G-designs of order n and index λ where G has 5 vertices or less.

D. G. Hoffman

Department of Discrete and Statistical Sciences
120 Math Annex, Auburn University, Alabama, USA 36849-5307

Kimberly S. Kirkpatrick

Department of Mathematics, University of Evansville
1800 Lincoln Ave, Evansville, Indiana, USA 47722-0001

Abstract

In this paper we construct G-designs of order n and index λ for a number of small graphs, G, where G has 4 or 5 vertices.

1 Introduction

A *G-design on a graph H* is a collection of subgraphs of H, each isomorphic to a graph G, that partition the edges of H. A *G-design of order n and index λ* is a G-design on the complete multi-graph on n vertices, λK_n. One problem in design theory is the *spectrum problem* for G, i.e. for what values of n and λ is there a G-design of order n and index λ? The spectrum problem has been solved for complete simple graphs on less than six vertices for all λ [9], stars for all λ [16], and paths [17], cycles of length at most 50 [13, 14], and various other small graphs for $\lambda = 1$ [1, 2]. G-designs on non-complete graphs have also been studied, for example, designs when G and H are both complete bipartite graphs [12]. Another example is when H is a complete graph of order n with a *hole* of size v, $K_n \backslash K_v$. This is a complete graph of order n from which the edges of a complete graph of order v have been removed. Doyen and Wilson first considered such designs with $G = K_3$ [8]. G-designs on graphs with holes have also been found when G is an n-cycle with $n \leq 6$ [4, 5, 6, 7] and K_3 with a pendant edge [11]. For this paper, we will consider G-designs on λK_n, where G is connected and has 5 or fewer vertices. This problem has been solved for many small graphs for $\lambda = 1$ [1, 2]. The case when G has 2 vertices is trivial. When G has 3 vertices there are 2 possibilities for G. Either G is a 2-star or G is a 3-cycle. Tarsi, among others, solved the case when G is a 2-star [16] and decomposing graphs into 3-cycles is equivalent to finding a Steiner Triple System which is a well studied problem in combinatorics [13, 15]. When G has 4 vertices there are 6 possible connected graphs. Either G is a 3-path [17], a 3-star [16], K_4 [9], a 4-cycle [14], $K_4\backslash\{e\}$ or a 3-cycle with a pendant edge. The first 4 cases have already been solved. We will consider the cases when G is $K_4\backslash\{e\}$, when G is a 3-cycle with a pendant edge and go on to consider many other cases when G has 5 vertices.

2 Necessary Conditions

There are 2 necessary conditions for a G-design of order n and index λ to exist. First, the number of edges of G, ϵ, must divide the number of edges of λK_n, therefore $2\epsilon|\lambda n(n-1)$. The second condition is on the degrees of the vertices. The degree of each vertex of λK_n must be divisible by the gcd of the degrees of the vertices of G, therefore $\gcd(\{d(v) : v \in V(G)\})|\lambda(n-1)$, where $d(v)$ indicates the degree of the vertex v. These conditions are certainly necessary, although in a number of cases these are not sufficient.

3 General Construction

For the general construction we find a G-design on the complete multipartite graph with k groups of size r. There are λ edges between vertices in different groups and no edges between vertices of the same group. These designs are also known as Group Divisible Designs. We will denote them by G-GDD(r, k, λ). For our purposes each group will have either $r = \epsilon$ or $r = 2\epsilon$ vertices. We will use quasigroups to form these GDD's.

A *Quasigroup* $(Q, \circ)$ is a set Q and a binary operation $\circ$ such that for every $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions $x, y \in Q$. A quasigroup is *commutative* if $x \circ y = y \circ x$ for all $x, y \in Q$ and a quasigroup is *idempotent* if $x \circ x = x$ for all $x \in Q$. A quasigroup has *holes of size 2* if there are subquasigroups, each of size 2, that partition Q (a quasigroup is idempotent if and only if it has holes of size 1).

Idempotent quasigroups exist for all orders except 2 and idempotent, commutative quasigroups exist for all odd orders. Quasigroups with holes of size 2, and commutative quasigroups with holes of size 2, exist for all even orders greater than or equal to 6 [15].

Using these GDD's, every edge between vertices in different groups will be contained in a block of G. Therefore, only the edges between vertices of the same group and edges between the vertices in the GDD and any added vertices need to be decomposed into copies of G to form a G-design.

The following theorem will be used extensively in the sequel. The proof is obvious.

Theorem 1 *If there is a G-design of order n and index λ_1 and a G-design of order n and index λ_2 then there is a G-design of order n and index $\lambda_1 + \lambda_2$.*

4 Triangle and a Stick

Theorem 2 *Let $G = K_3 \cup \{e\}$ (otherwise known as a triangle and a stick, see figure 1). Then there exists a G-design on λK_n if and only if $8|\lambda n(n-1)$, $n \geq 4$.*

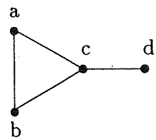

Figure 1: Block $(a, b, c) - d$ for Triangle and a Stick

We denote the copy of G in figure 1 by $(a, b, c) - d$ or $(b, a, c) - d$.

The necessity has already been proven, so we will now show the sufficiency. In order to do this we will need the follow lemma.

Lemma 1 *Let Q be an idempotent quasigroup of order k on the set $\{1, 2, ..., k\}$. Then the following blocks form a GDD$(4, k, 2)$: $((i, a)(j, a)(i \circ j, a+1)) - (i, a+3)$ for each ordered pair (i, j), $1 \leq i, j \leq k$, i and j not in the same hole of Q and for $a \in \mathbb{Z}_4$.*

(Here $\mathbb{Z}_m$ denotes the ring of integers (mod m).)

Proof of Theorem 1. In the proof we will consider 3 cases.

Case 1: $n \equiv 0, 1 (\text{mod } 8)$

When $n \equiv 0, 1(\text{mod } 8)$ the theorem implies no restriction on λ. By Theorem 1, if there is a design with $\lambda = 1$ then there is a design with all subsequent λ values. Thus, it suffices to find designs for $\lambda = 1$. Theorem 1 will be used implicitly throughout the sequel. This case was completely solved by Bermond and Schönheim [1].

Case 2: $n \equiv 4, 5 (\text{mod } 8)$

When $n \equiv 4, 5\,(\text{mod } 8)$ the theorem implies that $2|\lambda$ so it suffices to find designs for $\lambda = 2$.

Let $V = \{1, 2, ..., 2t + 1\} \times \{1, 2, 3, 4\}$ (together with ∞ in the 5(mod 8) case). On the points (i, a) for $1 \leq i \leq 2t + 1$ and $a \in \mathbb{Z}_4$ place a GDD$(4, 2t + 1, 2)$. On the points in each group (and ∞ if $n \equiv 5(\text{mod } 8)$) place a G-design on $2K_4$ or $2K_5$. These and all other necessary small G-designs may be found listed in the appendix.

Case 3: $n \equiv 2, 3, 6, 7 (\text{mod } 8)$

When $n \equiv 2, 3, 6, 7\,(\text{mod } 8)$ the theorem implies that $4|\lambda$ so it suffices to find designs for $\lambda = 4$.

First consider the case where $n \equiv 3, 7 \pmod 8$.

Use a commutative idempotent quasigroup, L, to form a K_3-design on $3K_n$ by letting $(a, b, a \circ b)$ be a triple for every unordered pair $\{a, b\}$ with $a, b \in K_n$, $a \neq b$. There will be $\frac{1}{2}n(n-1)$ such triples.

Form a bipartite graph $A(B, E)$ where B is the set of blocks of the K_3-design and E is the set of edges of K_n. Let A contain the edge (i, j) if and only if block i and edge j have exactly one vertex of K_n in common.

Each vertex of a block $b \in B$ is incident with $n-3$ other vertices of K_n (not in b) so the degree of each $b \in A$ is $3(n-3)$.

Each vertex x of an edge $e = (x, y)$ is contained in $3\left(\frac{n-1}{2}\right)$ blocks but 3 of these blocks contain the edge e. Therefore the number of blocks containing only the point x (and not y) is $3\left(\frac{n-3}{2}\right)$. Thus each $e \in A$ has degree $3(n-3)$.

The Marriage Theorem states that every regular bipartite graph has a perfect matching [3]; therefore, there is a perfect matching of A. This matching will pair up the triangles already formed from the $3K_n$ with the edges of a fourth K_n to form a G-design on $4K_n$.

Next consider the case where $n \equiv 2, 6 \pmod 8$.

Let $V = \{1, 2, ..., t\} \times \{1, 2, 3, 4\} \cup \{\infty_1, \infty_2\}$. On the points (i, a) for $1 \leq i \leq t$ and $a \in \mathbb{Z}_4$ place 2 copies of a GDD$(4, t, 2)$. On the points in the first group and the points ∞_1 and ∞_2 place a G-design on $4K_6$. On all subsequent groups and the points ∞_1 and ∞_2 place a G-design on $4K_6 \setminus 4K_2$ where ∞_1, ∞_2 are the points in the hole. $\square$

5 $K_4 \backslash \{e\}$

Theorem 3 *Let G be $K_4 \setminus \{e\}$ (see figure 2). Then there exists a G-design on λK_n if and only if $10 | \lambda n(n-1)$, $n \geq 4$, except K_5.*

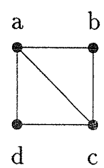

Figure 2: Block (a,b,c,d) for $K_4 - e$

We denote the copy of G in Figure 2 by (a, b, c, d) or (a, d, c, b).

In the proof of Theorem 3 we will need the following lemmas.

Lemma 2 *Let Q be a commutative, idempotent quasigroup of order $2t+1$ on the set $\{1, 2, ..., 2t+1\}$. Then the following blocks form a $G-GDD(5, 2t+1, 1)$: $((i,1),(i \circ j,2),(j,1),(i \circ j,3))$, for each unordered pair $\{i,j\}$, $1 \leq i < j \leq 2t+1$, i and j not in the same hole of Q and for $a \in \mathbb{Z}_5$.*

Lemma 3 *Let Q be a commutative quasigroup of order $2t$ with holes $2i-1, 2i$ for $1 \leq i \leq t$ on the set $1, 2, ..., 2t$. Then the following vlocks form a $G-GDD(10, t, 1)$: $((i,1),(i \circ j,2),(j,1),(i \circ j,3))$, for each unordered pair $\{i,j\}$, $1 \leq i < j \leq 2t$, i and j not in the same hole of Q and for $a \in \mathbb{Z}_5$.*

Proof of Theorem 2. The proof will be in 2 cases.

Case 1: $n \equiv 0, 1 (\text{mod } 5)$

When $n \equiv 0, 1\,(\text{mod } 5)$ the theorem implies no restriction on λ except when $n = 5$ so it suffices to find designs for $\lambda = 1$ for all $n > 5$ and designs on $2K_5$ and $3K_5$.

The case for $\lambda = 1$ was completely solved by Bermond, et al.[2]. Clearly, a design for K_5 does not exist.

The designs on $2K_5$ and $3K_5$ for are listed in the appendix.

Case 2: $n \equiv 2, 3, 4 (\text{mod } 5)$

When $n \equiv 2, 3, 4\,(\text{mod } 5)$ the theorem implies that $5|\lambda$ so it suffices to find designs for $\lambda = 5$.

First consider the case where $n \equiv 2, 3, 4\,(\text{mod } 10)$.

