v_1) = \Delta(G) - 1$, by $\Delta(G) \geq 7$, we can find a vertex $y \in N_G(v_1) \setminus \{u, w\}$ such that $\lambda(yv_1) \neq \lambda(f_0), \lambda(v_1v_2)$. Now we put: $\{uw, yv_1, f_0\} \to \beta \notin C$, $uv_2 \to \lambda(v_1v_2)$, $uv_1 \to \lambda(yv_1)$, $uwv_1[4]$, $uwv_2[5]$.□

Corollary 3.3 *If G is a 2-connected h_1-graph with $p(G) \geq 6$, then $\chi_{ef}(G) = \Delta(G)$.*

Corollary 3.4 *If G is an h_1-graph with $p(G) \geq 6$ and contains two Δ-vertices, then $\chi_{ef}(G) = \Delta(G)$.*

Theorem 3.5 *If G is an h_2-graph with $p(G) \geq 7$ and contains two adjacent Δ-vertices, then $\chi_{ef}(G) = \Delta(G)$.*

Proof Obviously we need only prove the bound $\chi_{ef}(G) \leq \Delta(G)$. We proceed by induction on $p(G)$. For $p(G) = 7, 8$, the theorem follows from enumeration. Assume that it is true for each h_2-graph with fewer than p vertices and two adjacent Δ-vertices. Let G be a graph satisfying the conditions of the theorem and $|V(G)| = p \geq 9$. Thus $\Delta(G) = p - 2 \geq 7$. By Lemma 2.10, we have three possibilities.

Case 1 There is a 2-vertex $u \in N_G(w_1) \cap N_G(w_2)$ such that $uw_1w_2 \in F(G)$. Form the graph $H = G - u$. Let f denote the face of G with u as a boundary vertex and $f \neq uw_1w_2$ and let f_0 denote the face of H which is subdivided into the union of f and uw_1w_2 in G. Since H is an h_2-graph with two adjacent Δ-vertices w_1 and w_2 and $\Delta(H) = \Delta(G) - 1$, by the induction assumption, H has a $(\Delta(G) - 1)$-EF coloring λ with a color set C. By $\Delta(G) - 1 \geq 6$, there must exist a vertex $x \in N_H(w_1) \setminus \{u, w_2\}$ such that $\lambda(xw_1) \neq \lambda(f_0)$. Based on λ, we construct a $\Delta(G)$-EF coloring σ of G as follows: $\{uw_2, xw_1\} \to \beta \notin C$, $uw_1 \to \lambda(xw_1)$, $f \to \lambda(f_0)$, $uw_1w_2[5]$.

Case 2 There is a 3-cycle yw_1w_2 such that its interior (or exterior) contains only a vertex u and three edges uy, uw_1 and uw_2 and $d_G(y) \leq 6$. Let $H = G - u$ and form a $(\Delta(G) - 1)$-EF coloring λ of H with a color set C. Based on λ, we further put: $\{yw_1, uw_2\} \to \beta \notin C$, $uw_1 \to \lambda(yw_1)$, $uy[6]$, $uyw_1[4]$, $uyw_2[5]$, $uw_1w_2[6]$.

Case 3 There are three vertices $u_1, u_2, u_3 \in N_G(w_1) \cap N_G(w_2)$ such that $d_G(u_1) \leq 5$, $d_G(u_2) \leq 4$, $d_G(u_3) \leq 5$ and the interior (or exterior) of the 4-cycle $u_1w_1u_3w_2u_1$ contains only u_2 and the edges incident to u_2. Again let $H = G - u$ and, by the induction assumption, H has a $(\Delta(G) - 1)$-EF coloring λ with a color set C. Based on λ, we can form a $\Delta(G)$-EF coloring σ of G as follows: first color u_2w_2 and w_1u_3 with a new color $\beta \notin C$ and then color u_2w_1 with $\lambda(w_1u_3)$. Further there exist some subcases.

If $u_1u_2, u_2u_3 \in E(G)$, we put: $u_1u_2[6]$, $u_2u_3[6]$, $u_1u_2w_1[4]$, $u_1u_2w_2[5]$, $u_2u_3w_2[5]$, $u_2u_3w_1[6]$.

If $u_1u_2 \notin E(G)$ and $u_2u_3 \in E(G)$ (for the converse case, we can give a similar proof), we put: $u_2u_3[5]$, $u_1w_1u_2w_2 \to \lambda(w_1u_3w_2u_1)$, $u_2u_3w_1[5]$, $u_2u_3w_2[6]$.

If $u_1u_2, u_2u_3 \notin E(G)$, we put: $u_1w_1u_2w_2 \to \lambda(w_1u_3w_2u_1)$, $w_1u_3w_2u_2[6]$. Now we have proved the theorem.□

Let $C = x_1x_2 \cdots x_{p-2}x_1$ be a cycle of length $p - 2(\geq 3)$. Add a new vertex u to the interior of C and another v to the exterior respectively, and then join both u and v to each x_i $(i = 1, 2, \cdots, p-2)$. Denote the resulting graph by $\widetilde{W}_p$.