Let $V = \{1, 2, ..., 2t\} \times \{1, 2, 3, 4, 5\} \cup \{\infty_1, ..., \infty_x\}$ for $x \in \{2, 3, 4\}$. On the points (i, a) for $1 \leq i \leq 2t$ and for $a \in \mathbb{Z}_5$ place 5 copies of a GDD(10,t,1) where the points $(2j-1, b)$ and $(2j, b)$ for $b \in \mathbb{Z}_5$ form the groups for $1 \leq j \leq t$. On the first group and the points $\infty_1, ..., \infty_x$ place a design on $5K_{10+x}$. On each subsequent group and the points $\infty_1, ..., \infty_x$ place a copy of a design on $5K_{10+x} \backslash 5K_x$. This design works for $t \geq 3$. Designs for $t = 0$ ($5K_4$), $t = 1$ ($5K_{12}$, $5K_{13}$, $5K_{14}$) and $t = 2$ ($5K_{22}$, $5K_{23}$, $5K_{24}$) may be found in the appendix.

Finally consider the case where $n \equiv$ 7,8,9 (mod 10).

Let $V = \{1, 2, ..., 2t+1\} \times \{1, 2, 3, 4, 5\} \cup \{\infty_1, ..., \infty_x\}$ for $x \in \{2, 3, 4\}$. On the points (i, a) for $1 \leq i \leq 2t+1$ and for $a \in \mathbb{Z}_5$ place 5 copies of a GDD(5,$2t+1$,1). On the first group and the points $\infty_1, ..., \infty_x$ place a design on $5K_{5+x}$. On each subsequent group and the points $\infty_1, ..., \infty_x$ place a copy of a design on $5K_{5+x} \backslash 5K_x$. □

6 3-Path and a Stick

Theorem 4 *Let G be a 3-path with an attached edge (also known as a 3-path and a stick, see figure 3). Then there exists a G-design on λK_n if and only if $8|\lambda n(n-1)$, $n \geq 5$.*

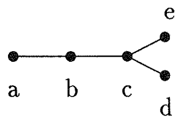

Figure 3: Block (a,b,c)-[d,e] for 3-Path and a Stick

We denote the copy of G in Figure 3 by $(a, b, c) - [d, e]$ or $(a, b, c) - [e, d]$.

Proof of Theorem 3. In the proof we will consider 3 cases.

Case 1: $n \equiv 0, 1(\text{mod } 8)$.
When $n \equiv 0, 1 \,(\text{mod } 8)$ the theorem implies no restriction on λ so it suffices to find designs for $\lambda = 1$.

This case was completely solved by Bermond, et al.[2].

Case 2: $n \equiv 4, 5(\text{mod } 8)$.
When $n \equiv 4, 5 \,(\text{mod } 8)$ the theorem implies that $2|\lambda$ so it suffices to find designs for $\lambda = 2$.

First consider the case where $n \equiv 4 \,(\text{mod } 8)$.

Let $n = 8t + 4$ and let $V = \{(i, 1) : 1 \leq i \leq 12\} \cup \{(j, 2) : 1 \leq j \leq 8\,(t - 1)\}$. Define the blocks of the G-design as follows:

Type 1: On the points $(i, 1)$, $1 \leq i \leq 12$ (hereafter known as level 1), place a G-design on $2K_{12}$. On the points $(j, 2)$, $1 \leq j \leq 8\,(t - 1)$ (hereafter known as level 2), place 2 G-designs on $K_{8(t-1)}$ found in Case 1.

Type 2: Partition the points on each level into groups of size 4. On each pair of groups between levels 1 and 2 place two G-designs on $K_{4,4}$.

Next consider the case where $n \equiv 5 \,(\text{mod } 8)$.

Let $n = 8t + 5$ and let $V = \{(i, 1) : 1 \leq i \leq 4\} \cup \{(j, 2) : 1 \leq j \leq 8t\} \cup \infty$. Define the blocks of the G-design as follows:

Type 1: On the points $(i, 1)$, $1 \leq i \leq 4$, and ∞ place a G-design on $2K_5$. On the points $(j, 2)$, $1 \leq j \leq 8t$, and ∞ place two G-designs on K_{8t+1} found in Case 1.

Type 2: These blocks are the same as the 4(mod 8) case in this section.

Case 3: $n \equiv 2, 3, 6, 7(\text{mod } 8)$.
When $n \equiv 2, 3, 6, 7 \,(\text{mod } 8)$ the theorem implies that $4|\lambda$ so it suffices to find designs for $\lambda = 4$.

Let $n = 4t + 2$, or $n = 4t + 3$, and let $V = \{(i, 1) : 1 \leq i \leq 6\} \cup \{(j, 2) : 1 \leq j \leq 4\,(t - 1)\}$ (together with ∞ for the 3,7(mod 8) case). Define the blocks of the G-design as follows:

Type 1: On the points $(i, 1)$, $1 \leq i \leq 6$ (hereafter known as level 1), (and ∞ in the 3,7(mod 8) case), place a G-design on $4K_6$ (or $4K_7$). On the points $(j, 2)$, $1 \leq j \leq 4(t-1)$ (hereafter known as level 2), (and ∞ in the 3,7(mod 8) case), place four G-designs on $K_{4(t-1)}$ (or $K_{4(t-1)+1}$) found in Case 1 if t is odd or 2 G-designs on $K_{4(t-1)}$ (or $K_{4(t-1)+1}$) found in Case 2 if t is even.

Type 2: Partition the points on level 1 into 3 groups of size 2 and partition the points on level 2 into $t-1$ groups of size 4. On each pair of groups between levels 1 and 2 place 4 copies of a G-design on $K_{4,2}$. □

7 Triangle and a 2-Path, Triangle and a 2-Star, Bull

Theorem 5 *Let $G_1 = K_3 \cup P_2$ (otherwise known as a triangle and a 2-path, see Figure 4), $G_2 = K_3 \cup K_{1,2}$ (otherwise known as a triangle and a 2-star, see Figure 5), and $G_3 = K_3 \cup \{e_1, e_2\}$ (otherwise known as the bull, see Figure 6). Then a G_1-design (respectively G_2- and G_3-design) on λK_n exists if and only if $10|\lambda n(n-1)$, $n \geq 5$, except K_5 for G_1 and G_2.*

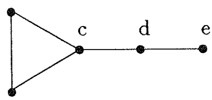

Figure 4: Block (a,b,c)-(d,e) for Triangle and a 2-Path

We denote the copy of G_1 in Figure 4 by $(a, b, c) - (d, e)$ or $(b, a, c) - (d, e)$.

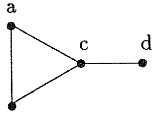

Figure 5: Block (a,b,c)-d,e for Triangle and a 2-Star

We denote the copy of G_2 in Figure 4 by $(a, b, c) - d, e$ or $(b, a, c) - d, e$.
We denote the copy of G_3 in Figure 6 by $d - (b, a, c) - e$ or $e - (c, a, b) - d$.
In the proof of Theorem 5 we will need the following lemmas.

Lemma 4 *Let Q be a commutative, idempotent quasigroup of order $2t \pm 1$ on the set $\{1, 2, ..., 2t \pm 1\}$. Then the following blocks form a G_1-, G_2- and G_3-GDD$(5, 2t \pm 1, 1)$ respectively:*

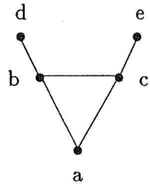

Figure 6: Block d-(b,a,c)-e for the Bull

$G_1: \quad ((j,a)(i \circ j, a+1)(i,a)) - ((j,a+2)(i,a-1)),\ i<j$
$G_2: \quad ((i,a)(j,a)(i \circ j, a+1)) - (i,a+3)(j,a+3)$
$G_3: \quad (j,a+2) - ((i,a)(i \circ j, a+1)(j,a)) - (i,a+2)$
for each unordered pair $\{i,j\}$, $1 \le i,j \le 2t \pm 1$, *i and j not in the same hole of Q and for* $a \in \mathbb{Z}_5$.

Lemma 5 *Let Q be a commutative quasigroup of order* $2t$ *with holes* $\{2i-1, 2i\}$ *for* $1 \le i \le t$ *on the set* $\{1,2,...,2t\}$. *Then the following blocks form a* G_1-, G_2- *and* $G_3-GDD(10,t,1)$ *respectively:*
G_1: $((j,a)(i \circ j, a+1)(i,a)) - ((j,a+2)(i,a-1)),\ i<j$
G_2: $((i,a)(j,a)(i \circ j, a+1)) - (i,a+3)(j,a+3)$
G_3: $(j,a+2) - ((i,a)(i \circ j, a+1)(j,a)) - (i,a+2)$
for each unordered pair $\{i,j\}$, $1 \le i,j \le 2t$, *i and j not in the same hole of Q and for* $a \in \mathbb{Z}_5$.

Proof of Theorem 5. We will consider 2 cases.

Case 1: $n \equiv 0,1 (\text{mod } 5)$

When $n \equiv 0,1\,(\text{mod } 5)$ the theorem implies no restriction on λ except when $n=5$ so it suffices to find designs for $\lambda = 1$ for all $n > 5$ ($n \ge 5$ for G_3) and designs on $2K_5$ and $3K_5$ with G_1 and G_2.

The case for $\lambda = 1$ was completely solved by Bermond, et al.[2]. Clearly, a design for K_5 does not exist for G_1 and G_2.

The designs on $2K_5$ and $3K_5$ for G_1 and G_2 are listed in the appendix.

Case 2: $n \equiv 2,3,4 (\text{mod } 5)$

When $n \equiv 2,3,4\,(\text{mod } 5)$ the theorem implies that $5|\lambda$ so it suffices to find designs for $\lambda = 5$.

First consider the case where $n \equiv 4\,(\text{mod } 10)$.

Let $V = \{1,2,...,2t\} \times \{1,2,3,4,5\} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$. On the points (i,a) for $1 \le i \le 2t$ and for $a \in \mathbb{Z}_5$ place 5 copies of a GDD(10,t,1) where the points $(2j-1,b)$ and $(2j,b)$ for $b \in \mathbb{Z}_5$ form the groups for $1 \le j \le t$. On the first group and the points $\infty_1,\infty_2,\infty_3,\infty_4$ place a design on $5K_{14}$. On each subsequent group and the points $\infty_1,\infty_2,\infty_3,\infty_4$ place 5 copies of a design on $K_{14}\backslash K_4$. This design works for

$t \geq 3$. A design for $t = 0$ ($5K_4$) clearly does not exist. Designs for $t = 1$ ($5K_{14}$) and $t = 2$ ($5K_{24}$) may be found in the appendix.

Next consider the case where $n \equiv 7,8,9 \pmod{10}$.

Let $V = \{1, 2, ..., 2t+1\} \times \{1, 2, 3, 4, 5\} \cup \{\infty_1, ..., \infty_x\}$ for $x \in \{2, 3, 4\}$. On the points (i, a) for $1 \leq i \leq 2t+1$ and for $a \in \mathbb{Z}_5$ place 5 copies of a GDD(5,$2t+1$,1). On the first group and the points $\infty_1, ..., \infty_x$ place a design on $5K_{5+x}$. On each subsequent group and the points $\infty_1, ..., \infty_x$ place 5 copies of a design on $K_{5+x} \backslash K_x$.

Finally consider the case where $n \equiv 2,3 \pmod{10}$.

Let $V = \{1, 2, ..., 2t-1\} \times \{1, 2, 3, 4, 5\} \cup \{\infty_1, ..., \infty_x\}$ for $x \in \{7, 8\}$ On the points (i, a) for $1 \leq i \leq 2t-1$ and for $a \in \mathbb{Z}_5$ place 5 copies of a GDD(5,$2t-1$,1). On the points $\infty_1, ..., \infty_x$ place a design on $5K_x$. On each group and the points $\infty_1, ..., \infty_x$ place 5 copies of a design on $K_{5+x} \backslash K_x$. $\square$

8 Square and a Stick

Theorem 6 *Let G be a 4-cycle with a pendant edge (see Figure 7). Then a G-design on λK_n exists if and only if $10|\lambda n(n-1)$, $n \geq 5$ except K_5.*

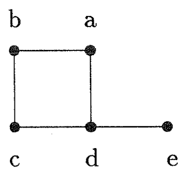

Figure 7: Block (a,b,c,d)-e

We denote the copy of G in Figure 7 by $(a, b, c, d) - e$ or $(c, b, a, d) - e$.

Proof of Theorem 6. In the proof we will consider 2 cases.

Case 1: $n \equiv 0, 1 \pmod 5$.

When $n \equiv 0, 1 \pmod 5$ the theorem implies no restriction on λ except when $n = 5$ so it suffices to find designs for $\lambda = 1$ when $n > 5$ and designs on $2K_5$ and $3K_5$.

The case for $\lambda = 1$ was completely solved by Bermond, et al. [2]. Certainly, a design for K_5 does not exist. The designs on $2K_5$ and $3K_5$ can be found in the appendix.

Case 2: $n \equiv 2, 3, 4 \pmod 5$.

When $n \equiv 2, 3, 4 \pmod 5$ the theorem implies that $5|\lambda$ so it suffices to find designs for $\lambda = 5$.

First consider the case where $n \equiv 2 (\mathrm{mod}\ 5)$.

Let $n = 5t + 2$ and let $V = \{(i,1) : 1 \le i \le 7\} \cup \{(j,2) : 1 \le j \le 5(t-1)\}$. Define the blocks of the design as follows:

Type 1: On the points $(i,1)$, $1 \le i \le 7$ (hereafter known as level 1), place a G-design on $5K_7$. On the points $(j,2)$, $1 \le j \le 5(t-1)$ (hereafter known as level 2), place 5 copies of a G-design on K_{5t-5} found in Case 1.

Type 2: Partition the points on level 2 into $t-1$ groups of size 5. Between each of these groups and points (1,1) and (2,1) place 5 copies of a G-design on $K_{2,5}$. Between each of the $t-1$ groups and the points $(i,1)$ for $3 \le i \le 7$ place 5 copies of a G-design on $K_{5,5}$.

Next consider the case where $n \equiv 3 (\mathrm{mod}\ 5)$.

Let $n = 5t + 3$ and let $V = \{(i,1) : 1 \le i \le 8\} \cup \{(j,2) : 1 \le j \le 5(t-1)\}$. Define the blocks of the design as follows:

Type 1: On the points $(i,1)$, $1 \le i \le 8$ (hereafter known as level 1), place a G-design on $5K_8$. On the points $(j,2)$, $1 \le j \le 5(t-1)$ (hereafter known as level 2), place 5 copies of a G-design on K_{5t-5} found in Case 1.

Type 2: Partition the points on level 2 into $t-1$ groups of size 5. Between each of these groups and points $(2i-1,1)$ and $(2i,1)$ for $1 \le i \le 4$, place 5 copies of a G-design on $K_{2,5}$.

Finally, consider the case where $n \equiv 4 (\mathrm{mod}\ 5)$.

Let $n = 5t + 4$ and let $V = \{(i,1) : 1 \le i \le 9\} \cup \{(j,2) : 1 \le j \le 5(t-1)\}$. Define the blocks of the design as follows:

Type 1: On the points $(i,1)$, $1 \le i \le 9$ (hereafter known as level 1), place a G-design on $5K_9$. On the points $(j,2)$, $1 \le j \le 5(t-1)$ (hereafter known as level 2), place 5 copies of a G-design on K_{5t-5} found in Case 1.

Type 2: Partition the points on level 2 into $t-1$ groups of size 5. Between each of these groups and points $(2i-1,1)$ and $(2i,1)$, for $i \in \{1,2\}$, place 5 copies of a G-design on $K_{2,5}$. Between each of these groups and the points $(i,1)$ for $5 \le i \le 9$ place 5 copies of a G-design on $K_{5,5}$. □

9 $K_4 - e$ and a Stick

Theorem 7 *Let $G_1 = \{K_4 - e\} \cup \{e\}$, type I (see Figure 8) and $G_2 = \{K_4 - e\} \cup \{e\}$, type II (see Figure 9). Then a G_1-design (respectively G_2-design) of λK_n exists if and only if $12 | \lambda n(n-1)$, $n \ge 5$.*

We denote the copies of G in Figures 8 and 9 by (a,b,c,d)-e or (c,b,a,d)-e.

We will need the following Lemmas in the proof of Theorem 7.

Lemma 6 *Let Q be a commutative, idempotent quasigroup of order $2t+1$ on the set $\{1, 2, ..., 2t+1\}$ (respectively let Q be a commutative quasigroup with holes $\{2i-1, 2i\}$ for $1 \le i \le t$ on the set $\{1, 2, ..., 2t\}$). Then there exists a $G_1-GDD(6, 2t+1, 1)$, (respectively a $G_1-GDD(12, t, 1)$).*

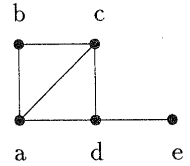

Figure 8: Block $(a, b, c, d) - e$ for $K_4 - e \cup \{e\}$ (type I)

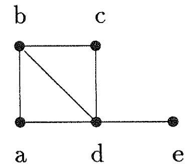

Figure 9: Block $(a, b, c, d) - e$ for $K_4 - e \cup \{e\}$ (type II)

Proof. First we will define $6t(2t+1)$ (respectively, $6t(2t-2)$) $K_4 - e$'s and the same number of sticks. Then we will match them up to form copies of G_1.

For each unordered pair $\{i, j\}$, $1 \leq i, j \leq 2t+1$, i and j not in the same hole of Q, define $((i, a)(i \circ j, a+1)(j, a)(i \circ j, a+2))$ to be a $K_4 - e$ for $a \in \mathbb{Z}_6$. There are $t(2t+1)$ (respectively, $t(2t-2)$) such unordered pairs for each value of a and thus $6t(2t+1)$ (respectively, $6t(2t-2)$) $K_4 - e$'s all together.

For each unordered pair $\{k, m\}$, $1 \leq k, m \leq 2t+1$, k and m not in the same hole of Q, define $((k, a), (m, a+3))$ to be a stick for $a \in \mathbb{Z}_6$. There are $t(2t+1)$ (respectively, $t(2t-2)$) such unordered pairs for each value of a and thus $6t(2t+1)$ (respectively, $6t(2t-2)$) sticks all together.

Form a bipartite graph $B(F, S)$ where F is the set of $K_4 - e$'s defined above and S is the set of sticks defined above. Let B contain the edge (f, s) if and only if the $K_4 - e$, f, and the stick, s, have exactly one of the vertices of degree 2 of f in common.

Each vertex of degree 2 of the $K_4 - e$, f, is in $2t$ (respectively, $2t-2$) unordered pairs with vertices that are not in the same hole. Therefore, each vertex of degree 2 of the $K_4 - e$, f, is incident with $2t$ (respectively, $2t-2$) sticks so each vertex, f of the bipartite graph B has degree $4t$ (respectively, $2(2t-2)$).

Likewise, each vertex of the stick, s, is in $2t$ (respectively, $2t-2$) unordered pairs with vertices that are not in the same hole. Therefore, each vertex of s is incident with $2t$ (respectively, $2t-2$) $K_4 - e$'s. Thus each vertex, s of the bipartite graph B has degree $4t$ (respectively, $2(2t-2)$).

The Marriage Theorem states that every regular bipartite graph has a perfect matching [3]; therefore, there is a perfect matching of B. This matching will pair up the $K_4 - e$'s with the remaining edges as sticks in order to form the G_1-GDD$(6, 2t+1, 1)$ (respectively, G_1-GDD$(12, t, 1)$). $\square$

Lemma 7 *Let Q be a commutative, idempotent quasigroup of order $2t+1$ on the set $\{1,2,...,2t+1\}$. Then the following blocks form a $G_2-GDD(6,2t+1,1)$: $((i \circ j, a+1)(i,a)(i \circ j, a+2)(j,a)) - (i, a+3)$ for each unordered pair $\{i,j\}$, $1 \leq i < j \leq 2t+1$, i and j not in the same hole of Q and for $a \in \mathbb{Z}_6$.*

Lemma 8 *Let Q be a commutative quasigroup of order $2t$ with holes $\{2i-1, 2i\}$ for $1 \leq i \leq t$ on the set $\{1,2,...,2t\}$. Then the following blocks form a $G_2-GDD(12,t,1)$: $((i \circ j, a+1)(i,a)(i \circ j, a+2)(j,a)) - (i, a+3)$ for each unordered pair $\{i,j\}$, $1 \leq i < j \leq 2t$, i and j not in the same hole of Q and for $a \in \mathbb{Z}_6$.*

Proof of Theorem 7. The proof will be in 4 cases.

Case 1: $n \equiv 0, 1, 4, 9 \pmod{12}$.

When $n \equiv 0, 1, 4, 9 \pmod{12}$ the theorem implies no restriction on λ so it suffices to find designs for $\lambda = 1$.

The case for $\lambda = 1$ was completely solved by Bermond, et al. [2].

Case 2: $n \equiv 3, 6, 7, 10 \pmod{12}$.

When $n \equiv 3, 6, 7, 10 \pmod{12}$ the theorem implies that $2|\lambda$ so it suffices to find designs for $\lambda = 2$.

First consider the case where $n \equiv 3 \pmod{12}$.

Let $V = \{1,2,...,2t\} \times \{1,2,3,4,5,6\} \cup \{\infty_1, \infty_2, \infty_3\}$. On the points (i,a) for $1 \leq i \leq 2t$ and $a \in \mathbb{Z}_6$ place 2 copies of a GDD(12,t,1) where the points $(2j-1, b)$ and $(2j, b)$ for $b \in \mathbb{Z}_6$ form the groups for $1 \leq j \leq t$. On the first group and the points $\infty_1, \infty_2, \infty_3$ place a G-design on $2K_{15}$. On all subsequent groups and the points $\infty_1, \infty_2, \infty_3$ place 2 copies of a G-design on $K_{15} \backslash K_3$.

This design works for $t \geq 3$. A design for $t = 0$ ($2K_3$) does not exist. Designs for $t = 1$ ($2K_{15}$) and $t = 2$ ($2K_{27}$) can be found in the appendix.

Now consider the case where $n \equiv 10 \pmod{12}$.

Let $V = \{1,2,...,2t+1\} \times \{1,2,3,4,5,6\} \cup \{\infty_1, ..., \infty_4\}$. On the points (i,a) for $1 \leq i \leq 2t+1$ and $a \in \mathbb{Z}_6$ place 2 copies of a GDD(6,$2t+1$,1). On the first group and the points $\infty_1, \infty_2, \infty_3, \infty_4$ place a G-design on $2K_{10}$. On all subsequent groups and the points $\infty_1, \infty_2, \infty_3, \infty_4$ place a copy of a G-design on $2K_{10} \backslash 2K_4$.

Finally consider the case where $n \equiv 6, 7 \pmod{12}$.