It is easily seen that $\widetilde{W}_p$ is an h_2-graph with two nonadjacent Δ-vertices. Moreover every h_2-graph G containing two nonadjacent Δ-vertices can be induced from $\widetilde{W}_p$ by removing some edges in $E(C)$, where $p = |V(G)|$.

Theorem 3.6 *If G is an h_2-graph with $p(G) \geq 7$ and contains two nonadjacent Δ-vertices, then $\chi_{ef}(G) = \Delta(G)$.*

Proof Given $p = |V(G)| \geq 7$, we first form a $(p-2)$-EF coloring λ of $\widetilde{W}_p$. Let $0, 1, \cdots, p-3$ denote $p-2$ colors and suppose that the following suffixes are taken modulo $p-2$. For $i = 1, 2, \cdots, p-2$, we put: $vx_i \to i-1$, $ux_i \to i-2$, $x_ix_{i+1} \to i+1$, $ux_ix_{i+1} \to i$, $vx_ix_{i+1} \to i+2$.

It is easily checked that λ is a $(p-2)$-EF coloring of $\widetilde{W}_p$ with the property that for each $i = 1, 2, \cdots, p-2$, the color $\lambda(vx_ix_{i+1})$ differs from each of $\lambda(ux_ix_{i+1})$, $\lambda(ux_{i-1}x_i)$, $\lambda(ux_{i+1}x_{i+2})$, $\lambda(ux_i)$ and $\lambda(ux_{i+1})$.

Next, according to the above discussion, we have $G \subseteq \widetilde{W}_p$ with $\Delta(G) = p(G)-2 = p-2$. Thus, based on λ, a $\Delta(G)$-EF coloring σ of G is formed as follows: for each edge $e = x_ix_{i+1} \in E(\widetilde{W}_p) \setminus E(G)$, we put $\sigma(vx_iux_{i+1}) = \lambda(vx_ix_{i+1})$. The other edges and faces of G are colored with the same colors as in λ. So we prove that $\chi_{ef}(G) \leq \Delta(G)$. But $\chi_{ef}(G) \geq \Delta(G)$ is trivial. Therefore $\chi_{ef}(G) = \Delta(G)$. This completes the proof.□

Theorem 3.7 *If G is an h_2-graph with $p(G) \geq 7$, then $\Delta(G) \leq \chi_{ef}(G) \leq \Delta(G)+1$, and $\chi_{ef}(G) = \Delta(G)+1$ iff G is an h_2^*-graph.*

Proof By induction on $p(G)$. If $p(G) = 7, 8$, the theorem follows by enumeration. Suppose that it is valid for $p-1$ and let G be an h_2-graph with $|V(G)| = p \geq 9$. If G is an h_2^*-graph, it follows from Lemma 3.1 that $\chi_{ef}(G) = \Delta(G) + 1$. Now suppose that G is not a h_2^*-graph, we show that $\chi_{ef}(G) = \Delta(G)$. By Corollary 2.2, G contains at most two Δ-vertices. However, when G contains two adjacent or nonadjacent Δ-vertices, the assertion has been verified in Theorems 3.5 or 3.6. Thus we need only consider the case in which G contains a unique Δ-vertex w. Let $N_G^c(w) = \{x\}$, then $d_G(x) \leq \Delta(G) - 1$. If $d_G(x) \leq 1$, we define the graph $H = G - x$. Clearly $\Delta(H) = \Delta(G) = p(G) - 2 = p(H) - 1 \geq 7$, so H is an h_1-graph. Since G is not an h_2^*-graph, H is not a h_1^*-graph. By Theorem 3.2, we prove easily $\chi_{ef}(G) = \chi_{ef}(H) = \Delta(H) = \Delta(G)$. So we may assume that $d_G(x) \geq 2$. In this case, by means of Lemma 2.9 and applying a similar discussion as in Theorem 3.2, we can complete the proof of theorem.□

Corollary 3.8 *If G is a 2-connected h_2-graph with $p(G) \geq 7$, then $\chi_{ef}(G) = \Delta(G)$.*

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Macmillan, New York (1976).

[2] O. V. Borodin, Simultaneous coloring of edges and faces of plane graphs, *Discrete Math.* **128** (1994), 21-33.

[3] Lin Cuiqin, Hu Guanzhang and Zhang Zhongfu, A six-color theorem for the edge-face coloring of plane graphs, *Discrete Math.* **141** (1995), 291-297.

[4] Recent advances in graph theory, Proc. Internat. Symp. Prague, (Academic, Praha) *Problem Section:* 543 (1975).

[5] Wang Weifan, The edge-face chromatic number of plane graphs with lower degrees, *Applied Math. – A J. of Chinese Universities* A 8(**3**) (1993), 300-307.

[6] Wang Weifan, Equitable colorings and total colorings of graphs, Doctoral Thesis, Nanjing University, 1997.

[7] A. Waller, Simultaneously coloring the edges and faces of plane graphs, *J. Combin. Theory* B **69**(2) (1997), 219-221.

(Received 23/10/97)