Let $V = \{1,2,...,2t+1\} \times \{1,2,3,4,5,6\}$ (together with ∞ for the 7 (mod 12) case). On the points (i,a) for $1 \leq i \leq 2t+1$ and $a \in \mathbb{Z}_6$ place 2 copies of a GDD(6,$2t+1$,1). On each of the groups (and the point ∞ in the 7(mod 12) case) place a G-design on $2K_6$ or $2K_7$.

Case 3: $n \equiv 5, 8 \pmod{12}$.

When $n \equiv 5, 8 \pmod{12}$ the theorem implies that $3|\lambda$ so it suffices to find designs for $\lambda = 3$.

Let $V = \{1,2,...,2t\} \times \{1,2,3,4,5,6\} \cup \{\infty_1,...,\infty_x\}$ for $x \in \{5,8\}$. On the points (i,a) for $1 \leq i \leq 2t$ and $a \in \mathbb{Z}_6$ place 3 copies of a GDD(12,t,1) where the points $(2j-1,b)$ and $(2j,b)$ for $b \in \mathbb{Z}_6$ form the groups for $1 \leq j \leq t$. On the points $\infty_1,...,\infty_x$ place a G-design on $3K_x$. On each group and the points $\infty_1,...,\infty_x$ place 3 copies of a G-design on $K_{12+x}\backslash K_x$. This design works for $t \geq 3$. Designs for $t=0$ ($3K_5$ and $3K_8$), $t=1$ ($3K_{17}$ and $3K_{20}$) and $t=2$ ($3K_{29}$ and $3K_{32}$) can be found in the appendix.

Case 4: $n \equiv 2, 11 (\text{mod } 12)$.

When $n \equiv 2, 11\,(\text{mod } 12)$ the theorem implies that $6|\lambda$ so it suffices to find designs for $\lambda = 6$.

First consider the case when $n \equiv 2\,(\text{mod } 12)$

Let $V = \{1,2,...2t\} \times \{1,2,3,4,5,6\} \cup \{\infty_1,\infty_2\}$. On the points (i,a) for $1 \leq i \leq 2t$ and $a \in \mathbb{Z}_6$ place 6 copies of a GDD(12,t,1) where the points $(2j-1,b)$ and $(2j,b)$ for $b \in \mathbb{Z}_6$ form the groups for $1 \leq j \leq t$. On the first group and the points ∞_1,∞_2 place a G-design on $6K_{14}$. On all subsequent groups and the points ∞_1,∞_2 place 6 copies of a G-design on $K_{14}\backslash K_2$. This design works for $t \geq 3$. A design when $t=0$ ($6K_2$) clearly does not exist. Designs for $t=1$ ($6K_{14}$) and $t=2$ ($6K_{26}$) can be found in the appendix.

Finally consider the case where $n \equiv 11\,(\text{mod } 12)$

Let $V = \{1,2,...,2t+1\} \times \{1,2,3,4,5,6\} \cup \{\infty_1,...,\infty_5\}$. On the points (i,a) for $1 \leq i \leq 2t+1$ and $a \in \mathbb{Z}_6$ place 6 copies of a GDD(6,$2t+1$,1). On the points $\infty_1,...,\infty_5$ place 2 copies of a G-design on $3K_5$. On each group and the points $\infty_1,...,\infty_5$ place 3 copies of a G-design on $2K_{11}\backslash 2K_5$. $\square$

10 $K_{2,3}$

Theorem 8 *Let $G = K_{2,3}$ (see Figure 10). Then a G-design on λK_n exists if and only if* $12|\lambda n\,(n-1)$*, $n \geq 5$, except K_9 and K_{12}.*

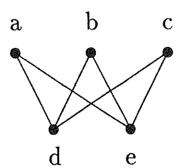

Figure 10: Block (a,b,c)-(d,e)

We denote the copy of G in Figure 10 above by (a,b,c)-(d,e), (b,c,a)-(d,e),...,etc.

We will need the following Lemma in the proof of Theorem 8.

Lemma 9 *Let $G = K_{2,3}$. If a G-design on λK_5 exists then $6|\lambda$.*

Proof. Let $c_1, ..., c_{10}$ be the $K_{2,3}$ subgraphs of K_5 and let $e_1, ..., e_{10}$ denote the edges of K_5. Let $A = [i,j]$ be a square matrix such that

$$a_{i,j} = \begin{cases} 1 & \text{if edge } i \text{ is used in } c_j \\ 0 & \text{otherwise} \end{cases}$$

Let $D = [d_i]$ be a column matrix with d_i being the number of times c_i is used in a G-design on λK_5. Let $\Lambda = [\lambda_i]$ be a column matrix with $\lambda_i = \lambda$.

Then $AD = \Lambda$ since each edge is used λ times. Let e_i be the edge between vertices a and b of the K_5. Then e_i must be contained in exactly 6 c_i's since both vertex a and vertex b must be in the partition of size two with vertices c, d and e. Since each edge is in 6 c_i's, A has 6 ones in each row. Let $d_i = \lambda/6$. This is certainly a solution to the equation $AD = \Lambda$. However, $|\det(A)| = 196$, so D is a unique solution. Therefore $6|\lambda$. □

Proof of Theorem 8. The proof will contain 4 cases.

Case 1: $n \equiv 0, 1, 4, 9 (\text{mod } 12)$.

When $n \equiv 0, 1, 4, 9 \pmod{12}$ the theorem implies no restriction on λ except for K_9 and K_{12} so it suffices to find designs for $\lambda = 1$ when $n > 12$, $2K_9$, $3K_9$, $2K_{12}$ and $3K_{12}$.

The case where $\lambda = 1$ was completely solved by Bermond, et al. [2]. designs on K_9 and K_{12} were proven not to exist. designs on $2K_9$, $3K_9$, $2K_{12}$ and $3K_{12}$ can be found in the appendix.

Case 2: $n \equiv 3, 6, 7, 10 (\text{mod } 12)$.

When $n \equiv 3, 6, 7, 10 \pmod{12}$ the theorem implies that $2|\lambda$ so it suffices to find designs for $\lambda = 2$.

Hoffman and Liatti have shown that a G-design on $K_{c,d}$ exists if $c, d \geq 2$ and $6|cd$ [12]. Therefore, there is a G-design on $K_{a,12t}$ for all $a \geq 2$ and $t \geq 1$.

Let $n = 12t + x$ where $x \in \{6, 7, 10, 15\}$ and let $V = \{(i,1) : 1 \leq i \leq x\} \cup \{(j,2) : 1 \leq j \leq 12t\}$. Define the blocks of the G-design as follows:

Type 1: On the points $(i,1)$, $1 \leq i \leq x$ (hereafter known as level 1), place the appropriate G-design on $2K_x$. On the points $(j,2)$, $1 \leq j \leq 12t$ (hereafter known as level 2), place 2 G-designs on K_{12t} found in Case 1.

Type 2: On the edges between the levels place 2 copies of a design on $K_{x,12t}$ [12].

Case 3: $n \equiv 8 (\text{mod } 12)$.

When $n \equiv 8 \pmod{12}$ the theorem implies that $3|\lambda$ so it suffices to find designs for $\lambda = 3$.

Let $n = 12t + 8$ and let $V = \{(i,1) : 1 \leq i \leq 8\} \cup \{(j,2) : 1 \leq j \leq 12t\}$. Define the blocks of the G-design as follows:

Type 1: On the points $(i,1)$, $1 \leq i \leq 8$ (hereafter known as level 1), place the G-design on $3K_8$. On the points $(j,2)$, $1 \leq j \leq 12t$ (hereafter known as level 2), place 3 G-designs on K_{12t} found in Case 1.

Type 2: On the edges between the levels place 3 copies of a design on $K_{x,12t}$ [12].

Case 4: $n \equiv 2, 5, 11 \pmod{12}$.

When $n \equiv 2, 5, 11 \pmod{12}$ Theorem 8 and Lemma 9 imply that $6|\lambda$ so it suffices to find designs for $\lambda = 6$.

Let $n = 12t + x$ where $x \in \{5, 11, 14\}$ and let $V = \{(i,1) : 1 \leq i \leq x\} \cup \{(j,2) : 1 \leq j \leq 12t\}$. Define the blocks of the G-design as follows:

Type 1: On the points $(i,1)$, $1 \leq i \leq x$ (hereafter known as level 1), place the appropriate G-design on $6K_x$. On the points $(j,2)$, $1 \leq j \leq 12t$ (hereafter known as level 2), place 6 G-designs on K_{12t} found in Case 1.

Type 2: On the edges between the levels place 6 copies of a design on $K_{x,12t}$ [12].

□

References

[1] J.-C. Bermond, and J. Schönheim, G-Decomposition of K_n Where G Has Four Vertices or Less. *Discrete Mathematics* 19 (1977) 113–120.

[2] J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau, Decomposition of Complete Graphs into Isomorphic Subgraphs with Five Vertices. *Ars Combinatoria* 10 (1980) 211–254.

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* North-Holland Publishing Company, New York (1979).

[4] D. E. Bryant, D. G. Hoffman and C. A. Rodger, 5-cycle systems with holes. *Designs, Codes and Cryptography* 8 (1996) 103–108.

[5] D. E. Bryant, C. A. Rodger and E. R. Spicer, Embedding m-cycle systems, and incomplete m-cycle systems: $m \leq 14$. *Discrete Mathematics* to appear.

[6] D. E. Bryant, C. A, Rodger, The Doyen-Wilson Theorem extended to 5-cycles. *Journal of Combinatorial Theory (A)* 68 (1994) 218–225.

[7] D. E. Bryant, C. A. Rodger, On the Doyen-Wilson Theorem for m-cycles. *Journal of Combinatorial Design* 2 (1994) 253–271.

[8] J. Doyen and R. M. Wilson, Embeddings of Steiner Triple Systems. *Discrete Mathematics* 5 (1973) 229-239.

[9] H. Hanani, The existence and construction of Balanced Incomplete Block Designs. *Annals of Mathematical Statistics* 32 (1961) 361–386.

[10] D. G. Hoffman, Unpublished Manuscript.

[11] D.G. Hoffman, K.S. Kirkpatrick, Another Doyen-Wilson Theorem. submitted *Ars Combinatoria.*

[12] D. G. Hoffman and M. Liatti, Bipartite Designs. *Journal of Combinatorial Designs* 3 (1995) 499–454.

[13] Rev. T. P. Kirkman, "On a Problem in Combinatorics". *Cambridge and Dublin Mathematical Journal* 2 (1847) 191–204.

[14] A. Kotzig, On the Decomposition of Complete Graphs into k-gons. *Mat. Fyz. Casop.* 15 (1965) 229–233.

[15] C. C. Linder and C. A. Rodger, *Design Theory* CRC Press, To Appear.

[16] M. Tarsi, Decomposition of Complete Multigraphs into Stars. *Discrete Mathematics* 26 (1979) 273–278.

[17] M. Tarsi, Decomposition of a complete multigraph into simple paths. *Journal of Combinatorial Theory* A 34 (1983), no. 1, 60–70.

APPENDIX

A Triangle and a Stick

- $2K_4$ on the vertex set $\mathbb{Z}_4$: $(4,2,1)-3$ $(2,1,3)-4$ $(3,2,4)-1$
- $2K_5$ on the vertex set $\mathbb{Z}_5$. Here is a base block to be developed cyclically (mod 5): $(0,1,3)-4$
- $4K_6$ on the vertex set $\mathbb{Z}_5 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 5): $(\infty,0,1)-3$ $(\infty,0,1)-3$ $(0,1,3)-4$
- $4K_{10}$ on the vertex set $\mathbb{Z}_9 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 9): $(0,1,3)-\infty$ $(0,1,3)-\infty$ $(0,1,4)-\infty$ $(0,4,5)-7$ $(0,2,6)-\infty$.
- $4K_6/4K_2$ on the vertex set $\mathbb{Z}_4 \cup \{\infty_1,\infty_2\}$ where ∞_1 and ∞_2 are the points in the hole: $(0,2,1)-3$ $(0,2,3)-1$ and here are some base blocks to be developed cyclically (mod 4): $(0,3,\infty_1)-2$ $(0,\infty_2,1)-\infty_1$ $(3,\infty_2,2)-0$

B 3-Path and a Stick

- $2K_5$ on the vertex set $\mathbb{Z}_5$. Here is a base block to be developed cyclically (mod 5): $(1,2,0)-[4,3]$
- $2K_{12}$ on the vertex set $\mathbb{Z}_{11} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 11): $(\infty,3,0)-[1,2]$ $(10,5,0)-[3,4]$ $(\infty,4,0)-[1,2]$

- $4K_6$ on the vertex set $\mathbb{Z}_5 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 5): $(\infty,1,0)-[2,4]$ $(1,4,\infty)-[2,3]$ $(4,3,0)-[2,1]$
- $4K_7$ on the vertex set $\mathbb{Z}_7$. Here are the base blocks to be developed cyclically (mod 7): $(4,3,0)-[1,2]$ $(3,5,0)-[1,4]$ $(6,3,0)-[1,2]$
- $K_{4,2}$ on the vertex set $\mathbb{Z}_4 \cup \{a,b\} : (a,0,b)-[2,3]$ $(b,1,a)-[2,3]$.
- $K_{4,4}$ on the vertex set $\mathbb{Z}_4 \times \mathbb{Z}_2$. Here is a base block to be developed cyclically (mod (4,–)): $((1,0),(2,1),(0,0))-[(0,1),(3,1)]$

C Triangle and a 2-Path

- $2K_5$ on the vertex set $\mathbb{Z}_5$: $(0,1,3)-(2,4)$ $(0,2,4)-(1,3)$ $(0,4,3)-(2,1)$ $(0,2,1)-(4,3)$
- $3K_5$ on the vertex set $\mathbb{Z}_5$: $(0,1,3)-(2,4)$ $(0,3,4)-(1,2)$ $(1,2,3)-(0,4)$ $(3,4,2)-(0,1)$ $(0,2,1)-(4,3)$ $(0,2,4)-(1,3)$
- $5K_7$ on the vertex set $\mathbb{Z}_7$. Here are the base blocks to be developed cyclically (mod 7): $(0,1,3)-(2,4)$ $(0,1,3)-(5,2)$ $(0,1,3)-(2,5)$
- $5K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $(0,1,\infty)-(2,3)$ $(0,1,3)-(\infty,2)$ $(0,1,3)-(5,2)$ $(0,1,3)-(5,2)$
- $5K_9$ on the vertex set $\mathbb{Z}_9$. Here are the base blocks to be developed cyclically (mod 9): $(0,1,3)-(7,6)$ $(0,1,3)-(7,5)$ $(0,1,3)-(7,4)$ $(0,1,3)-(7,2)$
- $5K_{14}$ on the vertex set $\mathbb{Z}_{13} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 13): $(0,1,3)-(5,2)$ $(0,5,11)-(8,9)$ $(0,1,4)-(8,7)$ $(0,1,6)-(3,8)$ $(0,2,6)-(11,9)$ $(0,4,\infty)-(1,7)$ $(0,\infty,5)-(11,7)$
- $5K_{24}$ on the vertex set $\mathbb{Z}_{23} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 23): $(0,3,10)-(9,20)$ $(0,5,13)-(10,8)$ $(0,1,8)-(4,7)$ $(0,2,10)-(5,9)$ $(0,2,6)-(12,7)$ $(0,1,\infty)-(2,8)$ $(0,1,3)-(\infty,2)$ $(0,4,9)-(2,10)$ $(0,6,11)-(3,10)$ $(0,9,11)-(2,12)$ $(0,3,9)-(19,10)$ $(0,4,11)-(10,21)$
- K_7/K_2 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1,\infty_2\}$ where the ∞'s are the points in the hole: $(3,4,\infty_1)-(2,\infty_2)$ $(\infty_2,4,1)-(\infty_1,0)$ $(2,4,0)-(3,\infty_2)$ $(2,3,1)-(0,\infty_2)$
- K_8/K_3 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1,\infty_2,\infty_3\}$ where the ∞'s are the points in the hole: $(3,4,\infty_1)-(2,\infty_2)$ $(\infty_2,4,0)-(\infty_1,1)$ $(0,\infty_3,1)-(\infty_2,3)$ $(2,4,\infty_3)-(3,1)$ $(0,3,2)-(1,4)$
- K_9/K_4 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$ where the ∞'s are the points in the hole: $(1,\infty_2,3)-(\infty_1,4)$ $(1,\infty_1,0)-(2,\infty_2)$ $(0,\infty_2,4)-(2,\infty_1)$ $(3,\infty_3,0)-(\infty_4,1)$ $(\infty_3,2,1)-(4,\infty_4)$ $(\infty_4,2,3)-(4,\infty_3)$

- K_{12}/K_7 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, ..., \infty_7\}$ where the ∞'s are the points in the hole: $(4, \infty_1, 3) - (\infty_2, 2)$ $(0, 2, \infty_1) - (1, \infty_2)$ $(4, \infty_2, 0) - (\infty_3, 2)$ $(4, \infty_3, 1) - (\infty_4, 2)$ $(1, 2, \infty_5) - (3, \infty_3)$ $(3, \infty_4, 0) - (\infty_5, 4)$ $(0, 1, \infty_6) - (4, \infty_4)$ $(2, \infty_6, 3) - (\infty_7, 0)$ $(2, 4, \infty_7) - (1, 3)$
- K_{13}/K_8 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, ..., \infty_8\}$ where the ∞'s are the points in the hole: $(4, \infty_1, 3) - (\infty_2, 2)$ $(0, 2, \infty_1) - (1, \infty_2)$ $(1, \infty_3, 0) - (\infty_2, 4)$ $(\infty_3, 4, 2) - (\infty_4, 0)$ $(3, \infty_4, 1) - (\infty_5, 2)$ $(0, 3, \infty_6) - (4, \infty_4)$ $(0, 4, \infty_5) - (3, \infty_3)$ $(1, \infty_6, 2) - (\infty_7, 4)$ $(2, \infty_8, 3) - (\infty_7, 1)$ $(1, 4, \infty_8) - (0, \infty_7)$
- K_{14}/K_4 on the vertex set $\mathbb{Z}_{10} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ where the ∞'s are the points in the hole: $(\infty_1, 9, 5) - (2, \infty_2)$ $(1, 0, 6) - (3, \infty_2)$ $(2, \infty_1, 7) - (4, \infty_2)$ $(3, 1, 8) - (5, \infty_2)$ $(4, 2, 9) - (6, \infty_2)$ $(5, 3, 0) - (7, \infty_2)$ $(6, 4, \infty_1) - (8, \infty_2)$ $(7, 5, 1) - (9, \infty_2)$ $(8, 6, 2) - (0, \infty_2)$ $(0, 8, 4) - (1, \infty_2)$ $(1, 2, \infty_3) - (6, 7)$ $(4, 5, \infty_3) - (0, \infty_4)$ $(8, \infty_3, 7) - (3, \infty_1)$ $(\infty_3, 9, 3) - (4, \infty_4)$ $(5, 6, \infty_4) - (1, \infty_1)$ $(\infty_4, 8, 9) - (0, \infty_1)$ $(2, 3, \infty_4) - (7, 9)$

D Triangle and a 2-Star

- $2K_5$ on the vertex set $\mathbb{Z}_5$: $(0, 4, 2) - 1, 3$ $(1, 4, 3) - 2, 0$ $(3, 4, 0) - 2, 1$ $(2, 4, 1) - 0, 3$
- $3K_5$ on the vertex set $\mathbb{Z}_5$: $(4, 1, 0) - 2, 3$ $(4, 3, 1) - 0, 2$ $(3, 1, 2) - 0, 4$ $(2, 4, 3) - 0, 1$ $(1, 2, 4) - 0, 3$ $(2, 3, 0) - 1, 4$
- $5K_7$ on the vertex set $\mathbb{Z}_7$. Here are the base blocks to be developed cyclically (mod 7): $(0, 1, 3) - 4, 5$ $(0, 1, 3) - 5, 6$ $(0, 1, 3) - 4, 6$
- $5K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $(0, 1, \infty) - 2, 3$ $(0, 1, 3) - \infty, 4$ $(0, 1, 3) - 5, 6$ $(0, 1, 3) - 5, 6$
- $5K_9$ on the vertex set $\mathbb{Z}_9$. Here are the base blocks to be developed cyclically (mod 9): $(0, 1, 3) - 4, 7$ $(0, 1, 3) - 7, 5$ $(0, 1, 3) - 7, 6$ $(0, 1, 3) - 7, 8$
- $5K_{14}$ on the vertex set $\mathbb{Z}_{13} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 13): $(0, 1, 3) - 8, 9$ $(0, 2, 5) - \infty, 11$ $(0, 1, 4) - \infty, 9$ $(0, 1, 6) - 8, 2$ $(0, 2, 6) - 5, \infty$ $(0, 6, 9) - \infty, 7$ $(0, 1, 5) - \infty, 8$
- $5K_{24}$ on the vertex set $\mathbb{Z}_{23} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 23): $(0, 3, 10) - 9, 4$ $(0, 5, 13) - 9, 2$ $(0, 1, 8) - 19, 2$ $(0, 2, 10) - 6, 1$ $(0, 2, 6) - 13, 1$ $(0, 1, \infty) - 2, 3$ $(0, 1, 3) - \infty, 13$ $(0, 4, 9) - 7, 1$ $(0, 6, 11) - 10, 3$ $(0, 9, 11) - 14, 1$ $(0, 3, 9) - 4, 2$ $(0, 4, 11) - 8, 2$
- K_7/K_2 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, \infty_2\}$ where the ∞'s are the points in the hole: $(1, 4, \infty_1) - 0, 3$ $(0, 1, \infty_2) - 2, 3$ $(1, 3, 2) - 0, \infty_1$ $(0, 3, 4) - 2, \infty_2$
- K_8/K_3 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3\}$ where the ∞'s are the points in the hole: $(\infty_1, 3, 4) - 0, \infty_3$ $(\infty_1, 0, 2) - 4, \infty_3$ $(\infty_2, 4, 1) - 2, \infty_1$ $(\infty_2, 2, 3) - 1, \infty_3$ $(\infty_3, 1, 0) - 3, \infty_2$

- K_9/K_4 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ where the ∞'s are the points in the hole: $(2,1,0) - \infty_2, \infty_1$ $(4,3,0) - \infty_3, \infty_4$ $(\infty_4, 2, 4) - \infty_3, \infty_2$ $(\infty_1, 4, 1) - \infty_4, \infty_3$ $(\infty_2, 1, 3) - \infty_1, \infty_4$ $(\infty_3, 3, 2) - \infty_1, \infty_2$
- K_{12}/K_7 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, ..., \infty_7\}$ where the ∞'s are the points in the hole: $(2,4,\infty_1) - 0,3$ $(0,2,\infty_2) - 3,4$ $(\infty_7, 3, 1) - \infty_1, \infty_2$ $(0,3,\infty_3) - 1,2$ $(1,2,\infty_4) - 0,3$ $(1, \infty_5, 4) - \infty_3, \infty_4$ $(\infty_5, 3, 2) - \infty_6, \infty_7$ $(1, \infty_6, 0) - \infty_5, \infty_7$ $(\infty_6, 3, 4) - \infty_7, 0$
- K_{13}/K_8 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, ..., \infty_8\}$ where the ∞'s are the points in the hole: $(0,3,\infty_2) - 2,4$ $(2,3,\infty_3) - 0,4$ $(\infty_1, 3, 1) - \infty_2, \infty_3$ $(0,1,\infty_4) - 2,3$ $(1,2,\infty_5) - 0,3$ $(2,\infty_1,4) - \infty_4, \infty_5$ $(\infty_7, 4, 0) - \infty_1, \infty_8$ $(\infty_6, 4, 1) - \infty_7, \infty_8$ $(\infty_6, 0, 2) - \infty_7, \infty_8$ $(\infty_8, 4, 3) - \infty_6, \infty_7$
- K_{14}/K_4 on the vertex set $\mathbb{Z}_{10} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ where the ∞'s are the points in the hole: $(\infty_1, 9, 5) - 2, \infty_2$ $(1,0,6) - 3, \infty_2$ $(2, \infty_1, 7) - 4, \infty_2$ $(3,1,8) - 5, \infty_2$ $(4,2,9) - 6, \infty_2$ $(5,3,0) - 7, \infty_2$ $(6,4,\infty_1) - 8, 0$ $(7,5,1) - 9, \infty_2$ $(8,6,2) - 0, \infty_2$ $(9,7,3) - \infty_1, \infty_2$ $(0,8,4) - 1, \infty_2$ $(\infty_3, 2, 1) - \infty_1, \infty_4$ $(4,5,\infty_3) - 0,8$ $(\infty_3, 6, 7) - 8, \infty_4$ $(\infty_4, 2, 3) - 4, \infty_3$ $(5,6,\infty_4) - 4,8$ $(\infty_4, 0, 9) - 8, \infty_3$

E Bull

- $5K_7$ on the vertex set $\mathbb{Z}_7$. Here are the base blocks to be developed cyclically (mod 7): $6 - (0,1,3) - 5$ $4 - (0,1,3) - 2$ $4 - (0,1,3) - 5$
- $5K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $3 - (0,\infty,1) - 4$ $3 - (0,2,\infty) - 1$ $2 - (0,1,3) - 4$ $2 - (0,1,3) - 4$
- $5K_9$ on the vertex set $\mathbb{Z}_9$. Here are the base blocks to be developed cyclically (mod 9): $4 - (0,1,3) - 2$ $4 - (0,1,3) - 5$ $4 - (0,1,3) - 6$ $4 - (0,1,3) - 7$
- $5K_{14}$ on the vertex set $\mathbb{Z}_{13} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 13): $6 - (0,1,3) - 8$ $4 - (0,2,6) - 10$ $2 - (0,1,4) - 6$ $3 - (0,1,5) - 8$ $1 - (0,5,11) - 10$ $6 - (0,\infty,3) - 8$ $6 - (0,5,\infty) - 1$
- $5K_{24}$ on the vertex set $\mathbb{Z}_{23} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 23): $6 - (0,3,10) - 9$ $6 - (0,5,13) - 12$ $7 - (0,1,8) - 6$ $7 - (0,2,10) - 13$ $8 - (0,2,6) - 3$ $8 - (0,1,\infty) - 2$ $9 - (0,1,3) - \infty$ $11 - (0,4,9) - \infty$ $9 - (0,6,11) - 7$ $10 - (0,9,11) - 6$ $11 - (0,3,9) - 5$ $10 - (0,4,11) - 16$
- K_7/K_2 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, \infty_2\}$ where the ∞'s are the points in the hole: $\infty_1 - (3,2,4) - 0$ $\infty_2 - (3,0,1) - 2$ $\infty_2 - (4,1,\infty_1) - 2$ $\infty_1 - (0,2,\infty_2) - 1$
- K_8/K_3 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3\}$ where the ∞'s are the points in the hole: $3 - (\infty_1, 0, 4) - 2$ $4 - (\infty_2, 3, 2) - 0$ $0 - (\infty_3, 3, 4) - 1$ $\infty_2 - (1, \infty_1, 2) - \infty_3$ $\infty_2 - (0,3,1) - \infty_3$

- K_9/K_4 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ where the ∞'s are the points in the hole: $4-(\infty_1,1,0)-\infty_2$ $4-(2,\infty_1,3)-\infty_2$ $2-(\infty_2,4,1)-\infty_3$ $\infty_4-(3,0,\infty_3)-4$ $3-(1,\infty_4,2)-\infty_3$ $3-(4,\infty_4,0)-2$
- K_{12}/K_7 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, ..., \infty_7\}$ where the ∞'s are the points in the hole: $4-(\infty_1,3,0)-\infty_2$ $\infty_2-(2,\infty_1,1)-\infty_3$ $\infty_3-(4,3,\infty_2)-1$ $0-(\infty_3,2,3)-\infty_4$ $2-(\infty_4,4,1)-\infty_5$ $\infty_4-(0,\infty_7,4)-\infty_5$ $3-(\infty_5,2,0)-\infty_6$ $1-(\infty_6,4,2)-\infty_7$ $0-(1,\infty_7,3)-\infty_6$
- K_{13}/K_8 on the vertex set $\mathbb{Z}_5 \cup \{\infty_1, ..., \infty_8\}$ where the ∞'s are the points in the hole: $\infty_2-(3,4,\infty_1)-0$ $\infty_2-(2,\infty_1,1)-\infty_3$ $\infty_3-(4,0,\infty_2)-1$ $3-(\infty_3,2,0)-\infty_4$ $4-(\infty_4,3,1)-\infty_5$ $\infty_4-(2,\infty_8,4)-\infty_5$ $0-(\infty_5,3,2)-\infty_6$ $3-(\infty_6,1,4)-\infty_7$ $\infty_6-(0,\infty_8,1)-\infty_7$ $2-(\infty_7,0,3)-\infty_8$
- K_{14}/K_4 on the vertex set $\mathbb{Z}_{10} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ where the ∞'s are the points in the hole: $0-(\infty_1,9,5)-2$ $\infty_2-(1,0,6)-3$ $\infty_2-(2,\infty_1,7)-4$ $\infty_2-(3,1,8)-5$ $\infty_2-(4,2,9)-6$ $\infty_2-(5,3,0)-7$ $\infty_2-(6,4,\infty_1)-8$ $\infty_2-(7,5,1)-9$ $\infty_2-(8,6,2)-0$ $\infty_2-(9,7,3)-\infty_1$ $\infty_2-(0,8,4)-1$ $9-(\infty_3,2,1)-\infty_1$ $3-(4,5,\infty_3)-0$ $6-(\infty_3,7,8)-\infty_4$ $1-(\infty_4,2,3)-\infty_3$ $7-(6,5,\infty_4)-4$ $7-(\infty_4,0,9)-8$

F Square and a Stick

- $2K_5$ on the vertex set $\mathbb{Z}_5$: $(0,1,2,3)-4$ $(0,2,1,4)-3$ $(0,4,2,3)-1$ $(0,2,4,1)-3$
- $3K_5$ on the vertex set $\mathbb{Z}_5$: $(0,1,2,3)-4$ $(1,3,0,4)-2$ $(1,0,2,3)-4$ $(4,0,3,2)-1$ $(2,0,1,4)-3$ $(4,0,2,1)-3$
- $5K_7$ on the vertex set $\mathbb{Z}_7$. Here are the base blocks to be developed cyclically (mod 7): $(0,1,3,4)-6$ $(0,2,5,3)-6$ $(0,1,2,3)-5$
- $5K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $(0,1,3,4)-\infty$ $(0,2,5,3)-6$ $(0,1,2,3)-\infty$ $(0,2,4,\infty)-1$
- $5K_9$ on the vertex set $\mathbb{Z}_9$. Here are the base blocks to be developed cyclically (mod 9): $(0,1,3,4)-6$ $(0,3,7,8)-5$ $(0,2,5,3)-7$ $(0,1,5,2)-6$
- $K_{2,5}$ on the vertex set $\{1,2\} \cup \{a,b,c,d,e\}$: $(a,2,b,1)-e$ $(d,1,c,2)-e$
- $K_{5,5}$ on the vertex set $\mathbb{Z}_5 \times \mathbb{Z}_2$. Here is a base block to be developed cyclically (mod (5,-)): $((0,1)\,(0,0)\,(1,1)\,(3,0))-(2,1)$

G $\mathbf{K_4-e}$ and a Stick (type I)

- $2K_6$ on the vertex set $\mathbb{Z}_5 \cup \{\infty\}$. Here is a base block to be developed cyclically (mod 5): $(0,\infty,1,2)-4$

- $2K_7$ on the vertex set $\mathbb{Z}_7$. Here is a base block to be developed cyclically (mod 7): $(0,1,3,4)-2$
- $2K_{10}$ on the vertex set $\mathbb{Z}_5 \times \mathbb{Z}_2$. Here are the base blocks to be developed cyclically (mod (5,–)): $((1,0)\,(2,1)\,(0,1)\,(0,0))-(2,0)$
$((0,1)\,(0,0)\,(2,0)\,(1,1))-(3,1)$ $((1,1)\,(4,0)\,(2,1)\,(0,0))-(1,0)$
- $2K_{15}$ on the vertex set $\mathbb{Z}_7 \times \mathbb{Z}_2 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod (7,–)): $((0,1)\,\infty\,(0,0)\,(2,1))-(3,1)$
$((0,0)\,(1,0)\,(6,1)\,(4,1))-(1,1)$ $((0,0)\,(1,1)\,(2,1)\,(6,0))-(5,1)$
$((0,0)\,(1,1)\,(5,1)\,(2,0))-(5,0)$ $((0,1)\,\infty\,(0,0)\,(3,0))-(5,0)$
- $2K_{27}$ on the vertex set $\mathbb{Z}_9 \times \mathbb{Z}_3$. Here are the base blocks to be developed cyclically (mod (9,–)): $((0,1)\,(1,2)\,(3,0)\,(7,0))-(3,2)$
$((0,1)\,(2,1)\,(6,2)\,(1,0))-(6,1)$ $((0,2)\,(3,2)\,(2,0)\,(3,0))-(4,1)$
$((0,1)\,(1,1)\,(4,1)\,(4,2))-(2,2)$ $((0,1)\,(2,0)\,(2,2)\,(6,2))-(6,1)$
$((8,1)\,(8,0)\,(0,2)\,(2,2))-(3,1)$ $((0,0)\,(2,0)\,(4,2)\,(1,1))-(4,1)$
$((0,0)\,(3,0)\,(2,2)\,(3,1))-(7,1)$ $((0,2)\,(1,2)\,(6,0)\,(2,1))-(7,2)$
$((0,1)\,(1,1)\,(6,0)\,(2,0))-(4,0)$ $((0,2)\,(3,2)\,(7,1)\,(3,0))-(2,1)$
$((0,1)\,(2,1)\,(0,0)\,(3,0))-(2,0)$ $((0,2)\,(1,2)\,(1,0)\,(4,2))-(6,1)$
- $3K_5$ on the vertex set $\mathbb{Z}_5$. Here is a base block to be developed cyclically (mod 5): $(0,3,2,1)-4$ (mod 5)
- $3K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $(0,1,3,\infty)-2$ $(0,1,3,2)-5$
- $3K_{17}$ on the vertex set $\mathbb{Z}_{17}$. Here are the base blocks to be developed cyclically (mod 17): $(0,1,3,8)-5$ $(0,4,10,9)-3$ $(0,1,3,8)-1$ $(0,4,10,15)-11$
- $3K_{20}$ on the vertex set $\mathbb{Z}_{19} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 19): $(0,1,3,\infty)-2$ $(0,4,9,12)-6$ $(0,6,14,18)-11$ $(0,8,9,7)-3$ $(0,6,8,5)-14$
- $3K_{29}$ on the vertex set $\mathbb{Z}_{29}$. Here are the base blocks to be developed cyclically (mod 29): $(0,1,3,7)-17$ $(0,1,3,8)-18$ $(0,4,10,24)-11$ $(0,5,12,23)-10$ $(0,9,21,22)-18$ $(0,13,15,18)-9$ $(0,9,23,12)-4$
- $3K_{32}$ on the vertex set $\mathbb{Z}_{31} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 31): $(\infty,0,10,7)-4$ $(0,5,11,23)-9$ $(0,14,29,22)-15$ $(0,6,20,15)-6$ $(0,1,3,7)-11$ $(0,13,26,30)-15$ $(0,11,19,6)-14$ $(0,1,10,12)-22$
- $6K_{14}$ on the vertex set $\mathbb{Z}_{13} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 13): $(0,1,3,\infty)-4$ $(0,5,8,4)-1$ $(0,2,5,\infty)-6$ $(0,1,4,10)-6$ $(0,4,6,5)-7$ $(0,1,7,5)-10$ $(0,6,7,11)-10$
- $6K_{26}$ on the vertex set $\mathbb{Z}_{25} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 25): $(0,1,3,7)-\infty$ $(0,1,3,8)-\infty$ $(0,1,3,9)-\infty$ $(0,10,18,14)-\infty$
$(0,11,17,5)-\infty$ $(0,12,21,10)-\infty$ $(0,10,13,8)-9$ $(0,4,12,18)-13$
$(0,5,6,4)-11$ $(0,12,5,14)-4$ $(0,10,11,8)-17$ $(0,9,6,4)-16$ $(0,7,9,19)-8$

- K_{14}/K_2 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$ where the ∞'s are the points in the hole: $(\infty_1, 9, 6, 4) - 0$ $(\infty_1, 0, 2, 11) - 8$ $(\infty_1, 3, 1, 5) - 8$ $(\infty_1, 7, 10, 8) - 6$ $(\infty_2, 7, 4, 8) - 0$ $(\infty_2, 10, 1, 11) - 3$ $(\infty_2, 5, 2, 6) - 10$ $(\infty_2, 3, 0, 9) - 11$ $(0, 7, 6, 1) - 4$ $(4, 5, 10, 11) - 7$ $(5, 6, 11, 0) - 10$ $(7, 3, 5, 9) - 1$ $(1, 8, 7, 2) - 4$ $(2, 9, 8, 3) - 6$ $(3, 4, 9, 10) - 2$
- K_{15}/K_3 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, \infty_2, \infty_3\}$ where the ∞'s are the points in the hole: $(\infty_1, 0, 5, 10) - 2$ $(8, 3, \infty_1, 1) - 6$ $(\infty_1, 4, 11, 6) - 5$ $(\infty_1, 7, 2, 9) - 5$ $(\infty_2, 0, 2, 4) - 8$ $(\infty_2, 3, 5, 7) - 0$ $(8, 10, \infty_2, 6) - 2$ $(\infty_2, 1, 11, 9) - 8$ $(\infty_3, 0, 1, 2) - 3$ $(\infty_3, 3, 4, 5) - 11$ $(\infty_3, 6, 7, 8) - 0$ $(\infty_3, 9, 10, 11) - 3$ $(4, 6, 10, 0) - 11$ $(3, 7, 9, 1) - 5$ $(0, 3, 6, 9) - 4$ $(1, 4, 7, 10) - 3$ $(2, 5, 8, 11) - 7$
- K_{17}/K_5 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ where the ∞'s are the points in the hole: $(\infty_1, 0, 11, 8) - 4$ $(\infty_1, 10, 1, 6) - 5$ $(\infty_1, 9, 3, 4) - 6$ $(\infty_1, 2, 7, 5) - 9$ $(\infty_2, 8, 5, 4) - 1$ $(\infty_2, 9, 6, 2) - 10$ $(\infty_2, 0, 1, 7) - 9$ $(\infty_2, 3, 11, 10) - 8$ $(\infty_3, 4, 9, 1) - 5$ $(\infty_3, 11, 5, 3) - 10$ $(\infty_3, 8, 2, 0) - 3$ $(\infty_3, 7, 6, 10) - 0$ $(\infty_4, 3, 6, 0) - 5$ $(\infty_4, 1, 8, 9) - 0$ $(\infty_4, 5, 2, 11) - 9$ $(\infty_4, 7, 10, 4) - 11$ $(\infty_5, 5, 10, 9) - 2$ $(\infty_5, 0, 4, 7) - 11$ $(\infty_5, 8, 6, 11) - 1$ $(\infty_5, 3, 1, 2) - 4$ $(8, 0, 7, 3) - 2$
- K_{20}/K_8 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, ..., \infty_8\}$ where the ∞'s are the points in the hole: $(\infty_1, 2, 7, 11) - \infty_3$ $(\infty_1, 10, 0, 4) - \infty_4$ $(\infty_1, 3, 1, 5) - \infty_5$ $(\infty_2, 8, 11, 2) - \infty_4$ $(\infty_2, 4, 6, 10) - \infty_6$ $(\infty_2, 5, 3, 9) - \infty_1$ $(\infty_3, 6, 8, 0) - \infty_2$ $(\infty_3, 1, 4, 7) - \infty_7$ $(\infty_3, 5, 9, 2) - \infty_6$ $(\infty_4, 0, 9, 8) - \infty_1$ $(\infty_4, 7, 10, 3) - \infty_8$ $(\infty_4, 6, 5, 11) - \infty_7$ $(\infty_5, 10, 2, 6) - \infty_1$ $(\infty_5, 1, 7, 9) - \infty_8$ $(\infty_5, 3, 11, 4) - \infty_7$ $(\infty_6, 7, 8, 1) - \infty_2$ $(\infty_6, 11, 0, 5) - \infty_8$ $(\infty_6, 9, 4, 3) - \infty_3$ $(\infty_7, 2, 5, 10) - \infty_3$ $(\infty_7, 9, 1, 0) - \infty_5$ $(\infty_7, 8, 3, 6) - \infty_6$ $(\infty_8, 11, 10, 1) - \infty_4$ $(\infty_8, 4, 2, 8) - \infty_5$ $(\infty_8, 0, 6, 7) - \infty_2$ $(11, 9, 6, 1) - 2$ $(8, 5, 4, 10) - 9$ $(3, 2, 0, 7) - 5$
- $2K_{10}/2K_4$ on the vertex set $\mathbb{Z}_6 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ where the ∞'s are the points in the hole: $(\infty_1, 2, 4, 5) - \infty_2$ $(\infty_1, 1, 2, 0) - \infty_2$ $(\infty_1, 3, 4, 1) - \infty_4$ $(\infty_2, 4, 0, 3) - \infty_3$ $(\infty_2, 2, 3, 1) - \infty_4$ $(\infty_2, 1, 5, 2) - \infty_4$ $(\infty_3, 2, 1, 0) - \infty_1$ $(\infty_3, 4, 2, 3) - \infty_1$ $(\infty_3, 4, 1, 5) - \infty_1$ $(\infty_4, 2, 0, 4) - \infty_2$ $(\infty_4, 3, 5, 0) - \infty_3$ $(\infty_4, 3, 4, 5) - \infty_3$ $(3, 1, 0, 5) - 2$
- $2K_{11}/2K_5$ on the vertex set $\mathbb{Z}_6 \cup \{\infty_1, ..., \infty_5\}$ where the ∞'s are the points in the hole: $(\infty_1, 0, 2, 3) - \infty_4$ $(\infty_1, 1, 0, 4) - \infty_2$ $(\infty_1, 3, 4, 2) - \infty_3$ $(\infty_2, 0, 1, 4) - \infty_5$
  $(\infty_2, 1, 2, 5) - \infty_1$ $(\infty_2, 0, 5, 3) - \infty_5$ $(\infty_3, 0, 4, 5) - \infty_5$ $(\infty_3, 5, 1, 3) - \infty_2$ $(\infty_3, 2, 3, 4) - \infty_4$ $(\infty_4, 0, 5, 1) - \infty_1$ $(\infty_4, 1, 4, 2) - \infty_2$ $(\infty_4, 0, 3, 5) - \infty_1$ $(\infty_5, 4, 5, 2) - \infty_4$ $(\infty_5, 1, 3, 0) - \infty_3$ $(\infty_5, 0, 2, 1) - \infty_3$

H $K_4 - e$ and a Stick (type II)

- $2K_6$ on the vertex set $\mathbb{Z}_5 \cup \{\infty\}$. Here is a base block to be developed cyclically (mod 5): $(\infty, 0, 1, 3) - 2$

- $2K_7$ on the vertex set $\mathbb{Z}_7$. Here is a base block to be developed cyclically (mod 7): $(4,0,1,3)-5$
- $2K_{10}$ on the vertex set $\mathbb{Z}_5 \times \mathbb{Z}_2$. Here are the base blocks to be developed cyclically (mod (5,–)): $((0,0)\,(1,0)\,(3,0)\,(0,1))-(2,0)$
$((1,0)\,(2,0)\,(4,0)\,(0,1))-(3,0)$ $((0,0)\,(1,1)\,(3,1)\,(0,1))-(4,1)$
- $2K_{15}$ on the vertex set $\mathbb{Z}_7 \times \mathbb{Z}_2 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod (7,–)): $((2,0)\,(0,0)\,\infty\,(0,1))-(2,1)$
$((3,1)\,(0,1)\,\infty\,(0,0))-(3,0)$ $((4,0)\,(6,0)\,(1,1)\,(0,1))-(3,1)$
$((6,1)\,(0,1)\,(1,0)\,(2,0))-(3,1)$ $((2,1)\,(0,1)\,(1,0)\,(5,0))-(4,0)$
- $2K_{27}$ on the vertex set $\mathbb{Z}_9 \times \mathbb{Z}_3$. Here are the base blocks to be developed cyclically (mod (9,–)): $((0,1)\,(1,2)\,(3,0)\,(7,0))-(5,0)$
$((0,1)\,(2,1)\,(6,2)\,(1,0))-(0,2)$ $((0,2)\,(3,2)\,(2,0)\,(3,0))-(1,2)$
$((0,1)\,(1,1)\,(4,1)\,(4,2))-(7,0)$ $((0,1)\,(2,0)\,(2,2)\,(6,2))-(1,0)$
$((8,1)\,(8,0)\,(0,2)\,(2,2))-(6,2)$ $((0,2)\,(2,2)\,(5,2)\,(0,1))-(6,0)$
$((0,0)\,(3,0)\,(2,2)\,(3,1))-(1,0)$ $((0,2)\,(1,2)\,(6,0)\,(2,1))-(7,0)$
$((0,1)\,(1,1)\,(6,0)\,(2,0))-(4,2)$ $((0,1)\,(3,0)\,(1,0)\,(4,0))-(6,2)$
$((1,2)\,(2,2)\,(0,1)\,(4,1))-(5,2)$ $((8,0)\,(0,1)\,(2,1)\,(5,1))-(1,2)$
- $3K_5$ on the vertex set $\mathbb{Z}_5$. Here is a base block to be developed cyclically (mod 5): $(1,0,3,2)-4$
- $3K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $(0,\infty,1,3)-6$ $(0,1,3,2)-5$
- $3K_{17}$ on the vertex set $\mathbb{Z}_{17}$. Here are the base blocks to be developed cyclically (mod 17): $(0,1,3,10)-7$ $(0,4,10,9)-6$ $(0,1,3,8)-5$ $(0,4,10,15)-11$
- $3K_{20}$ on the vertex set $\mathbb{Z}_{19} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 19): $(0,\infty,9,3)-12$ $(0,4,9,12)-3$ $(0,6,14,18)-13$ $(0,8,9,7)-2$ $(0,2,8,17)-14$
- $3K_{29}$ on the vertex set $\mathbb{Z}_{29}$. Here are the base blocks to be developed cyclically (mod 29): $(0,1,3,7)-15$ $(0,1,3,8)-22$ $(0,4,10,24)-12$ $(0,23,10,20)-8$ $(0,9,21,22)-12$ $(0,13,15,18)-8$ $(0,8,22,11)-7$
- $3K_{32}$ on the vertex set $\mathbb{Z}_{31} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 31): $(0,1,3,7)-\infty$ $(0,5,11,23)-\infty$ $(0,14,29,22)-\infty$
$(0,6,20,15)-5$ $(0,1,3,8)-19$ $(17,0,3,13)-1$ $(0,3,7,18)-8$ $(0,1,10,12)-9$
- $6K_{14}$ on the vertex set $\mathbb{Z}_{13} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 13): $(0,\infty,3,1)-5$ $(0,5,7,1)-6$ $(0,\infty,5,2)-7$ $(0,5,6,3)-9$ $(0,11,7,6)-8$ $(0,4,8,5)-11$ $(0,10,4,1)-7$
- $6K_{26}$ on the vertex set $\mathbb{Z}_{25} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 25): $(0,1,3,7)-\infty$ $(0,1,3,8)-\infty$ $(0,1,3,9)-\infty$ $(0,10,13,17)-\infty$
$(0,11,17,5)-\infty$ $(0,12,21,10)-\infty$ $(0,10,13,8)-7$ $(0,4,12,18)-15$
$(0,3,4,13)-9$ $(0,12,5,14)-9$ $(0,3,7,12)-5$ $(0,3,14,13)-5$ $(0,4,9,15)-6$

- K_{14}/K_2 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$ where the ∞'s are the points in the hole: $(2,11,8,\infty_1)-0$ $(10,1,4,\infty_1)-5$ $(9,7,3,\infty_1)-6$ $(10,0,3,\infty_2)-11$ $(9,6,4,\infty_2)-2$ $(8,5,7,\infty_2)-1$ $(8,10,2,6)-3$ $(1,6,7,0)-9$ $(2,0,8,4)-7$ $(3,5,9,1)-11$ $(8,1,2,7)-11$ $(9,8,3,2)-5$ $(4,9,10,3)-11$ $(5,4,11,10)-7$ $(0,5,6,11)-9$

- K_{15}/K_3 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, \infty_2, \infty_3\}$ where the ∞'s are the points in the hole: $(4,\infty_1,6,11)-0$ $(7,\infty_1,9,2)-3$ $(0,\infty_1,10,5)-6$ $(1,\infty_1,3,8)-9$ $(0,\infty_2,4,2)-6$ $(3,\infty_2,7,5)-9$ $(6,\infty_2,10,8)-4$ $(1,\infty_2,9,11)-5$ $(0,\infty_3,2,1)-5$ $(3,\infty_3,5,4)-9$ $(6,\infty_3,8,7)-11$ $(9,\infty_3,11,10)-2$ $(6,4,0,10)-3$ $(1,9,7,3)-11$ $(3,0,9,6)-1$ $(4,1,10,7)-0$ $(5,2,11,8)-0$

- K_{17}/K_5 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, ..., \infty_5\}$ where the ∞'s are the points in the hole: $(8,9,0,\infty_1)-10$ $(11,2,7,\infty_1)-3$ $(5,6,1,\infty_1)-4$ $(7,6,11,\infty_2)-1$ $(8,3,0,\infty_2)-2$ $(10,9,4,\infty_2)-5$ $(0,1,10,\infty_3)-2$ $(9,6,3,\infty_3)-11$ $(8,7,4,\infty_3)-5$ $(0,11,8,\infty_4)-9$ $(1,4,3,\infty_4)-6$ $(2,5,10,\infty_4)-7$ $(7,10,11,\infty_5)-9$ $(8,5,4,\infty_5)-6$ $(1,2,3,\infty_5)-0$ $(2+i,6+i,8+i,i)-i+7$ for $0 \leq i \leq 5$

- K_{20}/K_8 on the vertex set $\mathbb{Z}_{12} \cup \{\infty_1, ..., \infty_8\}$ where the ∞'s are the points in the hole: $(9,11,6,\infty_1)-0$ $(10,5,3,\infty_1)-1$ $(7,8,4,\infty_1)-2$ $(11,10,6,\infty_2)-4$ $(9,3,1,\infty_2)-7$ $(5,0,2,\infty_2)-8$ $(10,9,5,\infty_3)-11$ $(1,2,7,\infty_3)-0$ $(6,4,3,\infty_3)-8$ $(1,0,8,\infty_4)-3$ $(11,4,10,\infty_4)-9$ $(7,6,2,\infty_4)-5$ $(8,6,1,\infty_5)-9$ $(11,0,4,\infty_5)-5$ $(10,2,3,\infty_5)-7$ $(8,3,10,\infty_6)-0$ $(9,4,2,\infty_6)-11$ $(7,5,1,\infty_6)-6$ $(5,11,1,\infty_7)-2$ $(9,7,3,\infty_7)-6$ $(0,10,8,\infty_7)-4$ $(7,11,3,\infty_8)-10$ $(8,9,1,\infty_8)-2$ $(6,5,4,\infty_8)-0$ $(3,6,9,0)-7$ $(4,7,10,1)-8$ $(5,8,11,2)-9$

- $2K_{10}/2K_4$ on the vertex set $\mathbb{Z}_6 \cup \{\infty_1, ..., \infty_4\}$ where the ∞'s are the points in the hole: $(2,1,5,3)-4$ $(0,5,4,\infty_1)-3$ $(3,1,4,\infty_1)-2$ $(1,0,2,\infty_1)-5$ $(3,0,4,\infty_2)-5$ $(2,3,4,\infty_2)-1$ $(1,0,5,\infty_2)-2$ $(0,2,5,\infty_3)-1$ $(0,3,5,\infty_3)-4$ $(1,4,2,\infty_3)-3$ $(1,5,2,\infty_4)-0$ $(0,4,5,\infty_4)-3$ $(1,2,4,\infty_4)-3$

- $2K_{11}/2K_5$ on the vertex set $\mathbb{Z}_6 \cup \{\infty_1, ..., \infty_5\}$ where the ∞'s are the points in the hole: $(1,0,4,\infty_1)-5$ $(0,2,3,\infty_1)-1$ $(3,4,2,\infty_1)-5$ $(0,1,4,\infty_2)-2$ $(0,5,3,\infty_2)-4$ $(1,2,5,\infty_2)-3$ $(0,4,5,\infty_3)-1$ $(5,1,3,\infty_3)-2$ $(2,3,4,\infty_3)-0$ $(0,3,5,\infty_4)-2$ $(0,5,1,\infty_4)-4$ $(1,4,2,\infty_4)-3$ $(0,3,1,\infty_5)-4$ $(4,5,2,\infty_5)-3$ $(0,2,1,\infty_5)-5$

I $\mathbf{K_{2,3}}$

- $2K_6$ on the vertex set $\mathbb{Z}_5 \cup \{\infty\}$. Here is a base block to be developed cyclically (mod 5): $(3,0,\infty)-(1,2)$

- $2K_7$ on the vertex set $\mathbb{Z}_7$. Here is a base block to be developed cyclically (mod 7): $(2,3,4)-(1,5)$

- $2K_9$ on the vertex set $\mathbb{Z}_3 \times \mathbb{Z}_3$. Here are the base blocks to be developed cyclically (mod (3,–)): $((2,1)\,(1,0)\,(1,2))-((0,0)\,(0,1))$

$((0,1)\,(1,2)\,(0,0)) - ((0,2)\,(2,0))$ $((1,1)\,(0,2)\,(1,2)) - ((2,1)\,(1,0))$
$((1,2)\,(0,1)\,(1,1)) - ((0,2)\,(0,0))$

- $2K_{10}$ on the vertex set $\mathbb{Z}_5 \times \mathbb{Z}_2$. Here are the base blocks to be developed cyclically (mod (5,–)): $((1,1)\,(2,1)\,(0,0)) - ((0,1)\,(1,0))$ $((1,1)\,(2,1)\,(1,0)) - ((0,1)\,(3,0))$ $((2,0)\,(2,1)\,(3,1)) - ((0,0)\,(1,0))$
- $2K_{12}$ on the vertex set $\mathbb{Z}_{11} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 11): $(0,5,\infty) - (1,2)$ $(0,5,7) - (1,2)$
- $2K_{15}$ on the vertex set $\mathbb{Z}_7 \times \mathbb{Z}_2 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod (7,–)): $(\infty,(0,1)\,(1,1)) - ((3,0)\,(5,0))$ $(\infty,(3,0)\,(0,0)) - ((0,1)\,(2,1))$ $((0,1)\,(1,1)\,(3,1)) - ((0,0)\,(2,0))$ $((2,i)\,(3,i)\,(4,i)) - ((1,i)\,(5,i))$ for $i \in \mathbb{Z}_2$
- $3K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 7): $(1,2,3) - (0,\infty)$ $(4,5,6) - (0,3)$
- $3K_9$ on the vertex set $\mathbb{Z}_9$. Here are the base blocks to be developed cyclically (mod 9): $(2,3,4) - (0,6)$ $(1,3,6) - (2,4)$
- $3K_{12}$ on the vertex set $\mathbb{Z}_{11} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 11): $(1,3,5) - (0,\infty)$ $(1,4,5) - (0,3)$ $(7,8,9) - (0,4)$
- $6K_5$ on the vertex set $\mathbb{Z}_5$. Here is a base block to be developed cyclically (mod 5): $(1,2,3) - (4,0)$ $(2,3,0) - (1,4)$
- $6K_{11}$ on the vertex set $\mathbb{Z}_{11}$. Here are the base blocks to be developed cyclically (mod 11): $(0,1,2) - (3,7)$ $(0,1,2) - (3,6)$ $(0,1,2) - (3,5)$ $(0,1,2) - (3,10)$ $(0,1,3) - (7,10)$
- $6K_{14}$ on the vertex set $\mathbb{Z}_{13} \cup \{\infty\}$. Here are the base blocks to be developed cyclically (mod 11): $(0,4,3) - (\infty,2)$ $(0,1,2) - (\infty,6)$ $(0,1,2) - (3,7)$ $(0,1,2) - (4,6)$ $(0,1,2) - (3,5)$ $(0,2,4) - (3,5)$ $(0,2,4) - (6,8)$

(Received 6/1/97; revised 7/7/98)